Control Systems

Chapter 1: Introduction to Control Systems

Objectives

In this chapter we describe a general process for designing a control system.

A control system consisting of interconnected components is designed to achieve a desired purpose. To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history. These early systems incorporated many of the same ideas of feedback that are in use today.

Modern control engineering practice includes the use of control design strategies for improving manufacturing processes, the efficiency of energy use, advanced automobile control, including rapid transit, among others.

We also discuss the notion of a design gap. The gap exists between the complex physical system under investigation and the model used in the control system synthesis.

The iterative nature of design allows us to handle the design gap effectively while accomplishing necessary tradeoffs in complexity, performance, and cost in order to meet the design specifications.

Introduction

System – An interconnection of elements and devices for a desired purpose.

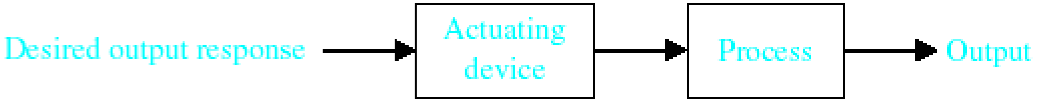
Control System – An interconnection of components forming a system configuration that will provide a desired response.

Process – The device, plant, or system under control. The input and output relationship represents the cause-and-effect relationship of the process.

Process to be controlled.

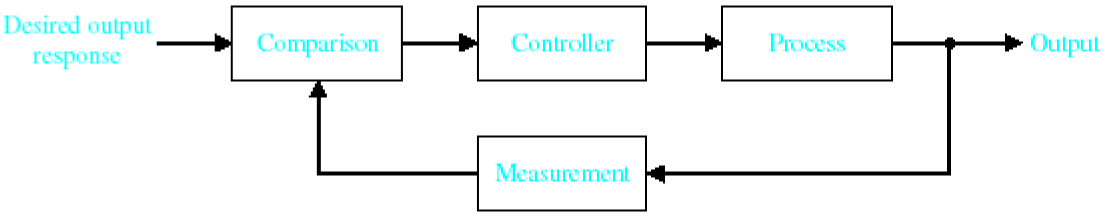
Introduction

Open-Loop Control Systems utilize a controller or control actuator to obtain the desired response.



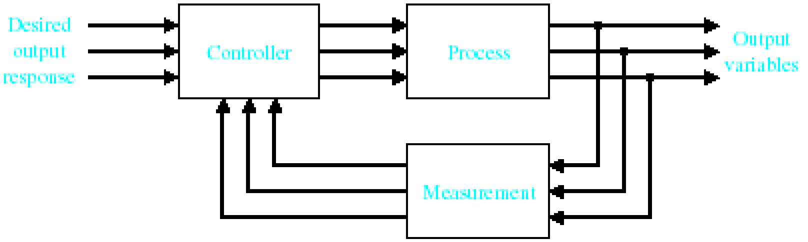
Open-loop control system (without feedback).

Closed-Loop Control Systems utilizes feedback to compare the actual output to the desired output response.



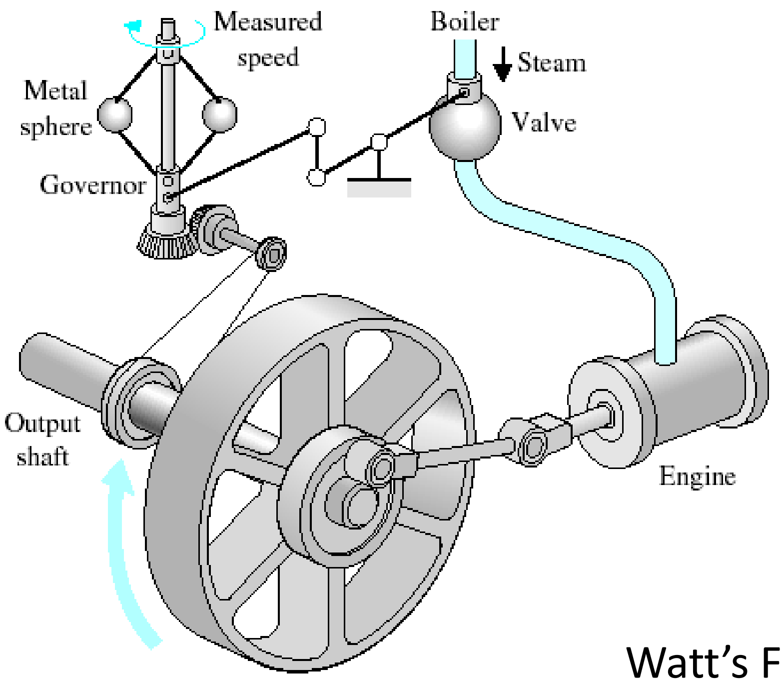
Closed-loop feedback control system (with feedback).

Multivariable Control System



History

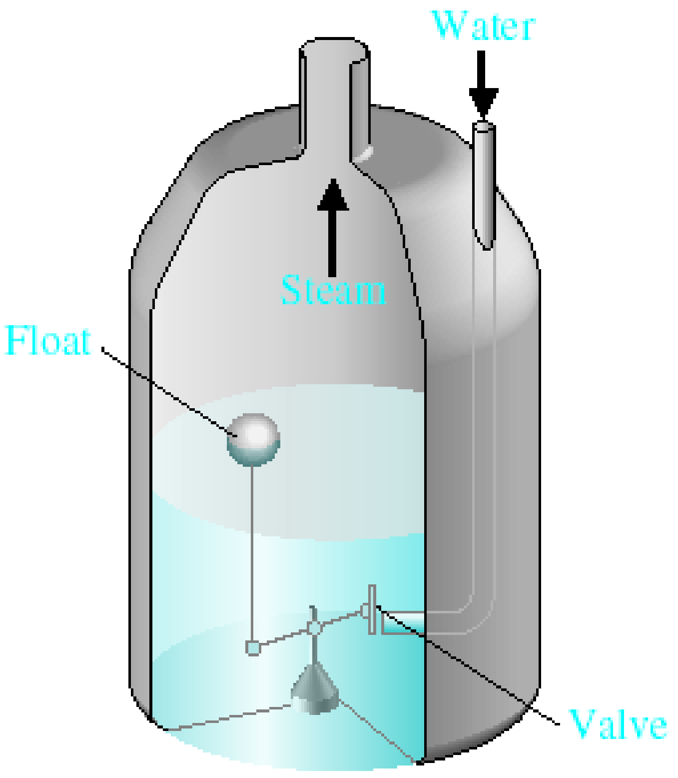
Greece (BC) – Float regulator mechanism
Holland (16th Century)– Temperature regulator



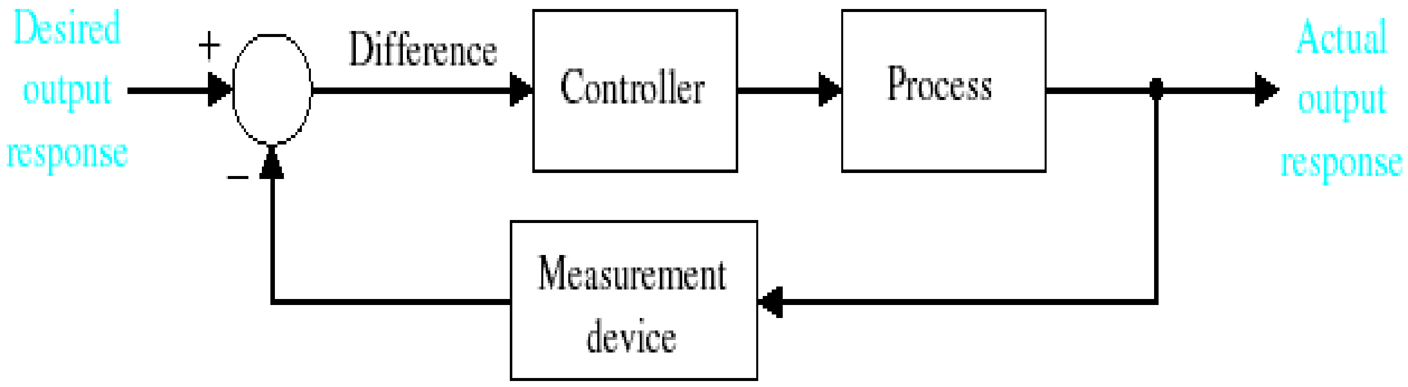
Watt's Flyball Governor
(18th century)

History

Water-level float regulator



History



Closed-loop feedback system.

History

18th Century James Watt's centrifugal governor for the speed control of a steam engine.

1920s Minorsky worked on automatic controllers for steering ships.

1930s Nyquist developed a method for analyzing the stability of controlled systems

1940s Frequency response methods made it possible to design linear closed-loop control systems

1950s Root-locus method due to Evans was fully developed

1960s State space methods, optimal control, adaptive control and

1980s Learning controls are begun to investigated and developed.

Present and on-going research fields. Recent application of modern control theory includes such non-engineering systems such as biological, biomedical, economic and socio-economic systems

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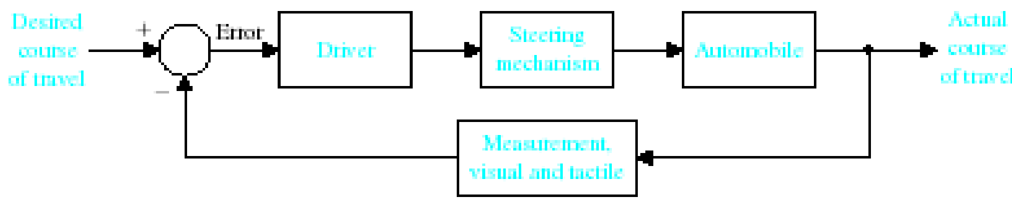
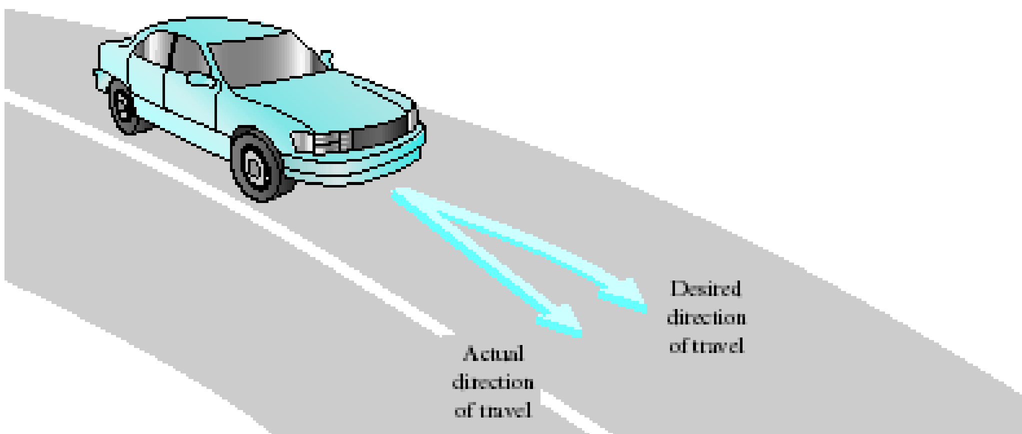
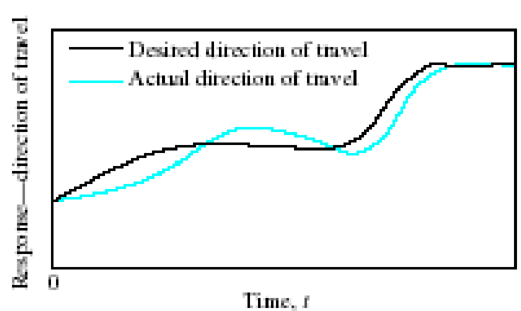
Examples of Modern Control Systems



(a)



(b)



(c)

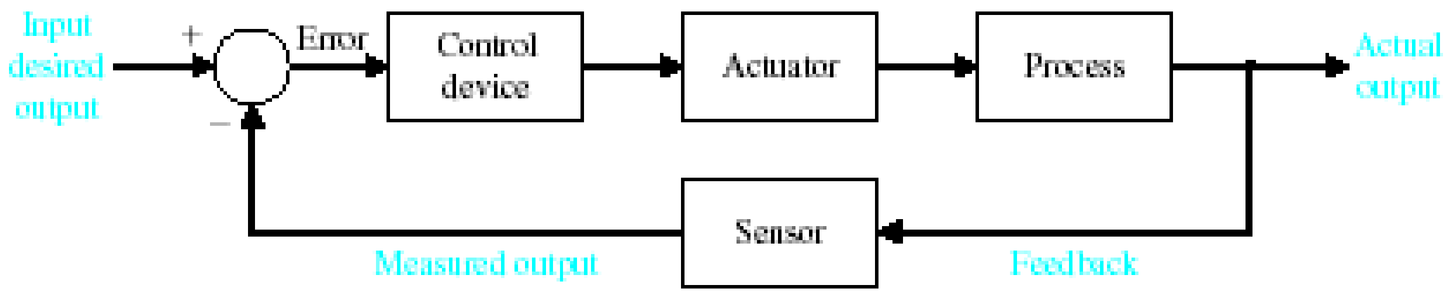
(a) Automobile steering control system.

(b) The driver uses the difference between the actual and the desired direction of travel

to generate a controlled adjustment of the steering wheel.

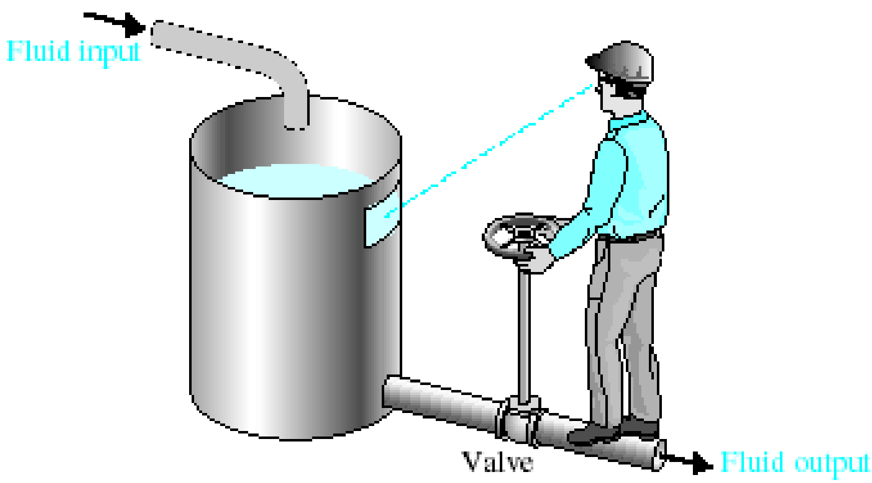
(c) Typical direction-of-travel response.

Examples of Modern Control Systems



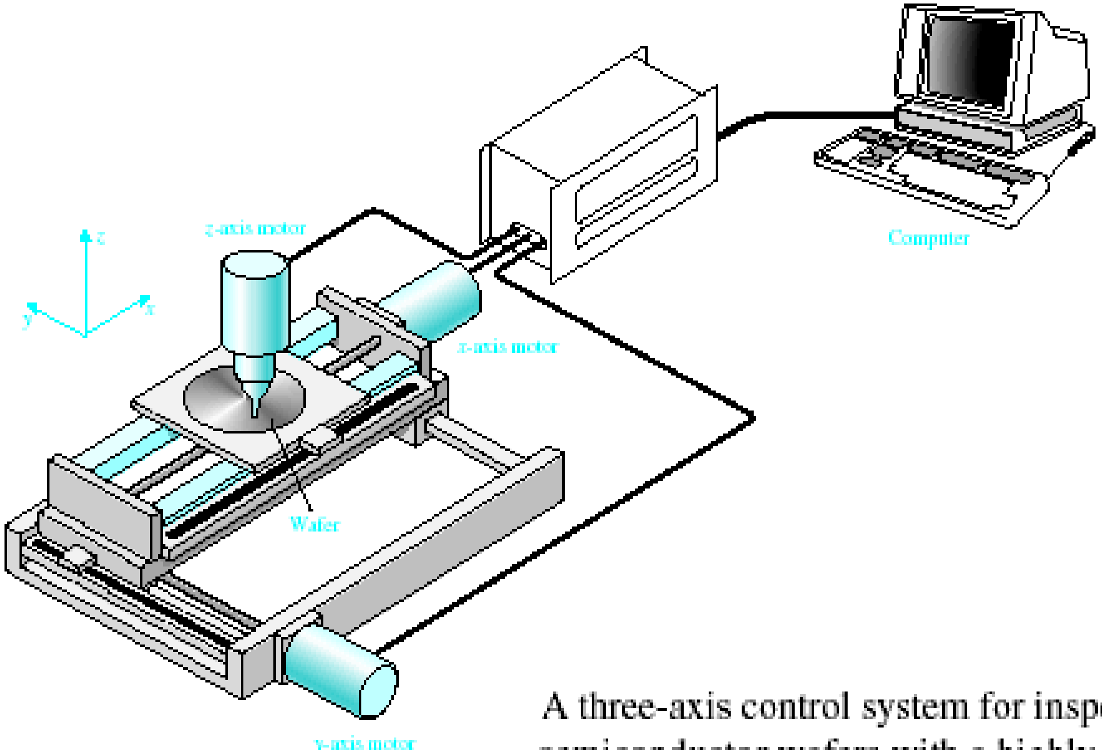
A negative feedback system block diagram depicting a basic closed-loop control system.
The control device is often called a "controller."

Examples of Modern Control Systems



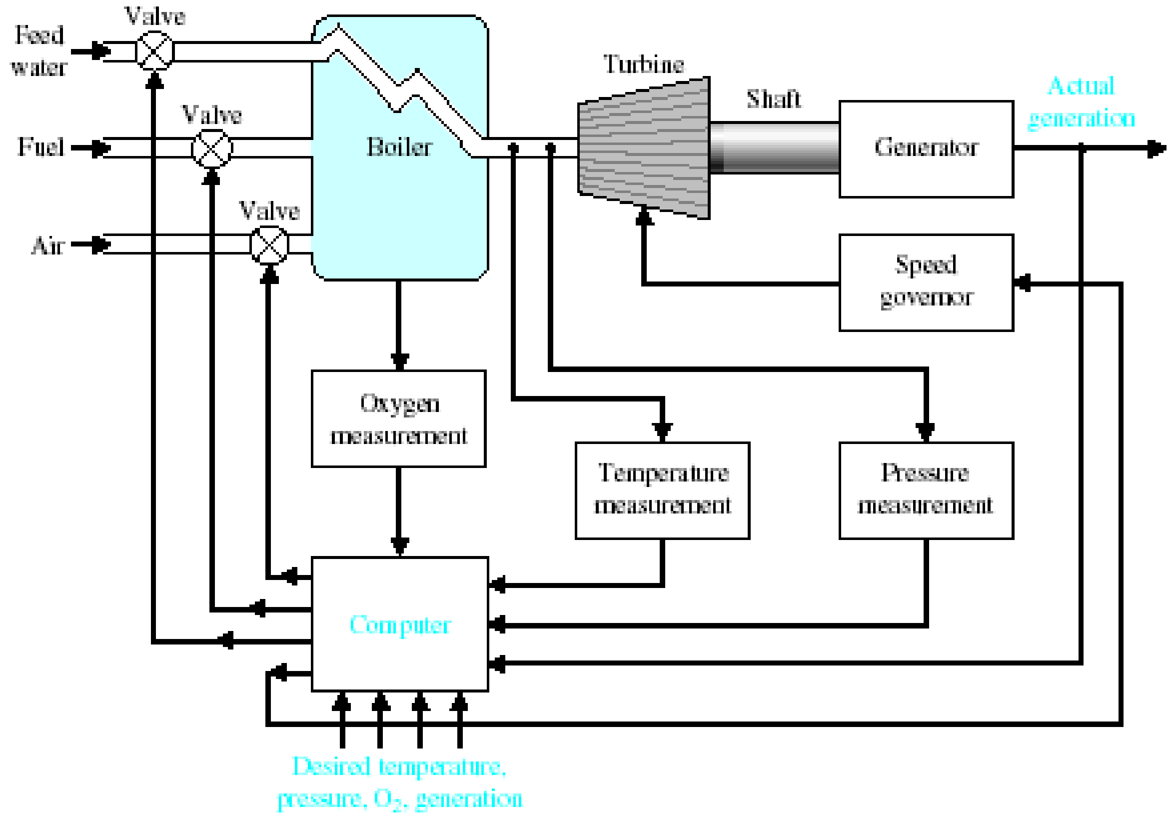
A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.

Examples of Modern Control Systems



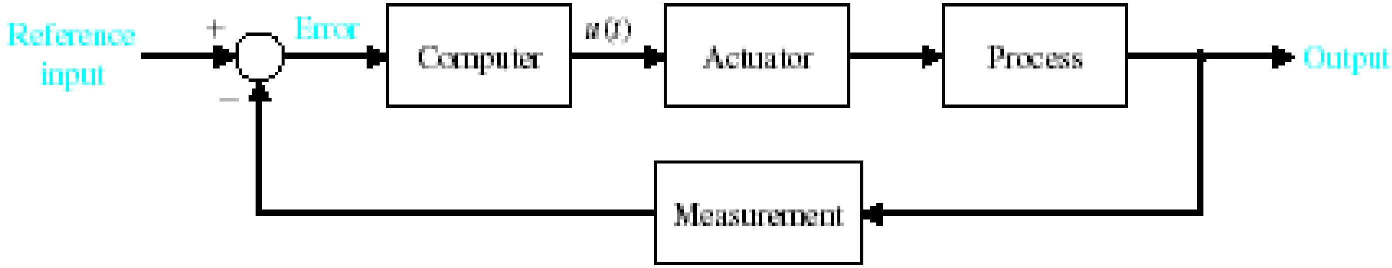
A three-axis control system for inspecting individual semiconductor wafers with a highly sensitive camera.

Examples of Modern Control Systems



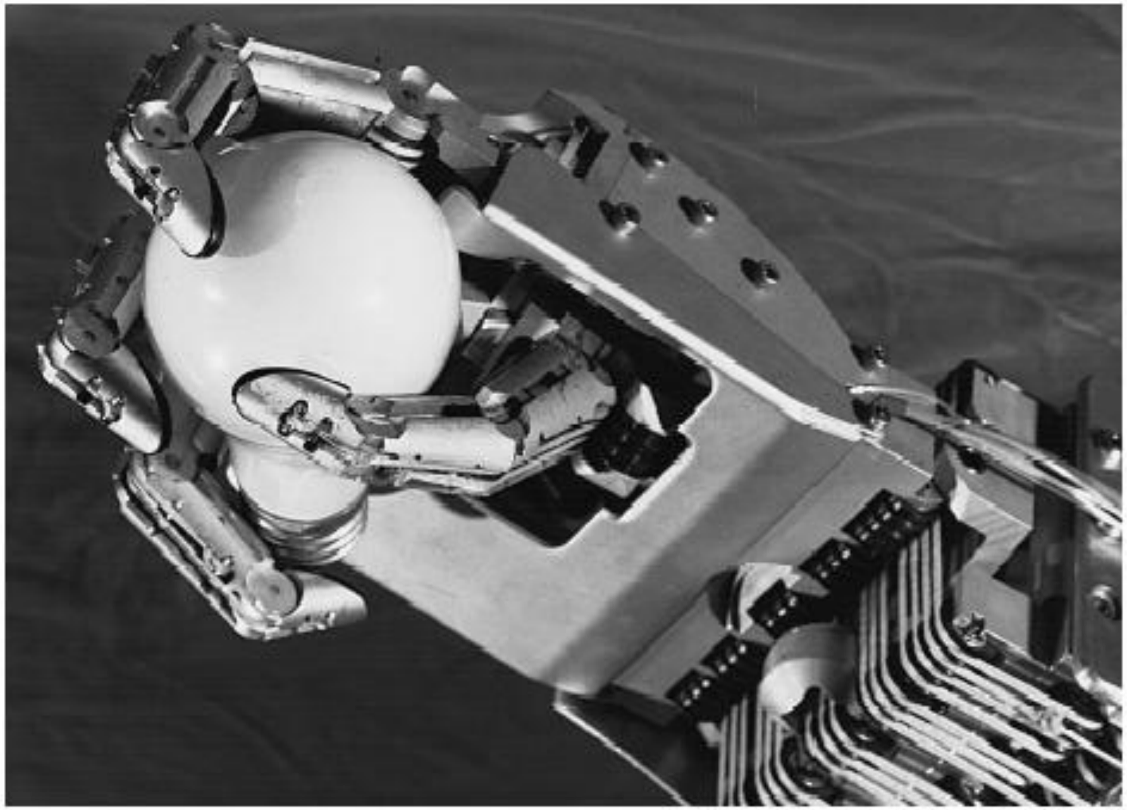
Coordinated control system for a boiler-generator.

Examples of Modern Control Systems



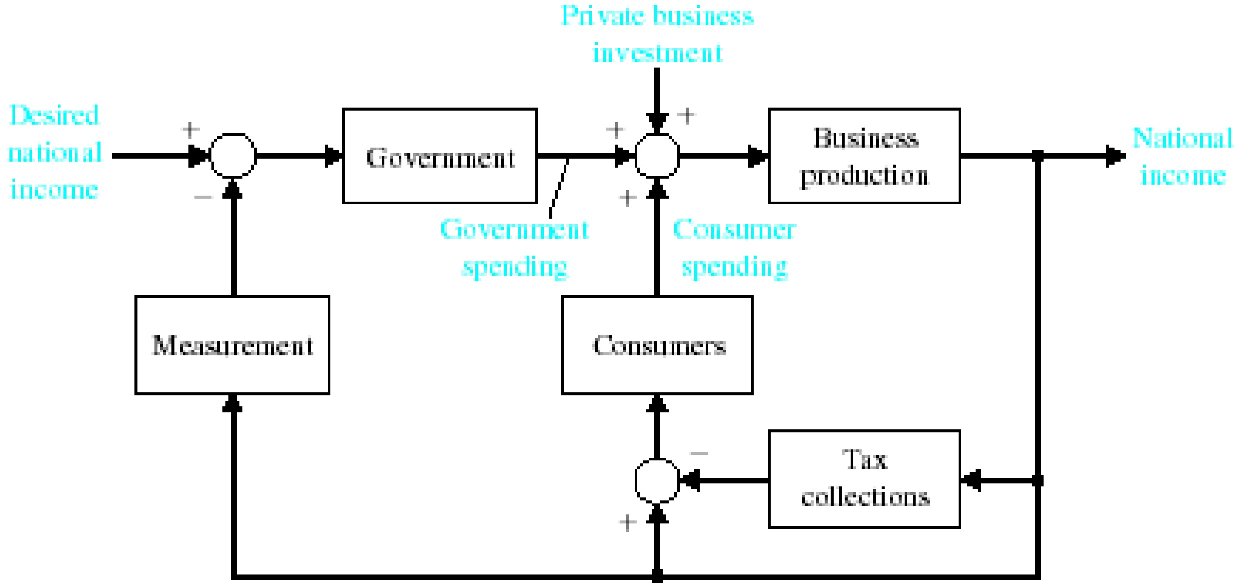
A computer control system.

Examples of Modern Control Systems



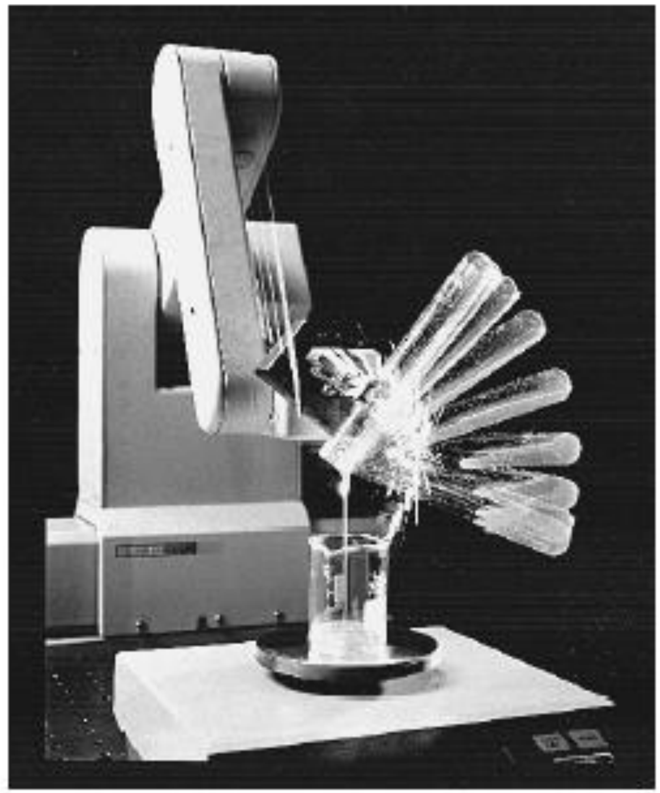
The Utah/MIT Dextrous Robotic Hand: A dextrous robotic hand having 18 degrees of freedom, developed as a research tool by the Center for Engineering Design at the University of Utah and the Artificial Intelligence Laboratory at MIT. It is controlled by five Motorola 68000 microprocessors and actuated by 36 high-performance electropneumatic actuators via high-strength polymeric tendons. The hand has three fingers and a thumb. It uses touch sensors and tendons for control. (Photograph by Michael Milochik. Courtesy of University of Utah.)

Examples of Modern Control Systems



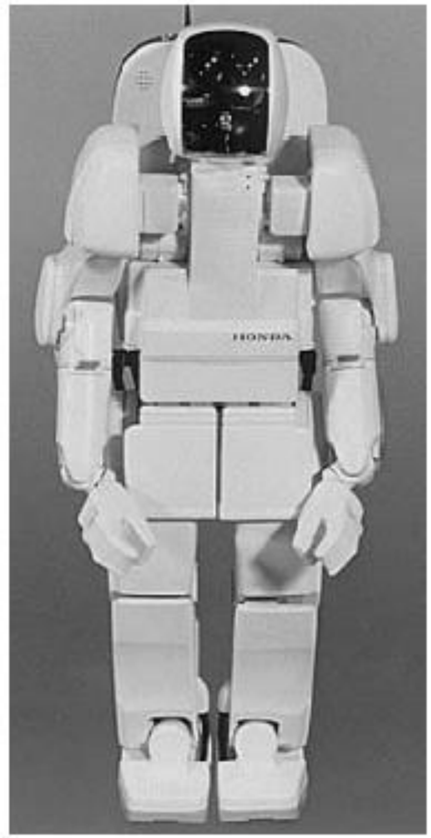
A feedback control system model of the national income.

Examples of Modern Control Systems



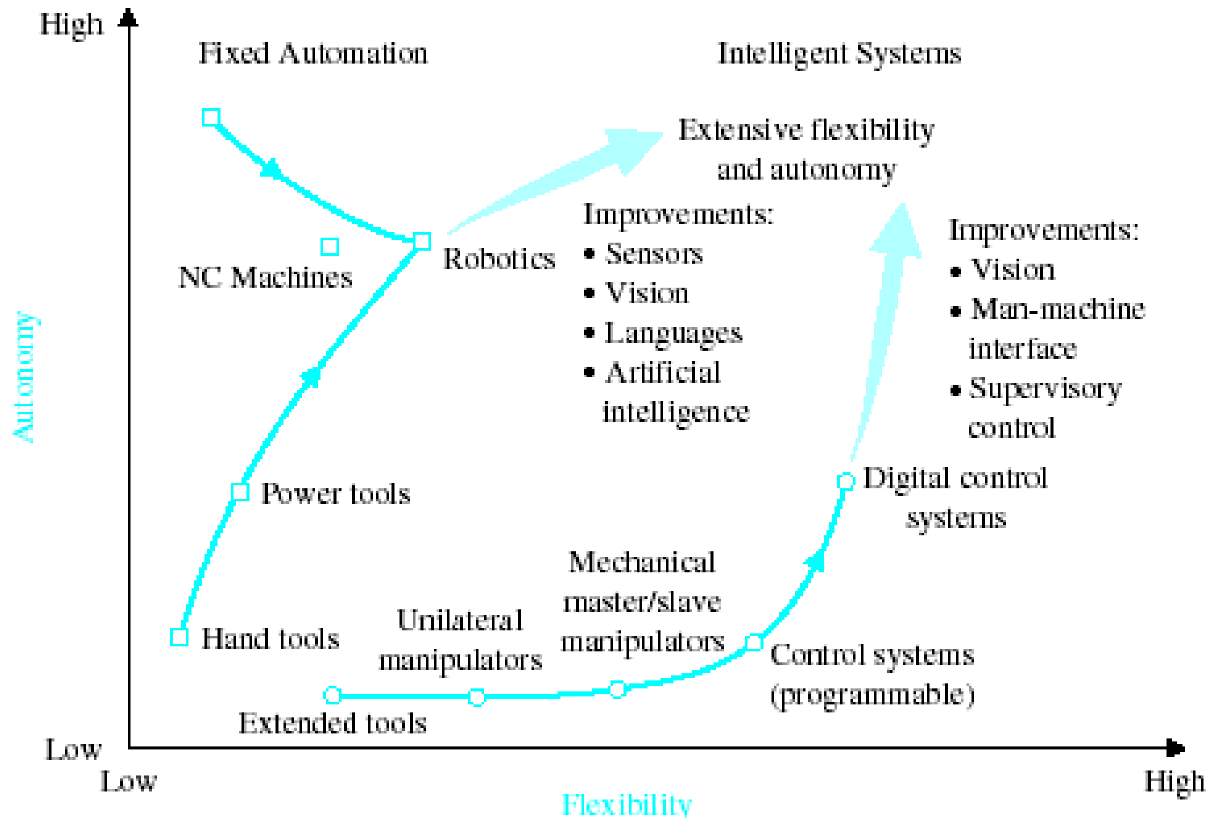
A laboratory robot used for sample preparation. The robot manipulates small objects, such as test tubes, and probes in and out of tight places at relatively high speeds [41].

The Future of Control Systems



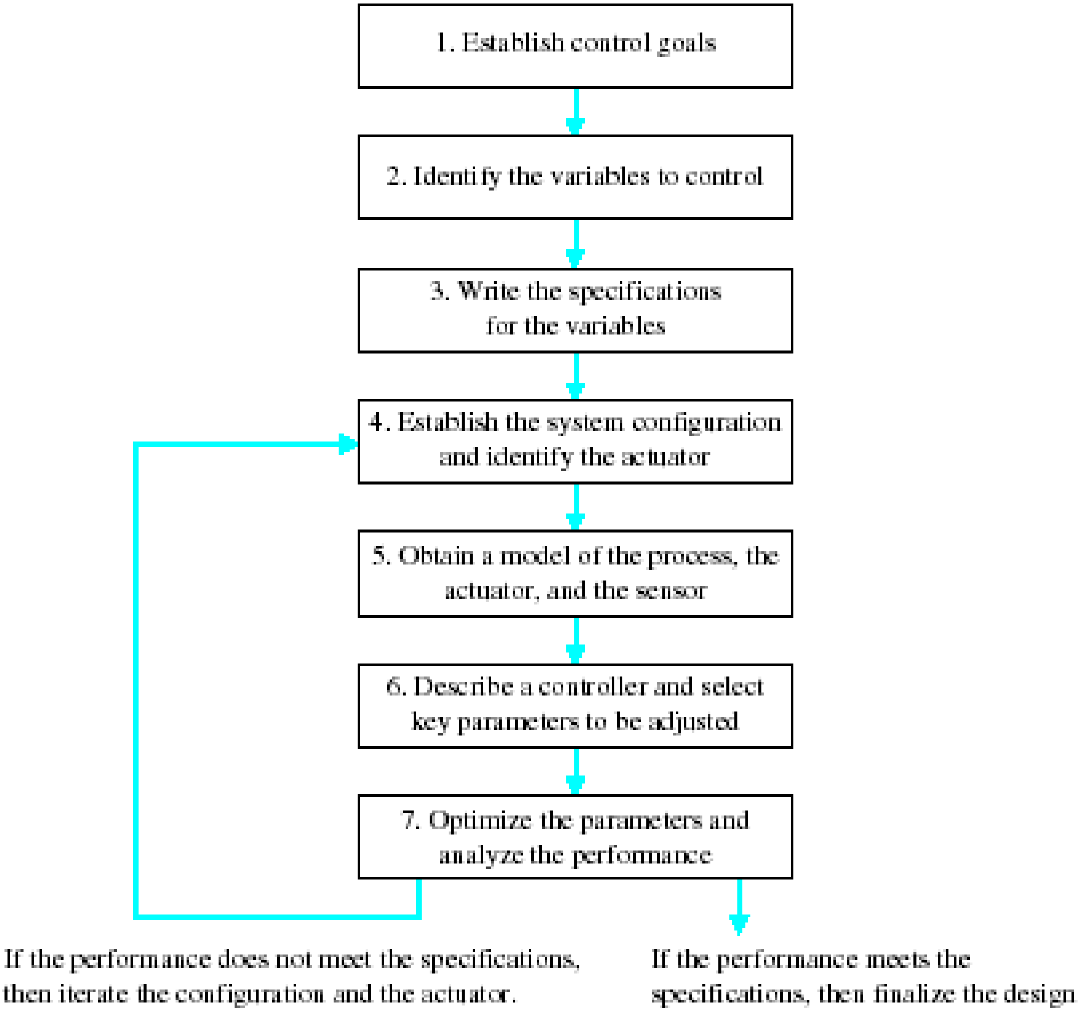
The Honda P3 humanoid robot. P3 walks, climbs stairs and turns corners.
Photo courtesy of American Honda Motor, Inc.

The Future of Control Systems

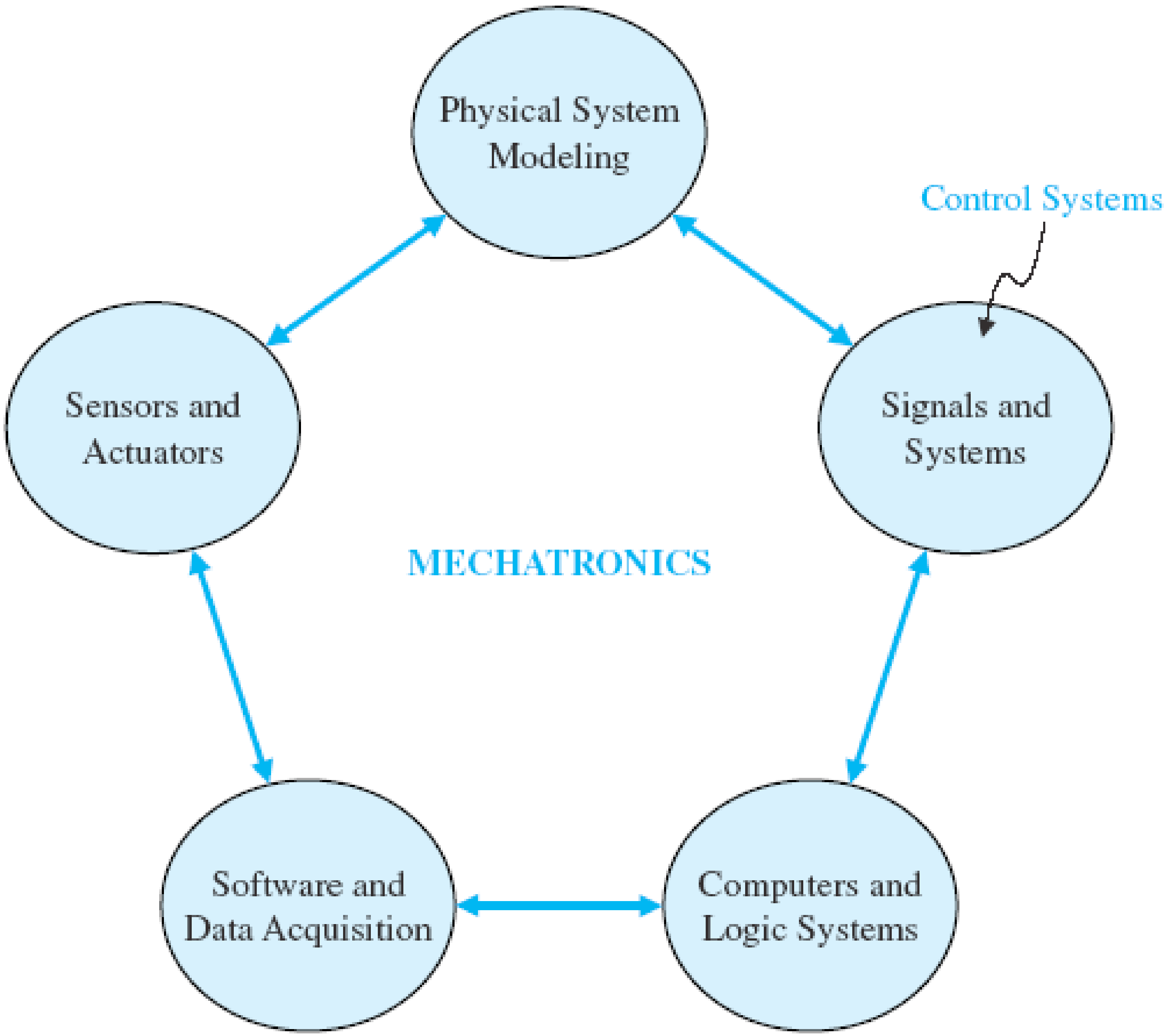


Future evolution of control systems and robotics.

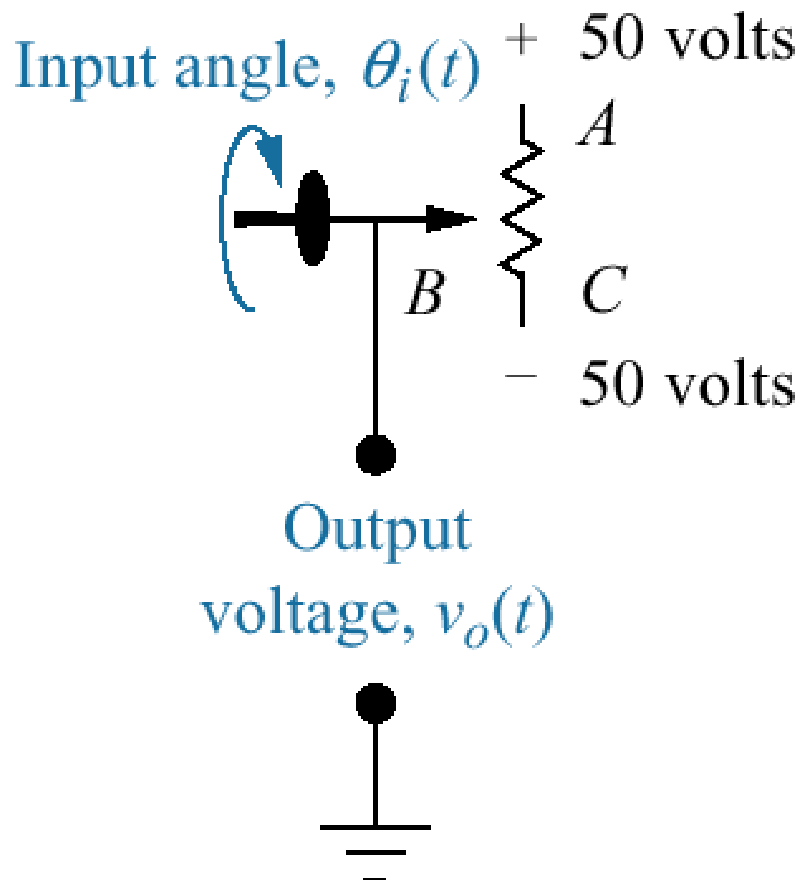
Control System Design



The control system design process.

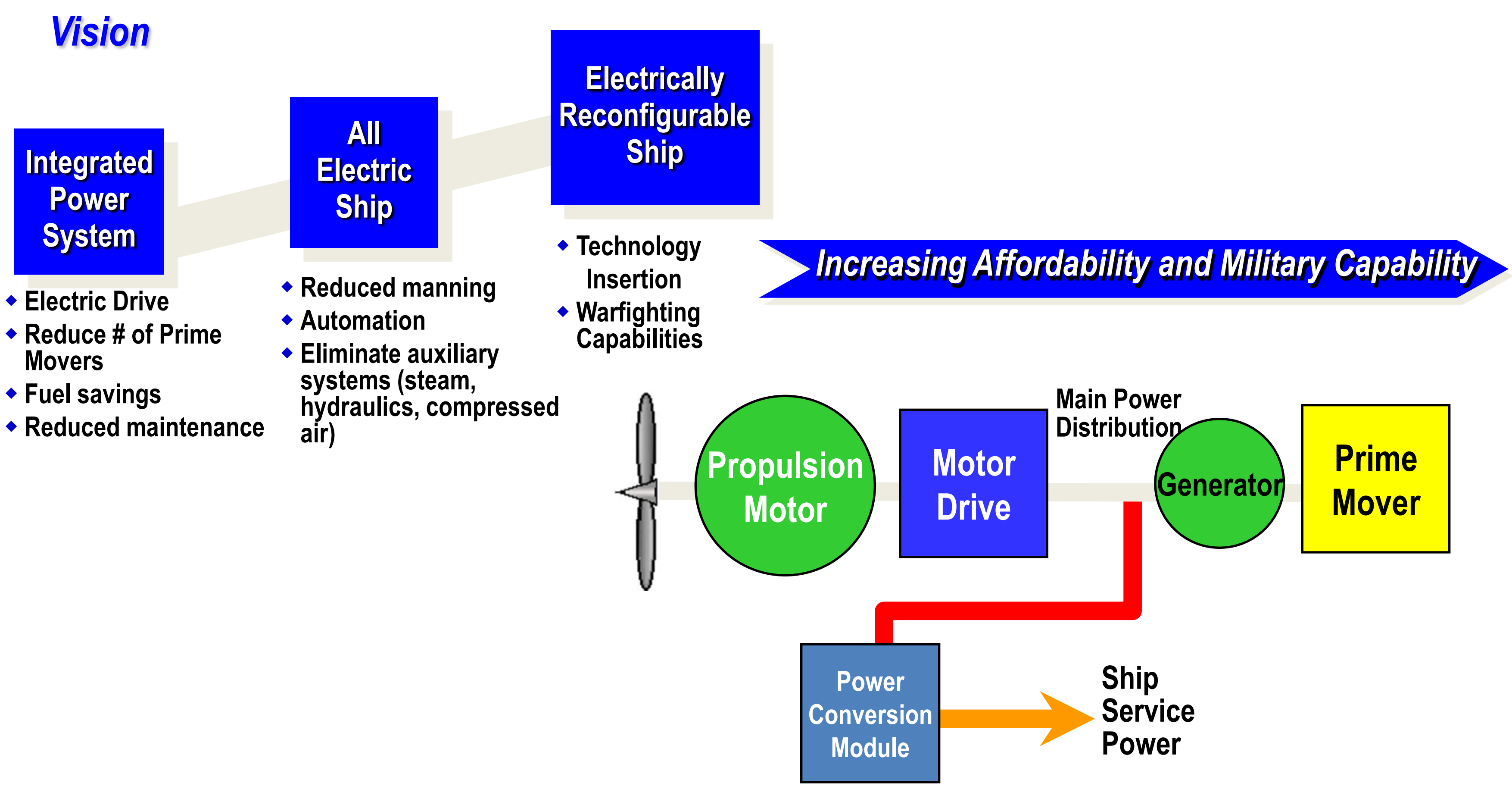

Physical System Modeling
Control Systems
Sensors and Actuators
Signals and Systems
MECHATRONICS
Software and Data Acquisition
Computers and Logic Systems

Design Example



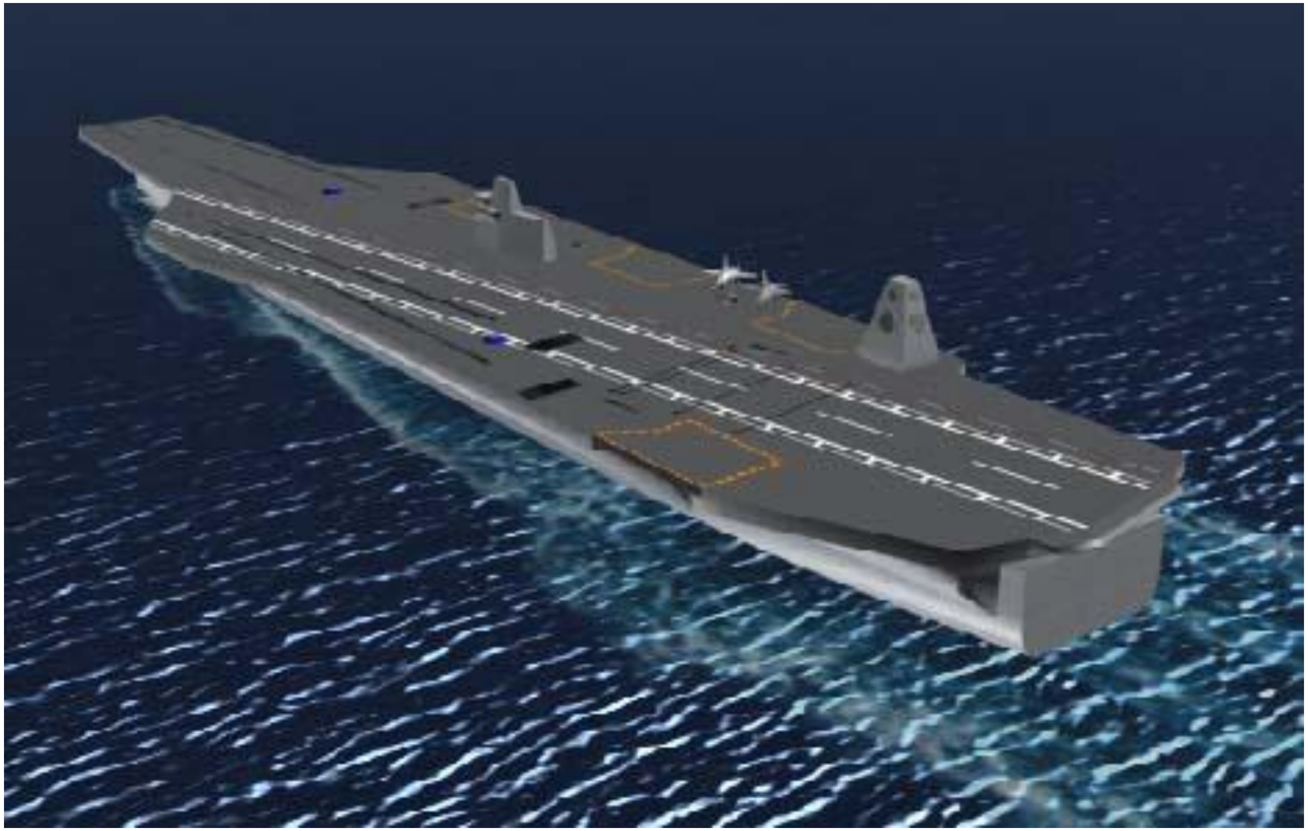
Design Example

ELECTRIC SHIP CONCEPT

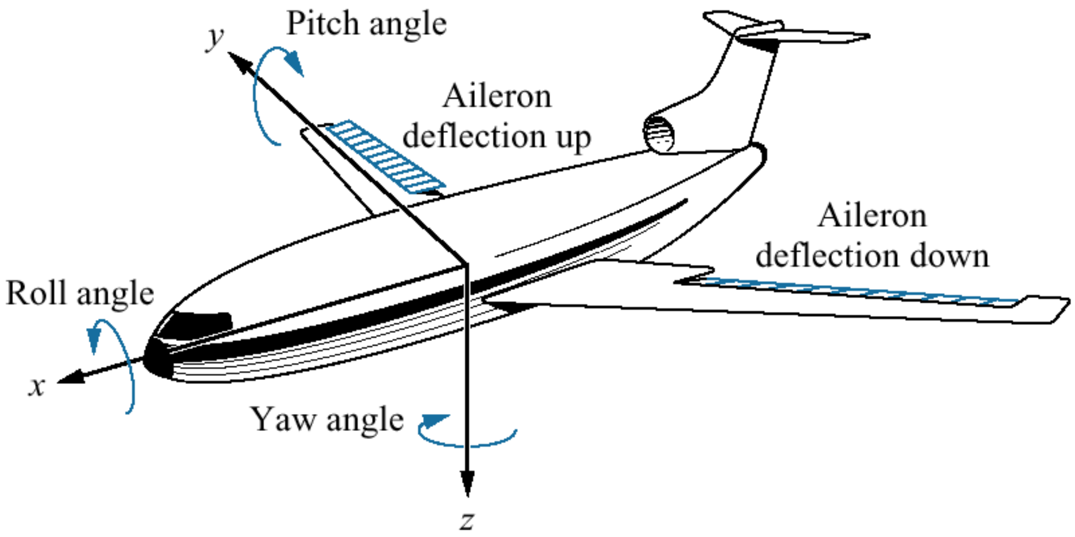


Design Example

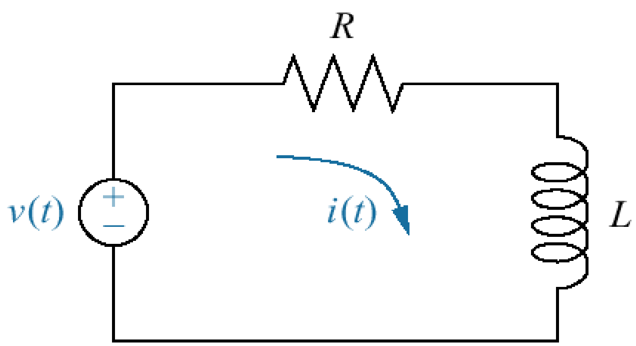
CVN(X) FUTURE AIRCRAFT CARRIER

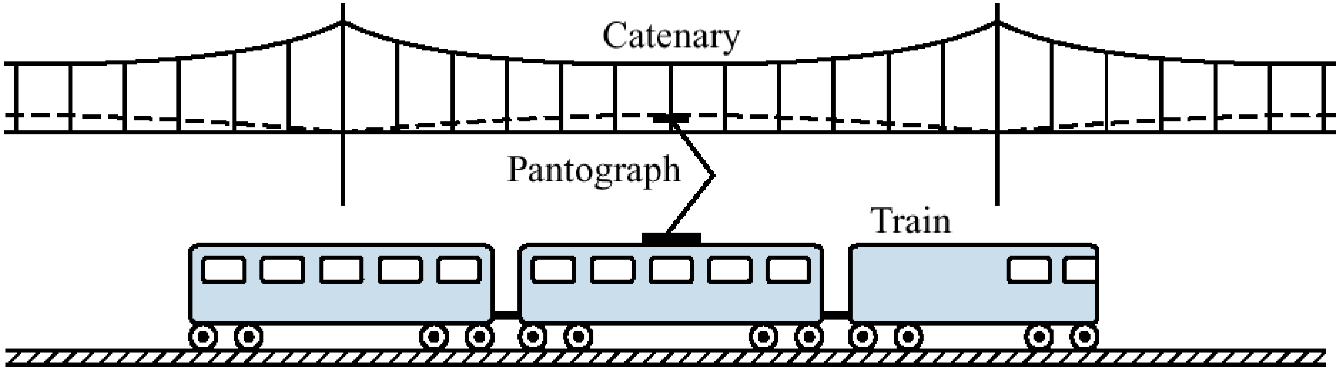


Design Example

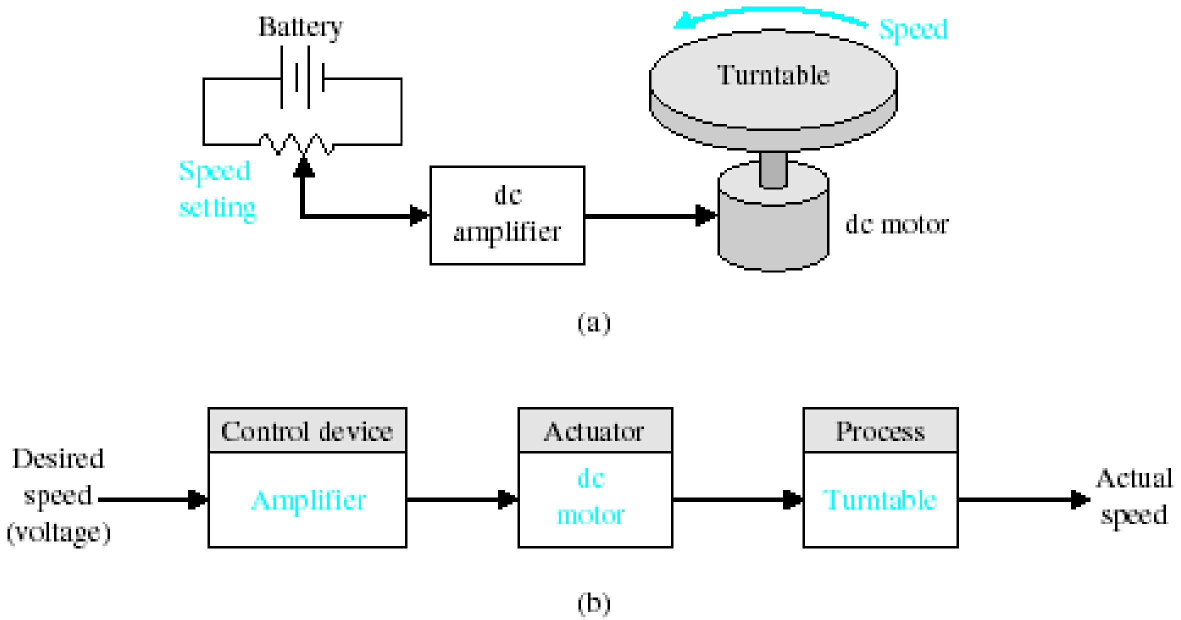


Design Example



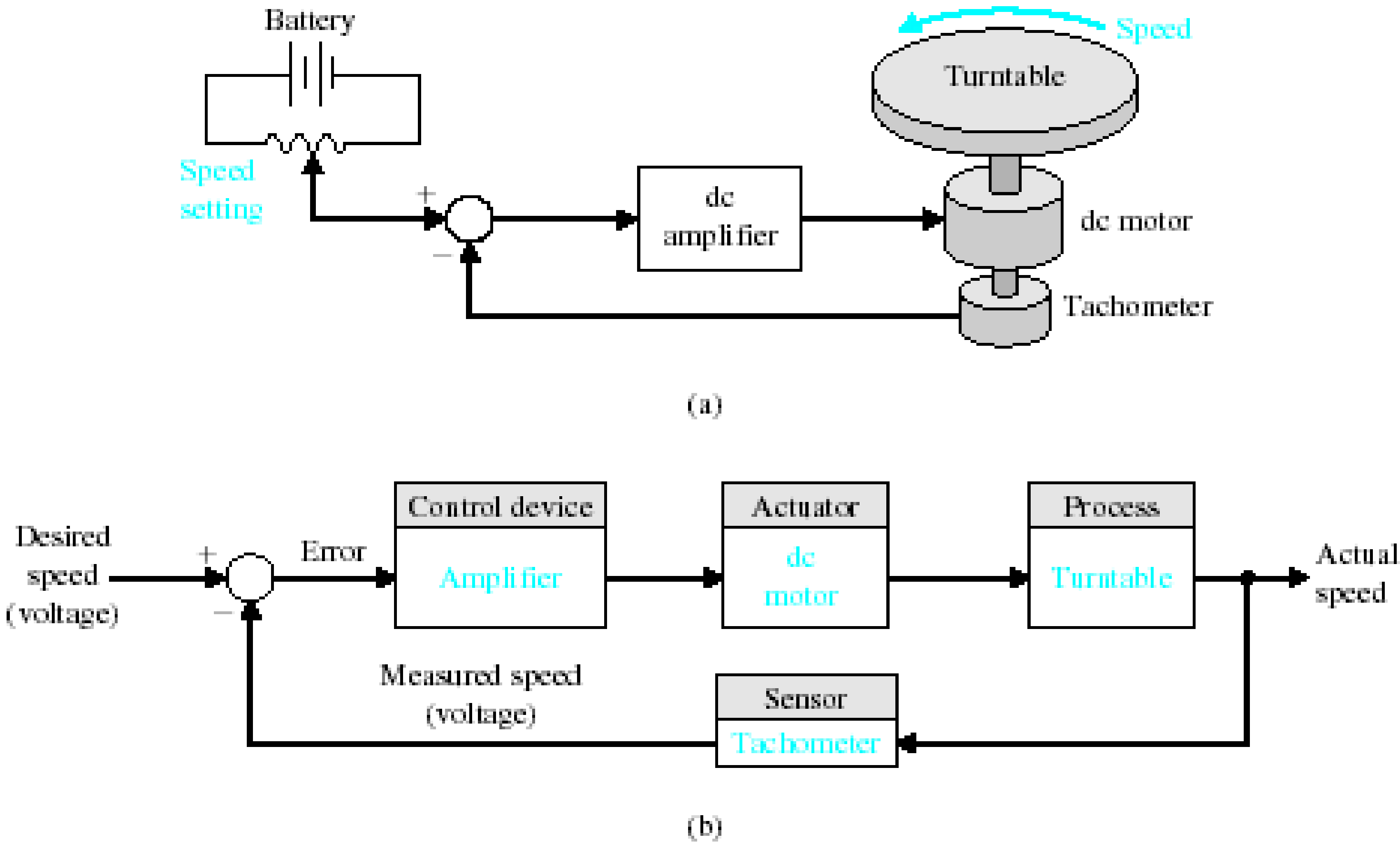


Design Example

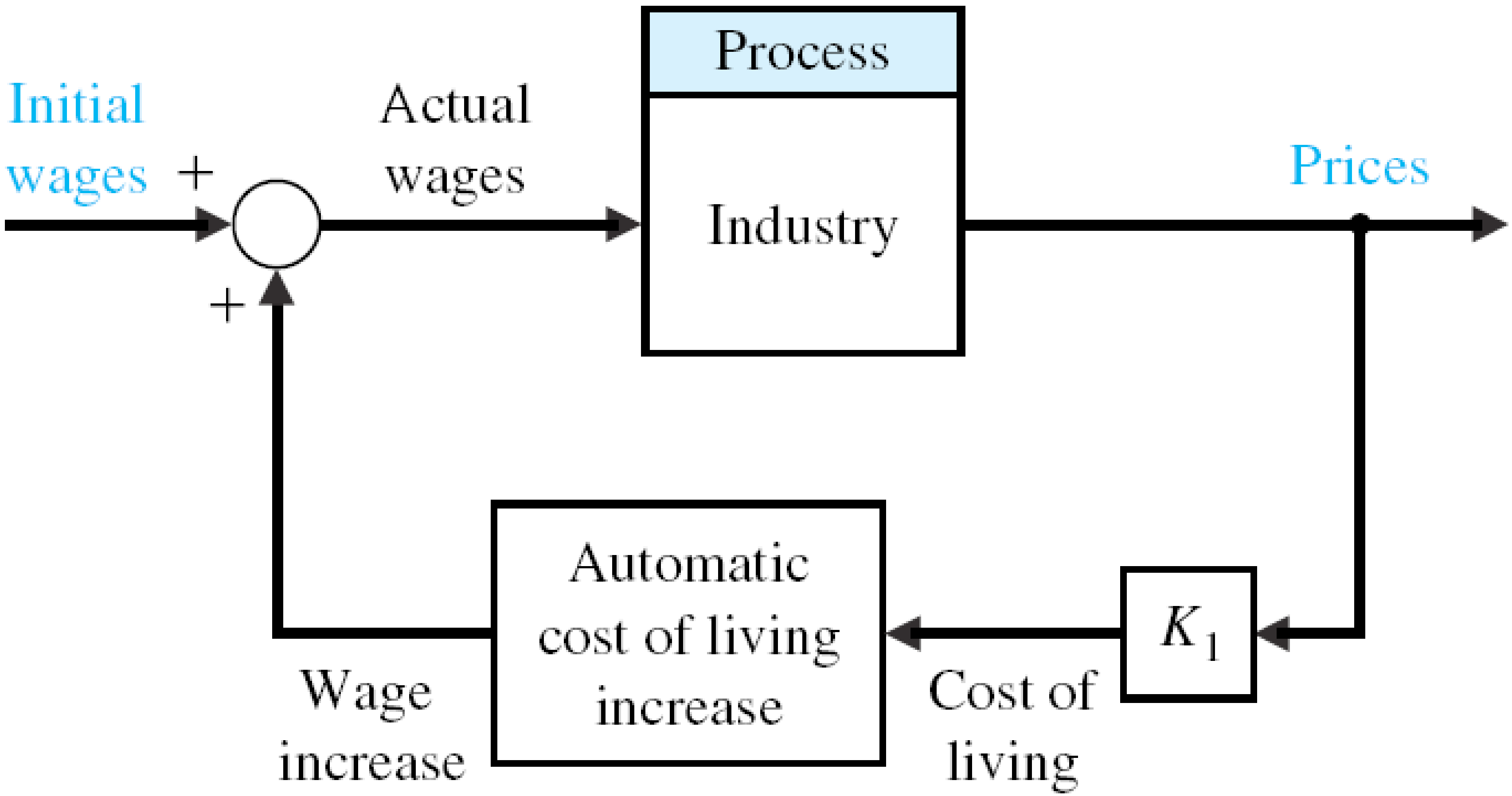


(a) Open-loop (without feedback) control of the speed of a turntable.
(b) Block diagram model.

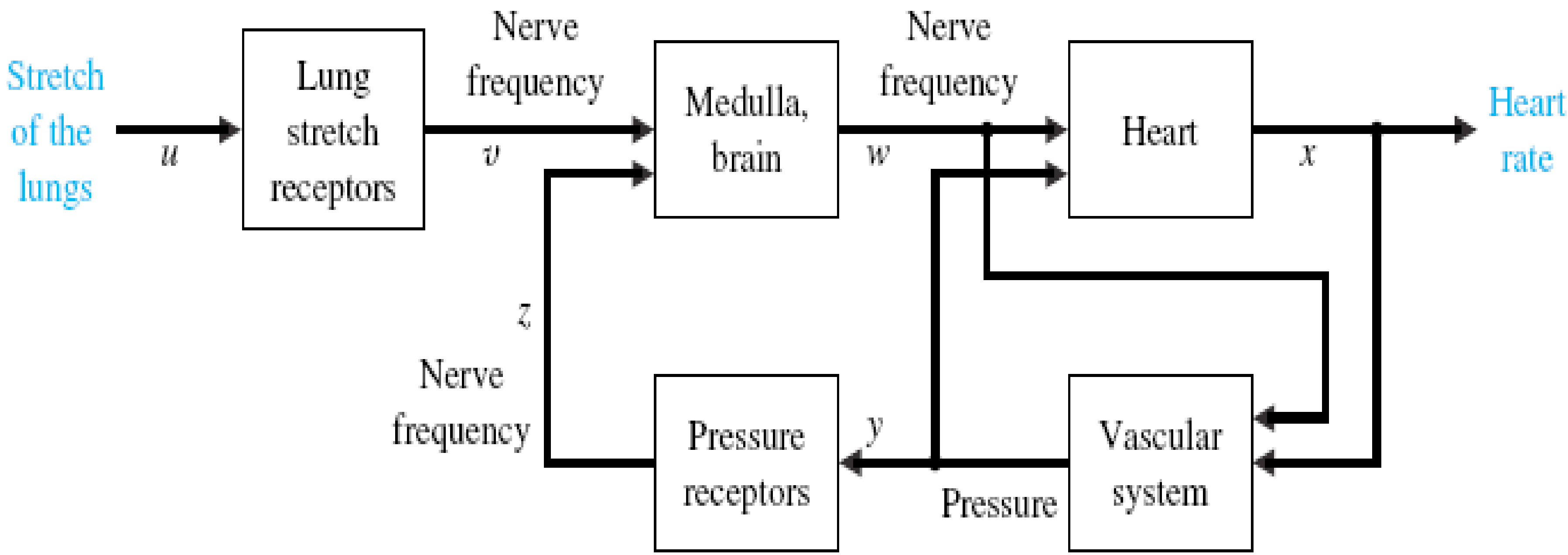
Design Example



(a) Closed-loop control of the speed of a turntable.
(b) Block diagram model.

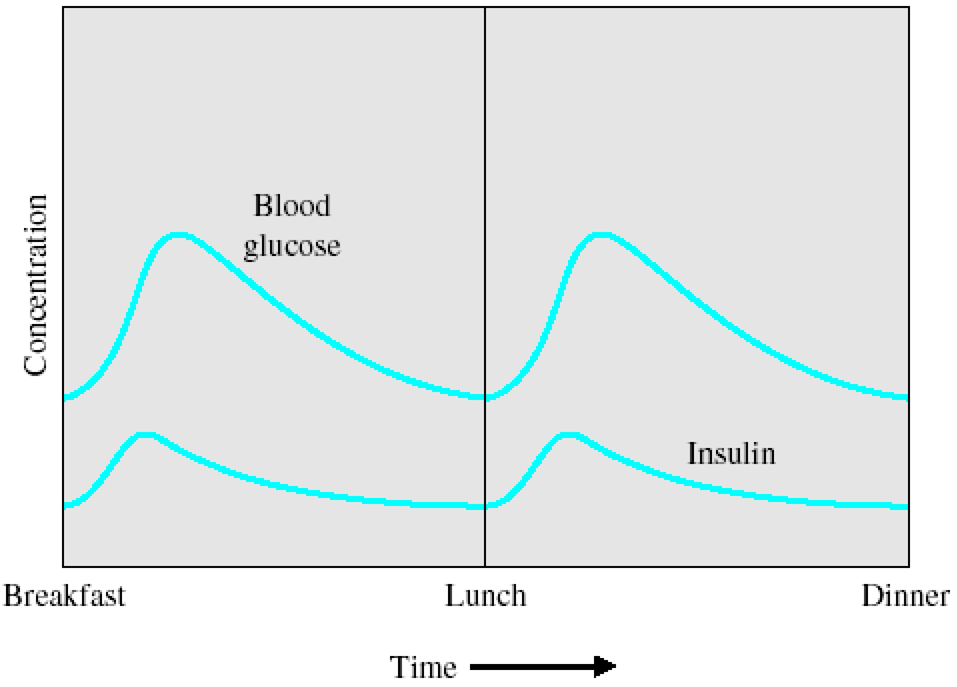


Initial wages
+
+
Actual wages
Process
Industry
Prices
Automatic cost of living increase
K_1
Wage increase
Cost of living



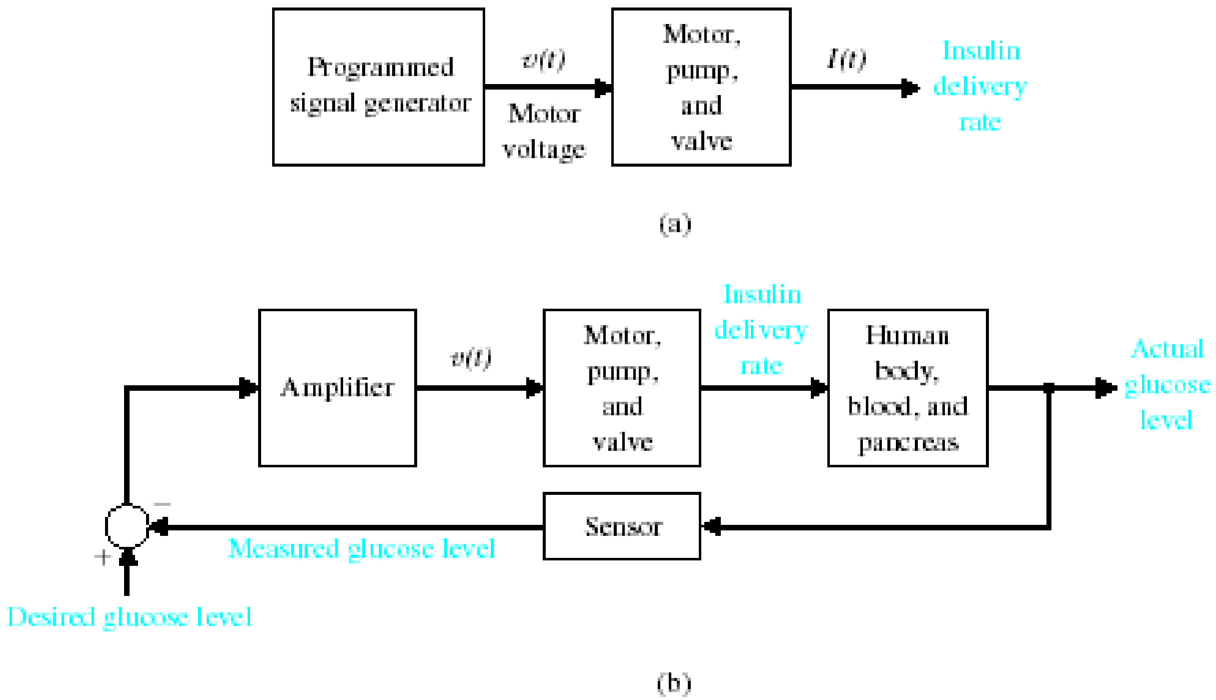
Stretch of the lungs
u
Lung stretch receptors
Nerve frequency
v
Medulla, brain
Nerve frequency
w
Heart
x
Heart rate
z
Nerve frequency
Pressure receptors
y
Pressure
Vascular system

Design Example



The blood glucose and insulin levels for a healthy person.

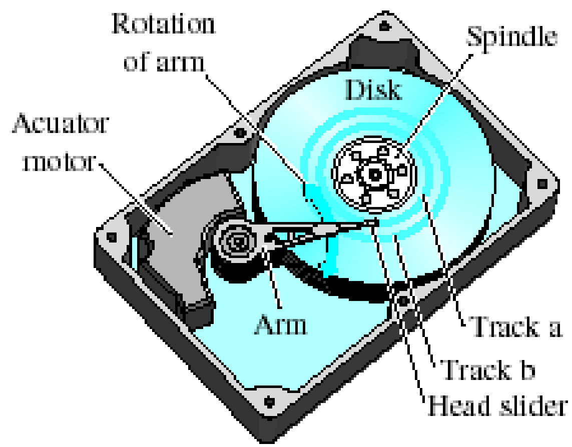
Design Example



(a) Open-loop (without feedback) control and (b) closed-loop control of blood glucose.

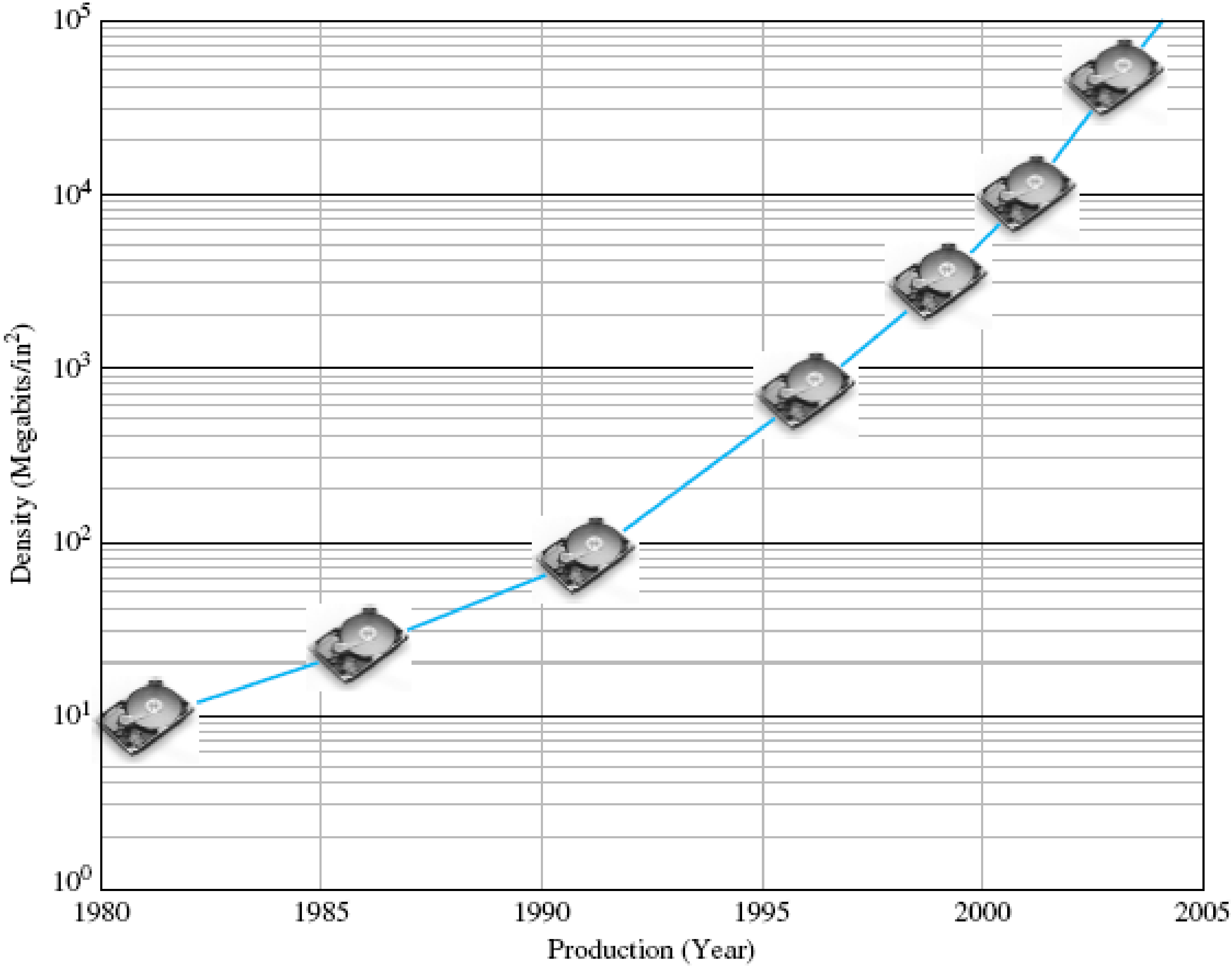
Sequential Design Example

(a)



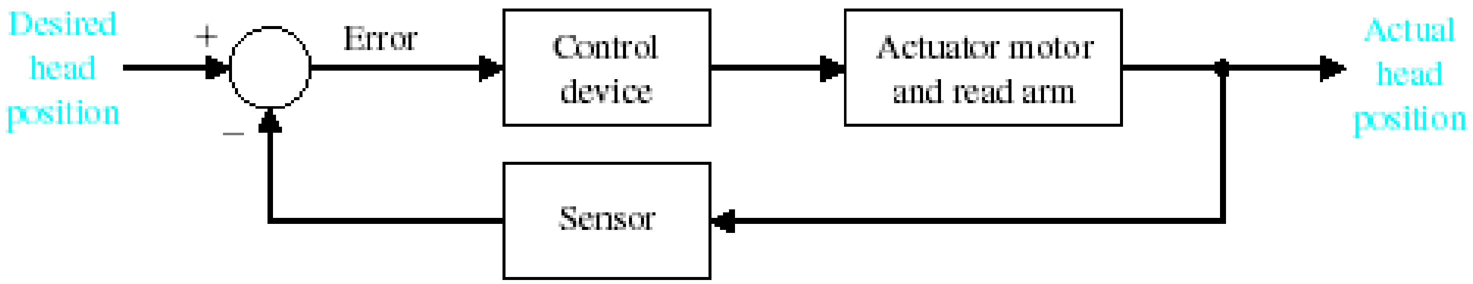
(b)

(a) A disk drive ©1999 Quantum Corporation. All rights reserved.
(b) Diagram of a disk drive.



10^5
10^4
10^3
10^2
10^1
10^0
Density (Megabits/in^2)
1980
1985
1990
1995
2000
2005
Production (Year)

Sequential Design Example



Closed-loop control system for disk drive.

Mathematical Models of Systems **Objectives**

We use quantitative mathematical models of physical systems to design and analyze control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems, including mechanical, hydraulic, and electrical. Since most physical systems are nonlinear, we will discuss linearization approximations, which allow us to use Laplace transform methods.

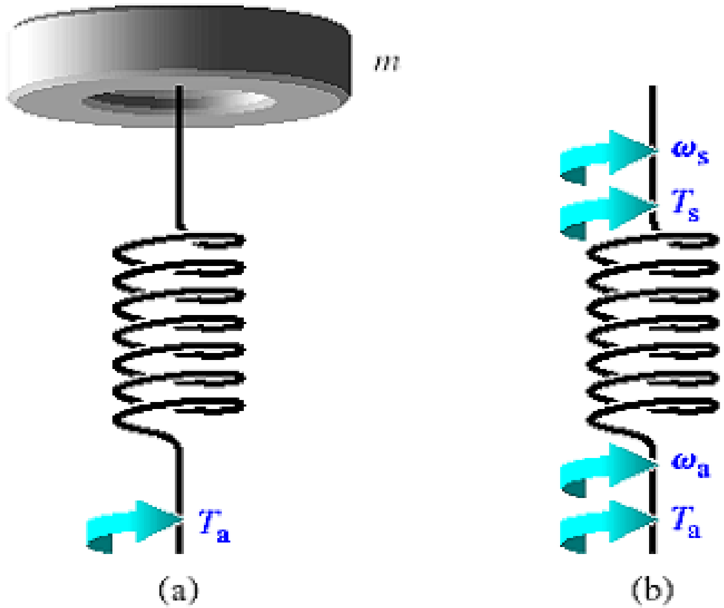
We will then proceed to obtain the input–output relationship for components and subsystems in the form of transfer functions. The transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams (and signal-flow graphs) are very convenient and natural tools for designing and analyzing complicated control systems

Introduction

Six Step Approach to Dynamic System Problems

Define the system and its components
Formulate the mathematical model and list the necessary assumptions
Write the differential equations describing the model
Solve the equations for the desired output variables
Examine the solutions and the assumptions
If necessary, reanalyze or redesign the system

Differential Equation of Physical Systems



(a) Torsional spring-mass system.
(b) Spring element.

$T_a(t) - T_s(t) = 0$

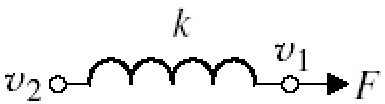
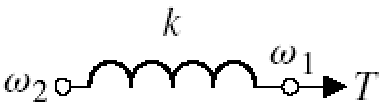
$T_a(t) = T_s(t)$

$\omega(t) = \omega_s(t) - \omega_a(t)$

$T_a(t)$ = through - variable

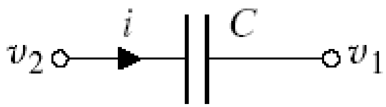
angular rate difference = across-variable

Differential Equation of Physical Systems

	Describing Equation	Energy or Power
Electrical Inductance 	$v_{21} = L \cdot \frac{d}{dt} i$	$E = \frac{1}{2} \cdot L \cdot i^2$
Translational Spring 	$v_{21} = \frac{1}{k} \cdot \frac{d}{dt} F$	$E = \frac{1}{2} \cdot \frac{F^2}{k}$
Rotational Spring 	$\omega_{21} = \frac{1}{k} \cdot \frac{d}{dt} T$	$E = \frac{1}{2} \cdot \frac{T^2}{k}$
Fluid Inertia 	$P_{21} = I \cdot \frac{d}{dt} Q$	$E = \frac{1}{2} \cdot I \cdot Q^2$

Differential Equation of Physical Systems

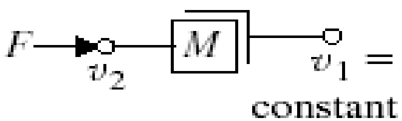
Electrical Capacitance



$$i = C \cdot \frac{d}{dt} v_{21}$$

$$E = \frac{1}{2} \cdot M \cdot {v_{21}}^2$$

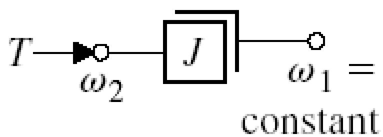
Translational Mass



$$F = M \cdot \frac{d}{dt} v_2$$

$$E = \frac{1}{2} \cdot M \cdot {v_2}^2$$

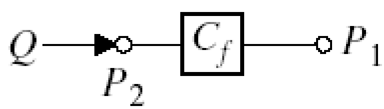
Rotational Mass



$$T = J \cdot \frac{d}{dt} \omega_2$$

$$E = \frac{1}{2} \cdot J \cdot {\omega_2}^2$$

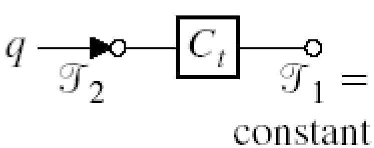
Fluid Capacitance



$$Q = C_f \cdot \frac{d}{dt} P_{21}$$

$$E = \frac{1}{2} \cdot C_f \cdot {P_{21}}^2$$

Thermal Capacitance

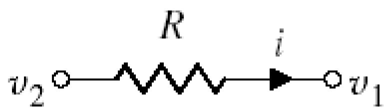


$$q = C_t \cdot \frac{d}{dt} T_2$$

$$E = C_t \cdot T_2$$

Differential Equation of Physical Systems

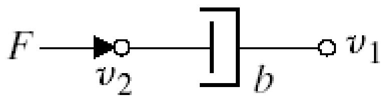
Electrical Resistance



$$i = \frac{1}{R} \cdot v_{21}$$

$$P = \frac{1}{R} \cdot v_{21}^{2}$$

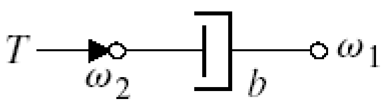
Translational Damper



$$F = b \cdot v_{21}$$

$$P = b \cdot v_{21}^{2}$$

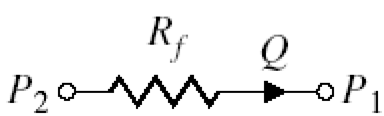
Rotational Damper



$$T = b \cdot \omega_{21}$$

$$P = b \cdot \omega_{21}^{2}$$

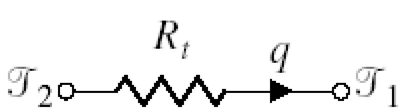
Fluid Resistance



$$Q = \frac{1}{R_f} \cdot P_{21}$$

$$P = \frac{1}{R_f} \cdot P_{21}^{2}$$

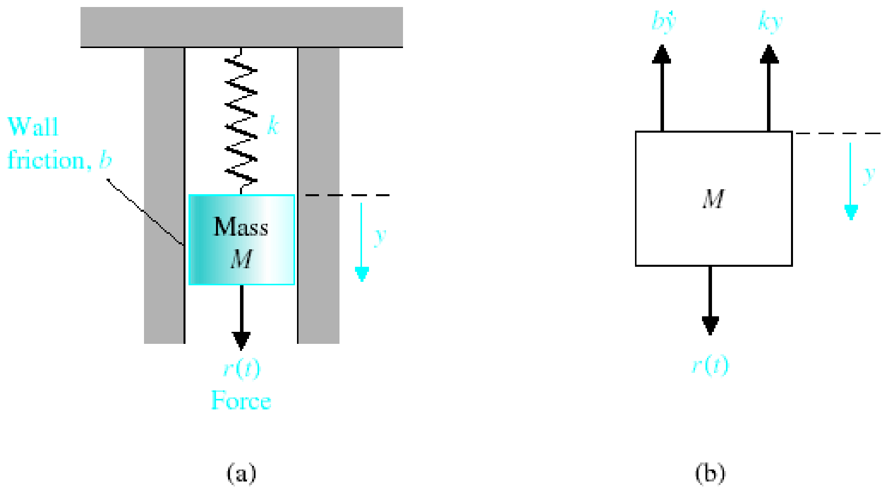
Thermal Resistance



$$q = \frac{1}{R_t} \cdot T_{21}$$

$$P = \frac{1}{R_t} \cdot T_{21}$$

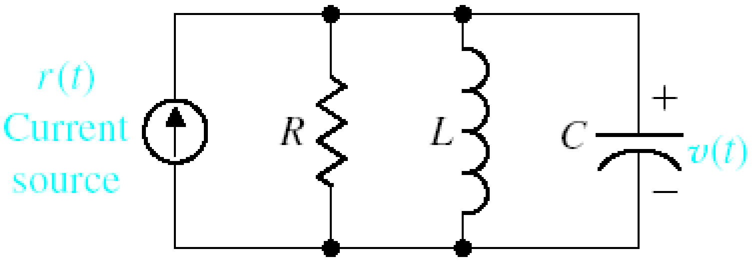
Differential Equation of Physical Systems



$$M\cdot\frac{d^2}{dt^2}y(t) + b\cdot\frac{d}{dt}y(t) + k\cdot y(t) = r(t)$$

(a) Spring-mass-damper system.
(b) Free-body diagram.

Differential Equation of Physical Systems

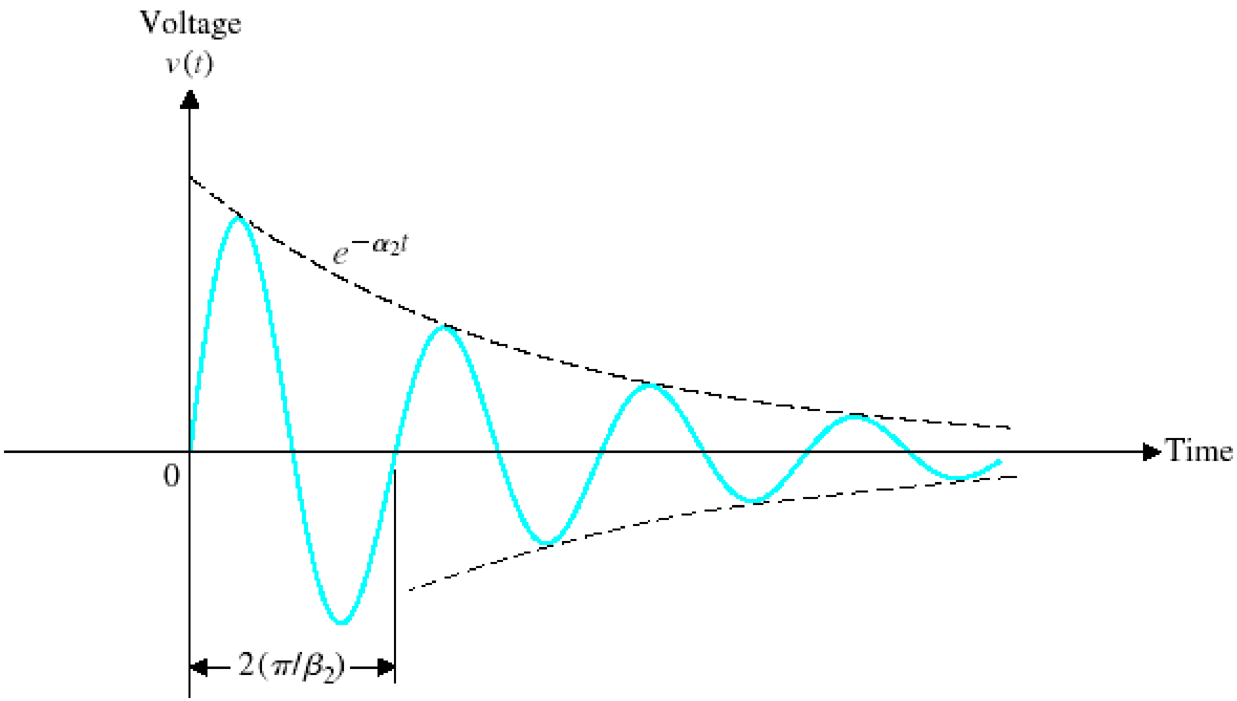


RLC circuit.

$$\frac{v(t)}{R} + C\cdot\frac{d}{dt}v(t) + \frac{1}{L}\cdot\int_0^t v(t)\,dt = r(t)$$

$$y(t) = K_1\cdot e^{-\alpha_1\cdot t}\cdot\sin(\beta_1\cdot t + \theta_1)$$

Differential Equation of Physical Systems



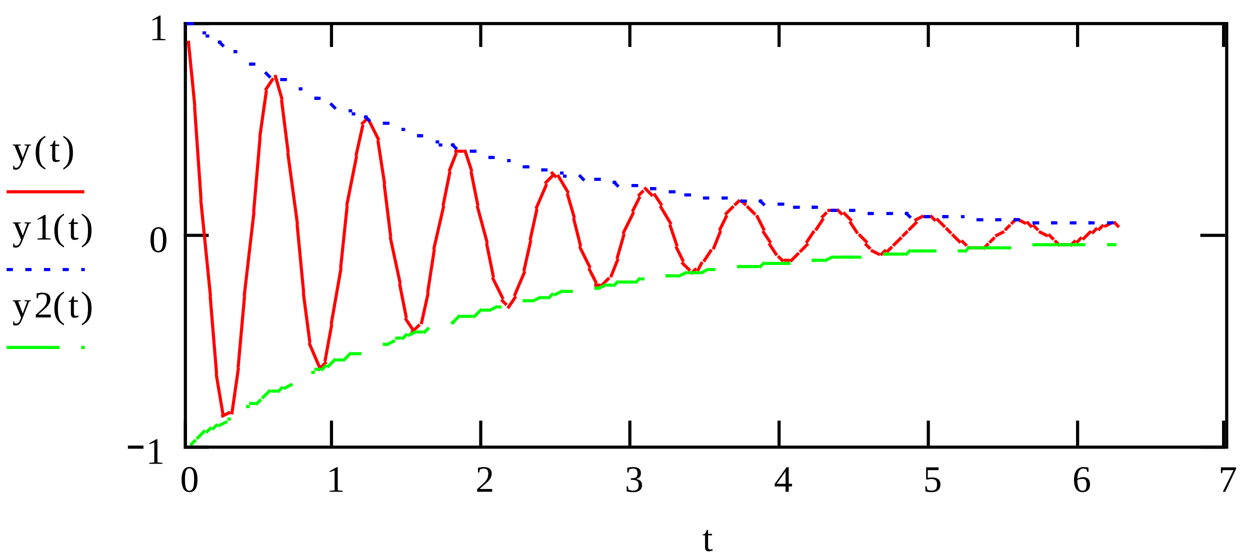
Typical voltage response for underdamped RLC circuit.

Differential Equation of Physical Systems

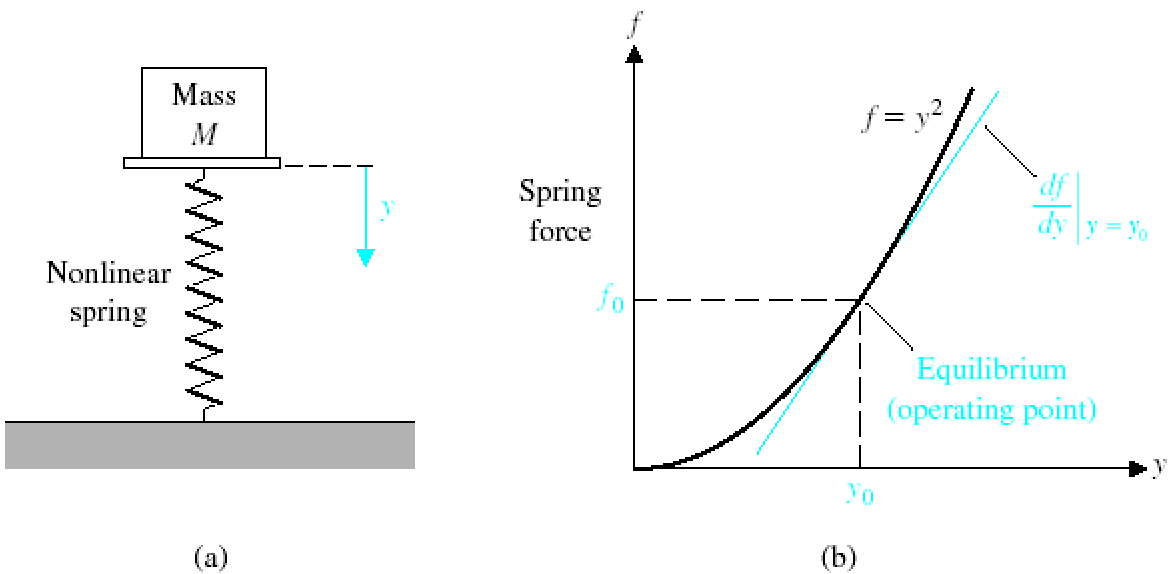
$K_2 := 1 \qquad \alpha_2 := .5 \qquad \beta_2 := 10 \qquad \theta_2 := 2$

$$y(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \cdot \sin\left(\beta_2 \cdot t + \theta_2\right)$$

$$y1(t) := K_2 \cdot e^{-\alpha_2 \cdot t} \qquad y2(t) := -K_2 \cdot e^{-\alpha_2 \cdot t}$$



Linear Approximations



(a) A mass sitting on a nonlinear spring.
(b) The spring force versus y.

Linear Approximations

Linear Systems - Necessary condition

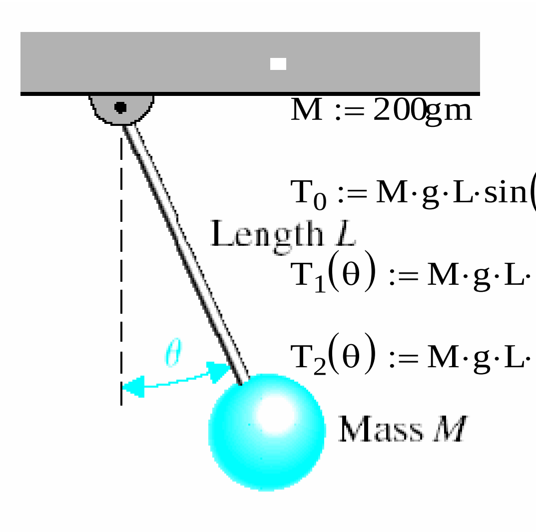
Principle of Superposition

Property of Homogeneity

Taylor Series

http://www.maths.abdn.ac.uk/%7Eigc/tch/ma2001/notes/node46.html

Linear Approximations – Example 2.1



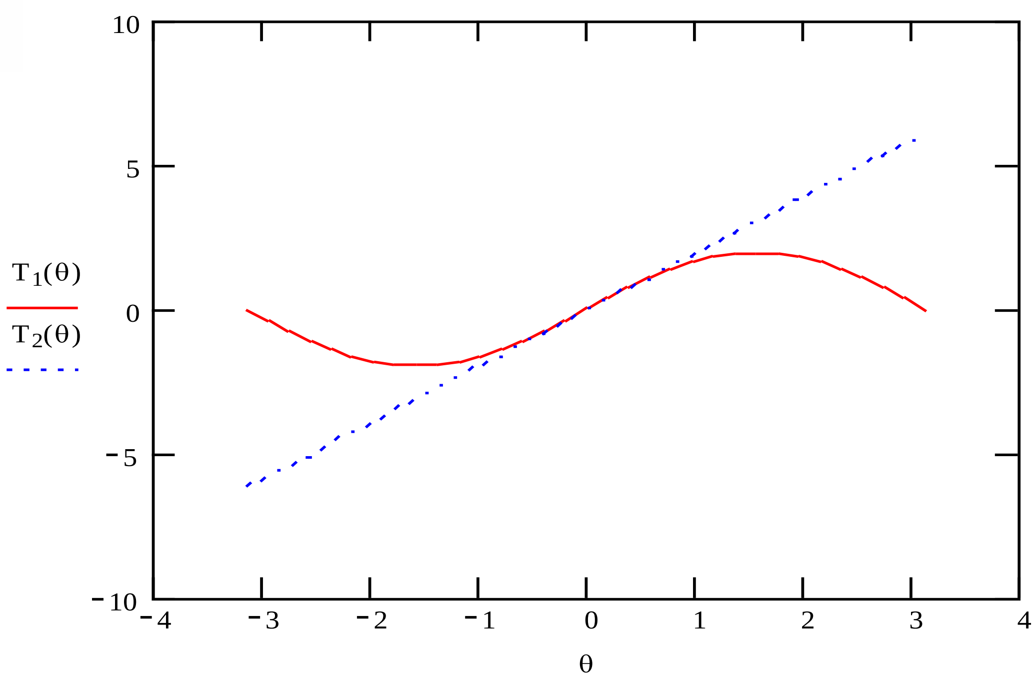
Pendulum oscillator.

$M := 200gm \qquad g := 9.8\frac{m}{s^2} \qquad L := 100cm \qquad \theta_0 := 0rad \qquad \theta := -\pi, \frac{-15\pi}{16} .. \pi$

$T_0 := M \cdot g \cdot L \cdot \sin(\theta_0)$

$T_1(\theta) := M \cdot g \cdot L \cdot \sin(\theta)$

$T_2(\theta) := M \cdot g \cdot L \cdot \cos(\theta_0) \cdot (\theta - \theta_0) + T_0$



Students are encouraged to investigate linear approximation accuracy for different valu θ_0

The Laplace Transform

Historical Perspective - Heaviside’s Operators

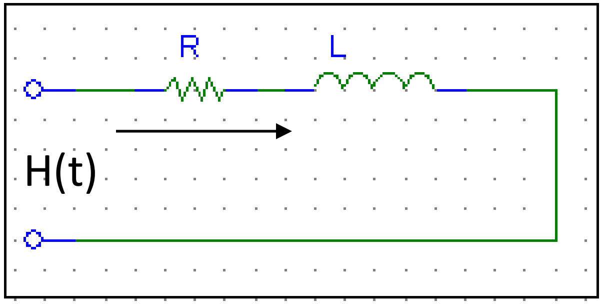
Origin of Operational Calculus (1887)

Historical Perspective - Heaviside's Operators
Origin of Operational Calculus (1887)

$$p = \frac{d}{dt}$$

$$\frac{1}{p} = \int_0^t 1\,du$$

$v =$



$$i = \frac{v}{Z(p)}$$

$$Z(p) = R + L\cdot p$$

Expanded in a power series

$$i = \frac{1}{R + L\cdot p}\cdot H(t) = \frac{1}{L\cdot p\left(1 + \frac{R}{L\cdot p}\right)}\cdot H(t) = \frac{1}{R}\cdot\left[\frac{R}{L}\cdot\frac{1}{p} - \left(\frac{R}{L}\right)^2\cdot\frac{1}{p^2} + \left(\frac{R}{L}\right)^3\cdot\frac{1}{p^3}\,.....\right]\cdot H(t)$$

$$\frac{1}{p^n}\cdot H(t) = \frac{t^n}{n!}$$

$$i = \frac{1}{R}\cdot\left[\frac{R}{L}\cdot t - \left[\left(\frac{R}{L}\right)^2\cdot\frac{t^2}{2!} + \left(\frac{R}{L}\right)^3\cdot\frac{t^3}{3!} - ..\right]\right]$$

$$i = \frac{1}{R}\cdot\left[1 - e^{-\left(\frac{R}{L}\right)\cdot t}\right]$$

(*) Oliver Heaviside: Sage in Solitude, Paul J. Nahin, IEEE Press 1987.

The Laplace Transform

Definition

$$L(f(t)) = \int_0^{\infty} f(t)\cdot e^{-s\cdot t}\,dt \qquad = F(s)$$

Here the complex frequency is $s = \rho + j\cdot w$

The Laplace Transform exists when

$$\int_0^{\infty} f(t)\cdot e^{-s\cdot t}\,dt < \infty$$

this means that the integral converg

The Laplace Transform

Determine the Laplace transform for the functions

a) $f_1(t) := 1$ for $t \ge 0$

$$F_1(s) := \int_0^{\infty} e^{-s \cdot t}\, dt \quad = \quad -\frac{1}{s} \cdot e^{-(s \cdot t)} \qquad \frac{1}{s}$$

b) $f_2(t) = e^{-(a \cdot t)}$

$$F_2(s) = \int_0^{\infty} e^{-(a \cdot t)} \cdot e^{-(s \cdot t)}\, dt \quad = \quad -\frac{1}{s+1} \cdot e^{-[(s+a) \cdot t]} \qquad F_2(s) = \frac{1}{s+a}$$

The Laplace Transform

Evaluate the laplace transform of the derivative of a function

$$L\left(\frac{d}{dt}f(t)\right) = \int_0^{\infty} \frac{d}{dt}f(t)\cdot e^{-(s\cdot t)}\,dt$$

by the use of $\int u\,dv \quad = \quad u\cdot v - \int v\,du$

where $u = e^{-(s\cdot t)}$ $\qquad dv = df(t)$

and, from which

$du = -s\cdot e^{-(s\cdot t)}\cdot dt$ $\qquad$ and $\qquad v = f(t)$

we obtain

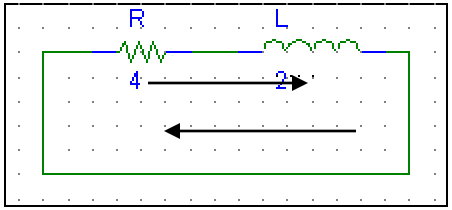
$$\int_0^{\infty} u\,dv \quad = \quad f(t)\cdot e^{-(s\cdot t)} \quad - \int_0^{\infty} f(t)\cdot\left[-s\cdot e^{-(s\cdot t)}\right]dt$$

$$= \; -f(0+) + \quad s\cdot\int_0^{\infty} f(t)\cdot e^{-(s\cdot t)}\,dt$$

→ $L\cdot\left(\frac{d}{dt}f(t)\right)$ = sF(s) - f(0+) note that the initial condition is included in the transform

The Laplace Transform

Practical Example - Consider the circuit.



The KVL equation is

$$4\cdot i(t) + 2\cdot\frac{d}{dt}i(t) = 0$$

assume i(0+) = 5 A

Applying the Laplace Transform, we have

$$\int_{0}^{\infty}\left(4\cdot i(t) + 2\cdot\frac{d}{dt}i(t)\right)\cdot e^{-(s\cdot t)}\,dt = 0 \qquad 4\cdot\int_{0}^{\infty} i(t)\cdot e^{-(s\cdot t)}\,dt + 2\cdot\int_{0}^{\infty}\frac{d}{dt}i(t)\cdot e^{-(s\cdot t)}\,dt = 0$$

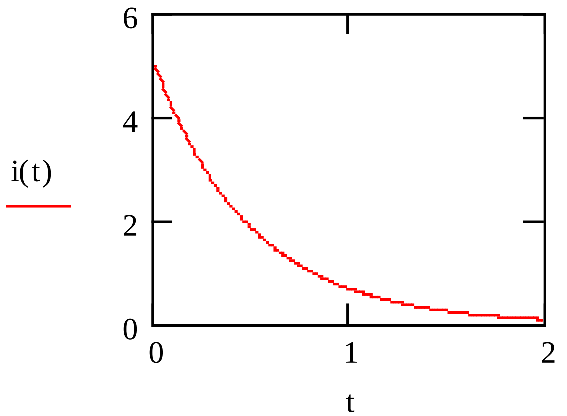
$$4\cdot I(s) + 2\cdot(s\cdot I(s) - i(0)) = 0 \qquad 4\cdot I(s) + 2\cdot s\cdot I(s) - 10 = 0$$

$$I(s) := \frac{5}{s+2}$$

transforming back to the time domain, with our present knowledge of Laplace transform, we may say that

$$t \equiv (0, 0.01 .. 2)$$

$$i(t) \equiv 5\cdot e^{-(2\cdot t)}$$



The Laplace Transform

The Partial-Fraction Expansion (or Heaviside expansion theorem)

Suppose that

$$F(s) = \frac{s + z1}{(s + p1)\cdot(s + p2)}$$

or

$$F(s) = \frac{K1}{s + p1} + \frac{K2}{s + p2}$$

The partial fraction expansion indicates that F(s) consists of a sum of terms, each of which is a factor of the denominator. The values of K1 and K2 are determined by combining the individual fractions by means of the lowest common denominator and comparing the resultant numerator coefficients with those of the coefficients of the numerator before separation in different terms.

Evaluation of Ki in the manner just described requires the simultaneous solution of n equations. An alternative method is to multiply both sides of the equation by (s + pi) then setting s= - pi, the right-hand side is zero except for Ki so that

$$Ki = \frac{(s + pi)\cdot(s + z1)}{(s + p1) + (s + p2)} \qquad s = -pi$$

The Laplace Transform

Property	Time Domain	Frequency Doma
1. Time delay	$f(t-T)\cdot u(t-T)$	$e^{-(s\cdot T)}\cdot F(s)$
2. Time scaling	$f(at)$	$\frac{1}{a}\cdot F\left(\frac{s}{a}\right)$
3. Frequency differentiation	$t\cdot f(t)$	$-\frac{d}{ds}F(s)$
4. Frequency shifting	$f(t)\cdot e^{-(a\cdot t)}$	$F(s+a)$
5. Frequency Integration	$\frac{f(t)}{t}$	$\int_0^{\infty} F(s)\,ds$
6. Initial-value Theorem	$Lim(f(t)) = f(0)$ t -> 0	$Lim(s\cdot F(s))$ s -> infinite
7. Final-value Theorem	$Lim(f(t))$ t -> infinite	$Lim(s\cdot F(s))$ s -> 0

The Laplace Transform

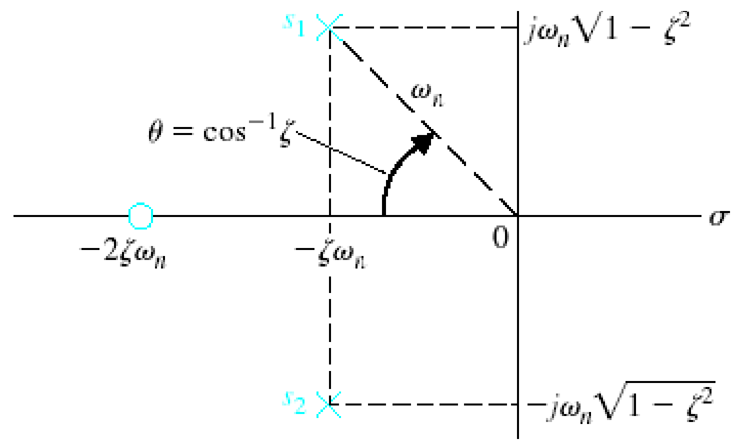
Useful Transform Pairs

The Laplace Transform

Consider the mass-spring-damper system

$$Y(s) = \frac{(Ms + b)\cdot yo}{Ms^2 + bs + k} \qquad \text{equation 2.21}$$

An s-plane plot of the poles and zeros of Y(s).



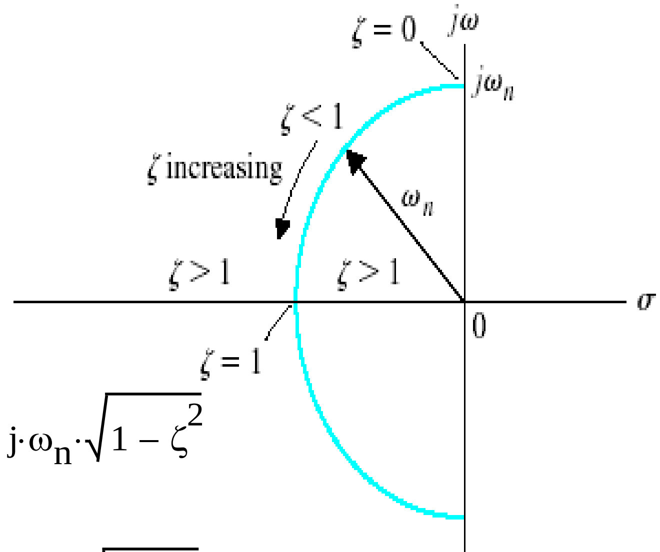
The locus of roots as z varies with vn constant.

$$y(s) = \frac{\left(s + \frac{b}{M}\right)\cdot(y_o)}{\left[s^2 + \left(\frac{b}{M}\right)\cdot s + \frac{k}{M}\right]} = \frac{(s + 2\cdot\zeta\cdot\omega_n)}{s^2 + 2\cdot\zeta\cdot\omega_n + \omega_n^2}$$

$$s1 = -(\zeta\cdot\omega_n) + \omega_n\cdot\sqrt{\zeta^2 - 1}$$

$$\omega_n = \sqrt{\frac{k}{M}} \qquad \zeta = \frac{b}{(2\cdot\sqrt{k\cdot M})}$$

$$s2 = -(\zeta\cdot\omega_n) - \omega_n\cdot\sqrt{\zeta^2 - 1}$$



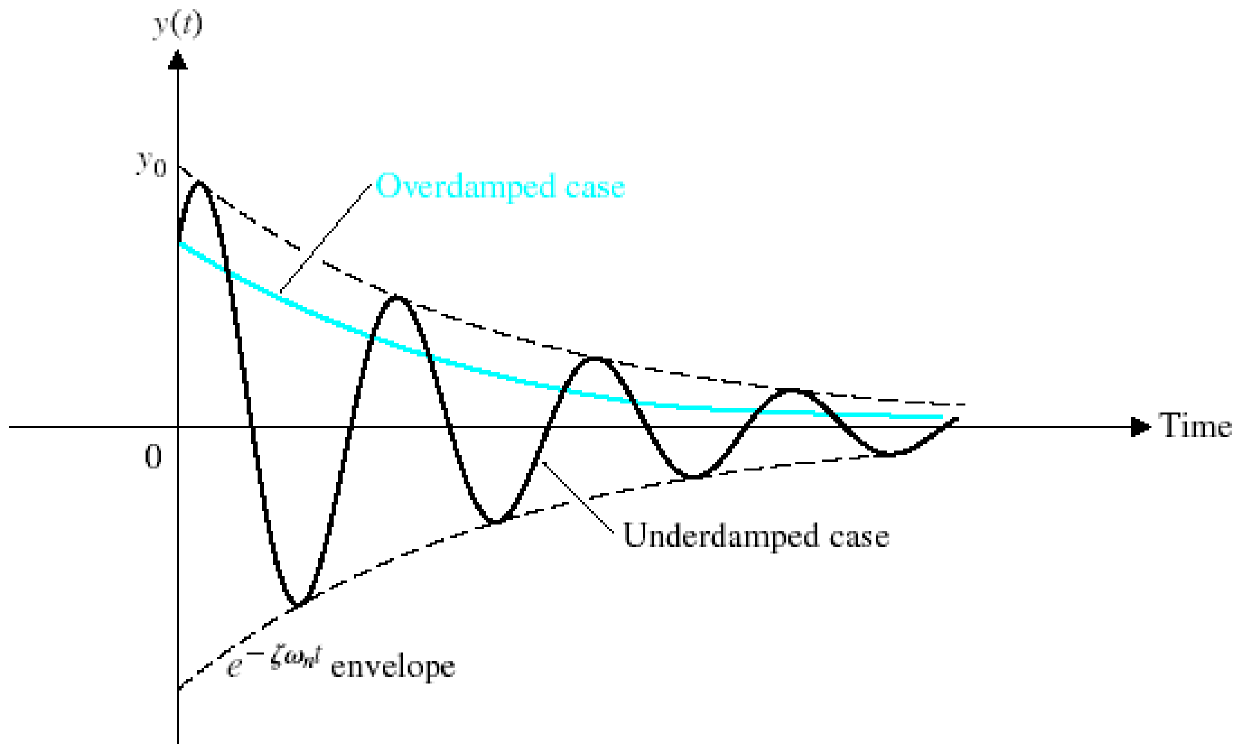
Roots

Real
Real repeated
Imaginary (conjugates)
Complex (conjugates)

$$s1 = -(\zeta\cdot\omega_n) + j\cdot\omega_n\cdot\sqrt{1 - \zeta^2}$$

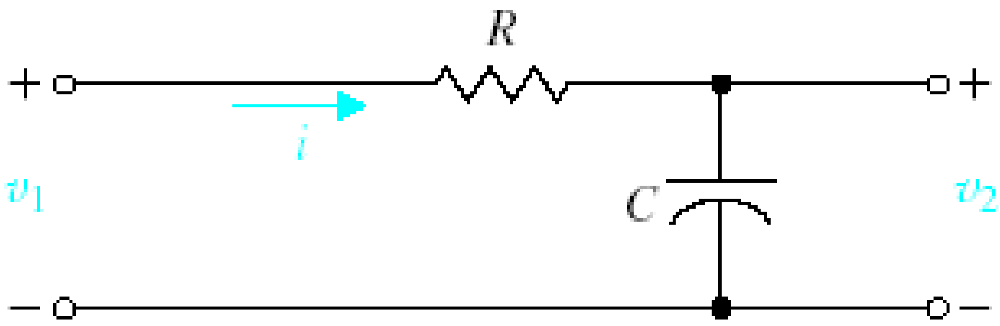
$$s2 = -(\zeta\cdot\omega_n) - j\cdot\omega_n\cdot\sqrt{1 - \zeta^2}$$

The Laplace Transform



Response of the spring-mass-damper system.

The Transfer Function of Linear Systems



An RC network.

$$V_1(s) = \left(R + \frac{1}{Cs}\right) \cdot I(s) \qquad Z_1(s) = R$$

$$V_2(s) = \left(\frac{1}{Cs}\right) \cdot I(s) \qquad Z_2(s) = \frac{1}{Cs}$$

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

The Transfer Function of Linear Systems

Example 2.2

$$\frac{d^2}{dt^2}y(t) + 4\cdot\frac{d}{dt}y(t) + 3\cdot y(t) = 2\cdot r(t)$$

Initial Conditions: $Y(0) = 1$ $\quad \frac{d}{dt}y(0) = 0$ $\quad r(t) = 1$

The Laplace transform yields:

$$\left(s^2\cdot Y(s) - s\cdot y(0)\right) + 4\cdot(s\cdot Y(s) - y(0)) + 3Y(s) = 2\cdot R(s)$$

Since R(s)=1/s and y(0)=1, we obtain:

$$Y(s) = \frac{(s+4)}{\left(s^2 + 4s + 3\right)} + \frac{2}{s\cdot\left(s^2 + 4s + 3\right)}$$

The partial fraction expansion yields:

$$Y(s) = \left[\frac{\frac{3}{2}}{(s+1)} + \frac{\frac{-1}{2}}{(s+3)}\right] + \left[\frac{-1}{(s+1)} + \frac{\frac{1}{3}}{(s+3)}\right] + \frac{\frac{2}{3}}{s}$$

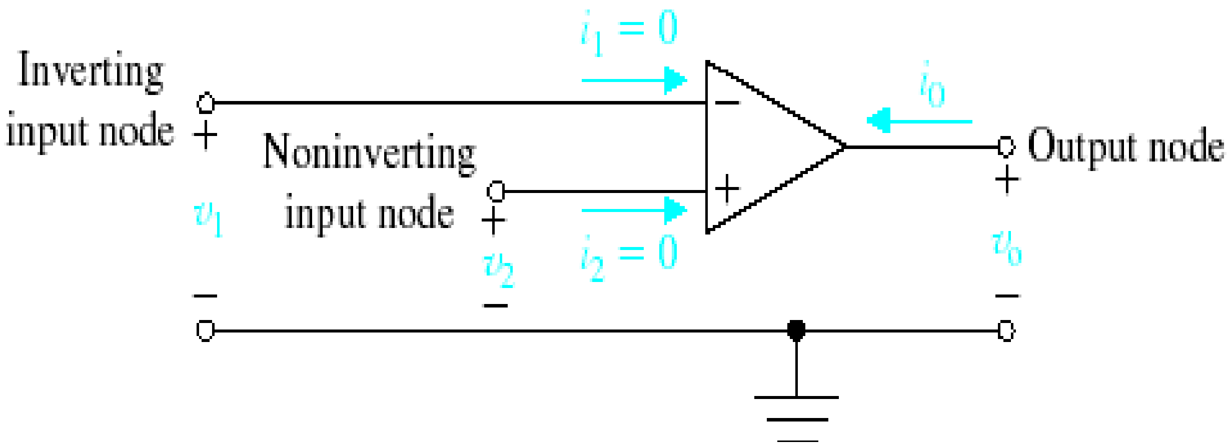
Therefore the transient response is:

$$y(t) = \left(\frac{3}{2}\cdot e^{-t} - \frac{1}{2}\cdot e^{-3\cdot t}\right) + \left(-1e^{-t} + \frac{1}{3}\cdot e^{-3t}\right) + \frac{2}{3}$$

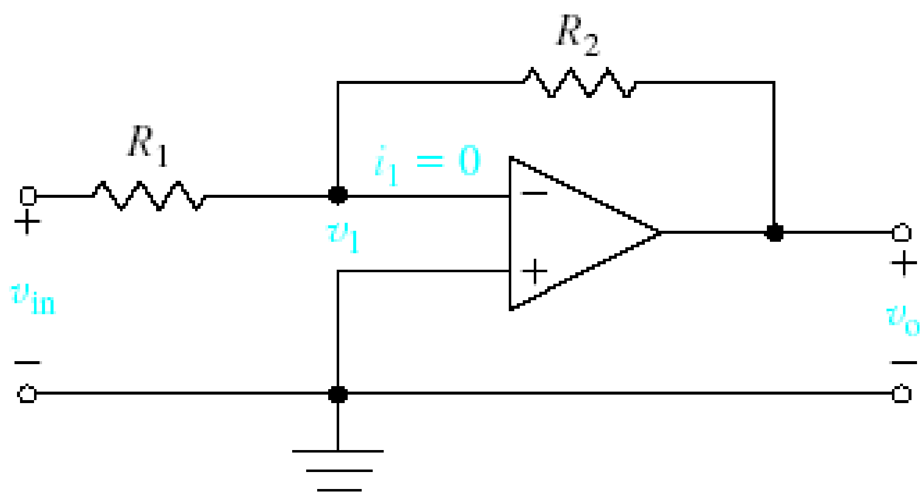
The steady-state response is:

$$\lim_{t\to\infty} y(t) = \frac{2}{3}$$

The Transfer Function of Linear Systems

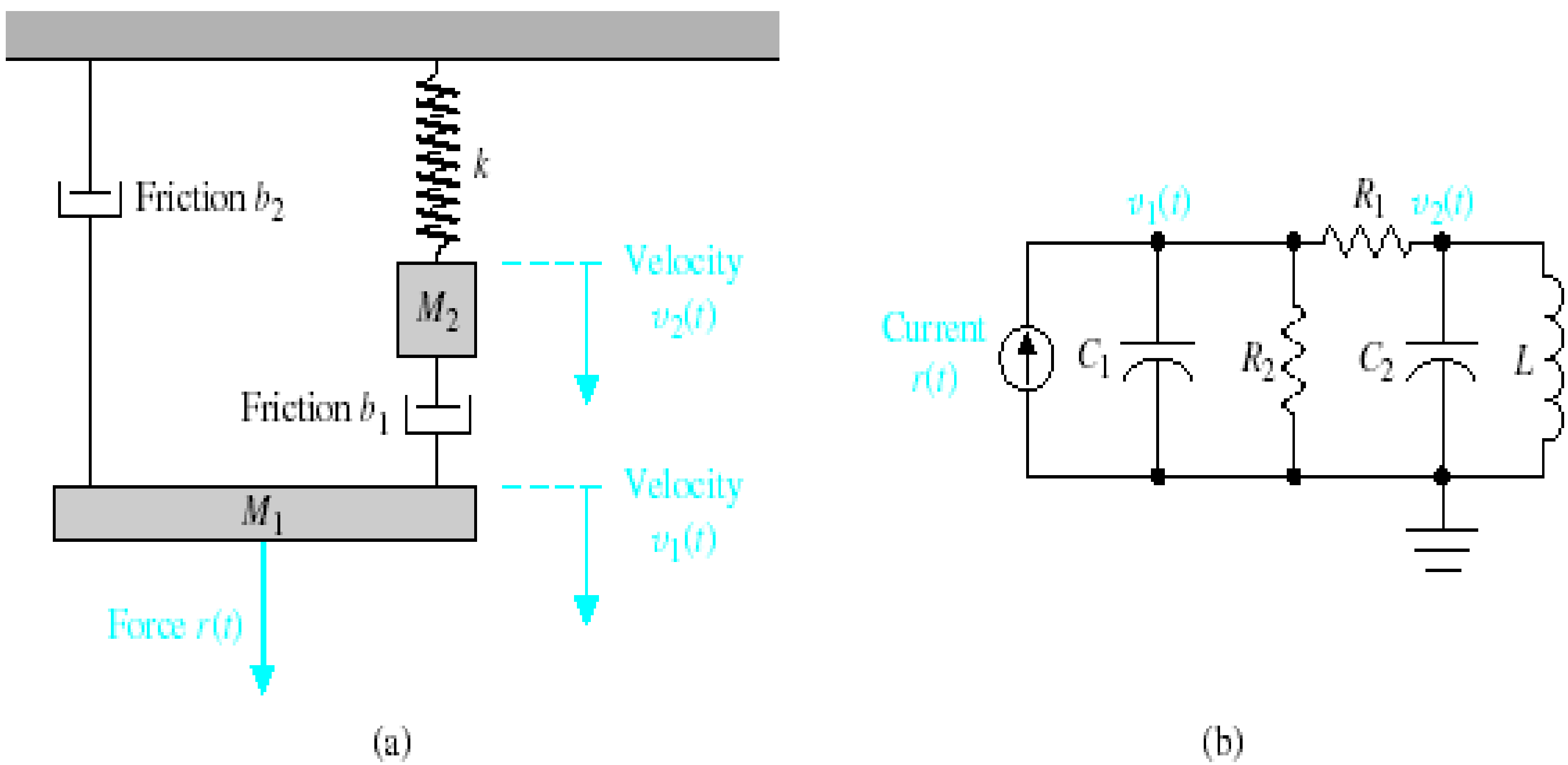


The ideal op-amp



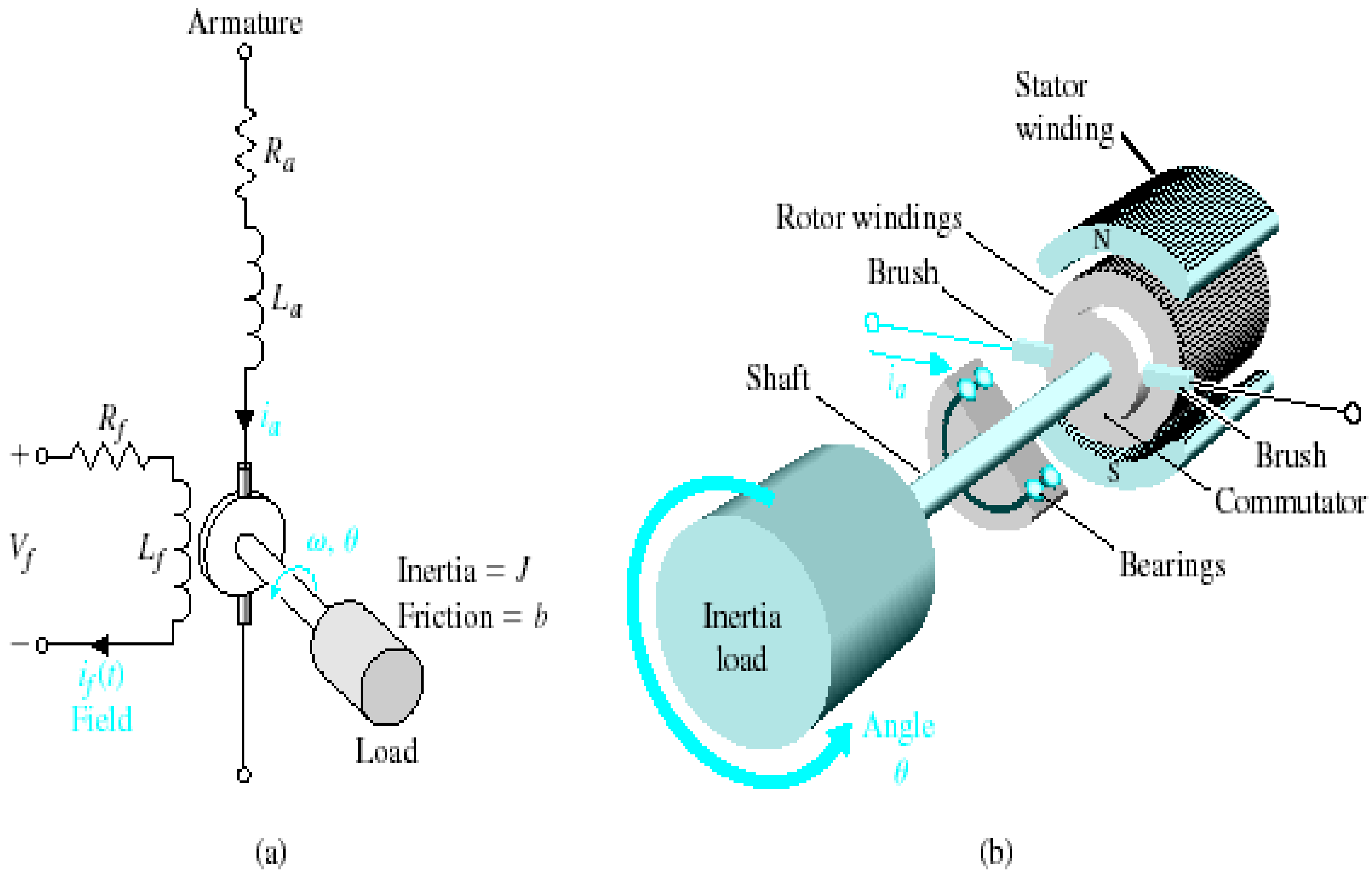
An inverting amplifier operating with ideal conditions.

The Transfer Function of Linear Systems



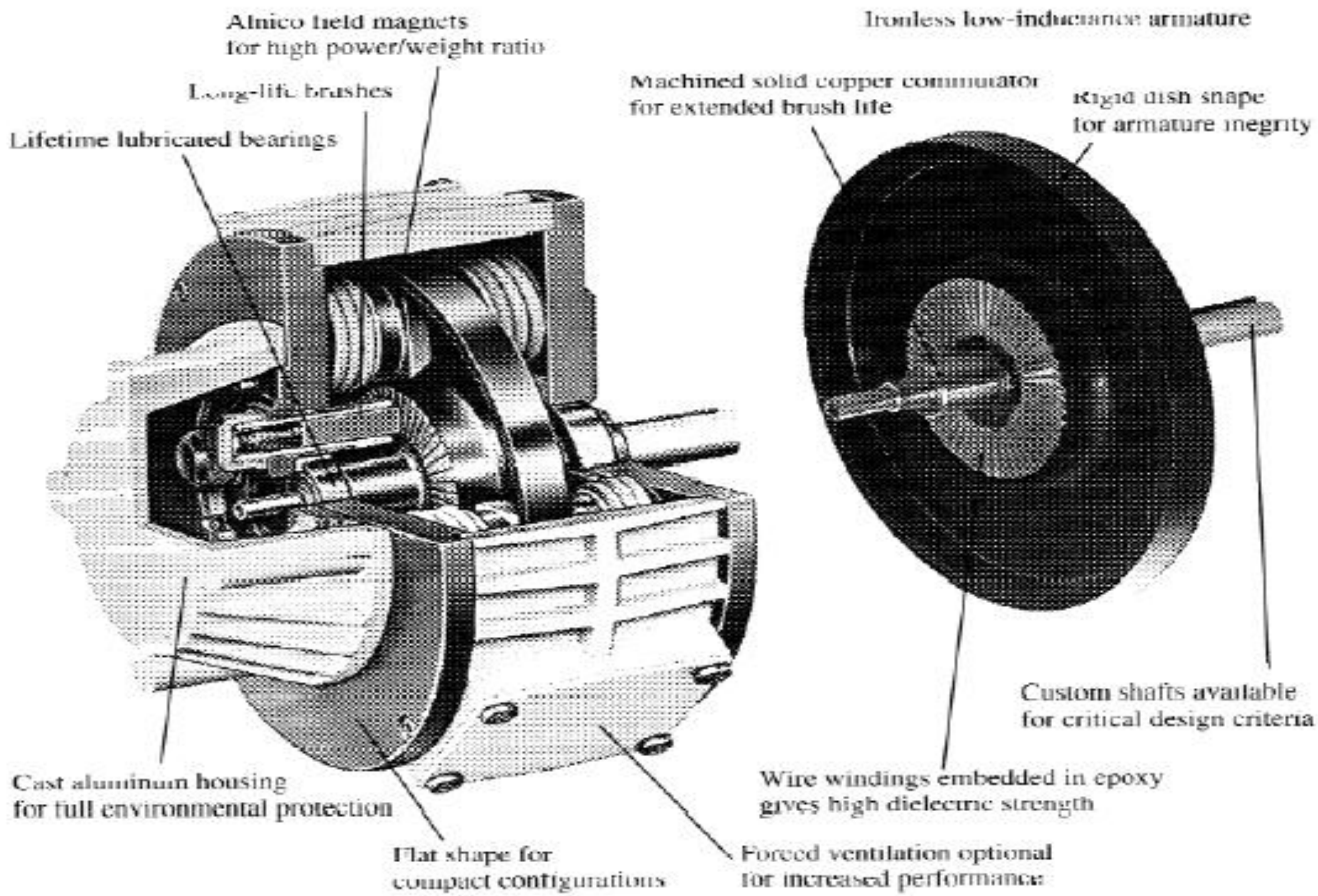
(a) Two-mass mechanical system. (b) Two-node electric circuit analog C1 = M1, C2 = M2, L = 1/k, R1 = 1/b1, R2 = 1/b2.

The Transfer Function of Linear Systems



A dc motor (a) wiring diagram
and (b) sketch.

The Transfer Function of Linear Systems



A pancake dc motor with a flat-wound armature and a permanent magnet rotor. These motors are capable of providing high torque with a low rotor inertia. A typical mechanical time constant is in the range of 15 ms. (Courtesy of Mavilor Motors.)

The Transfer Function of Linear Systems

$\phi = K_f \cdot i_f$

$T_m = K_1 \cdot K_f \cdot i_f(t) \cdot i_a(t)$

field controled motor - Lapalce Transfc

$T_m(s) = \left(K_1 \cdot K_f \cdot I_a\right) \cdot I_f(s)$

$V_f(s) = \left(R_f + L_f \cdot s\right) \cdot I_f(s)$

$T_m(s) = T_L(s) + T_d(s)$

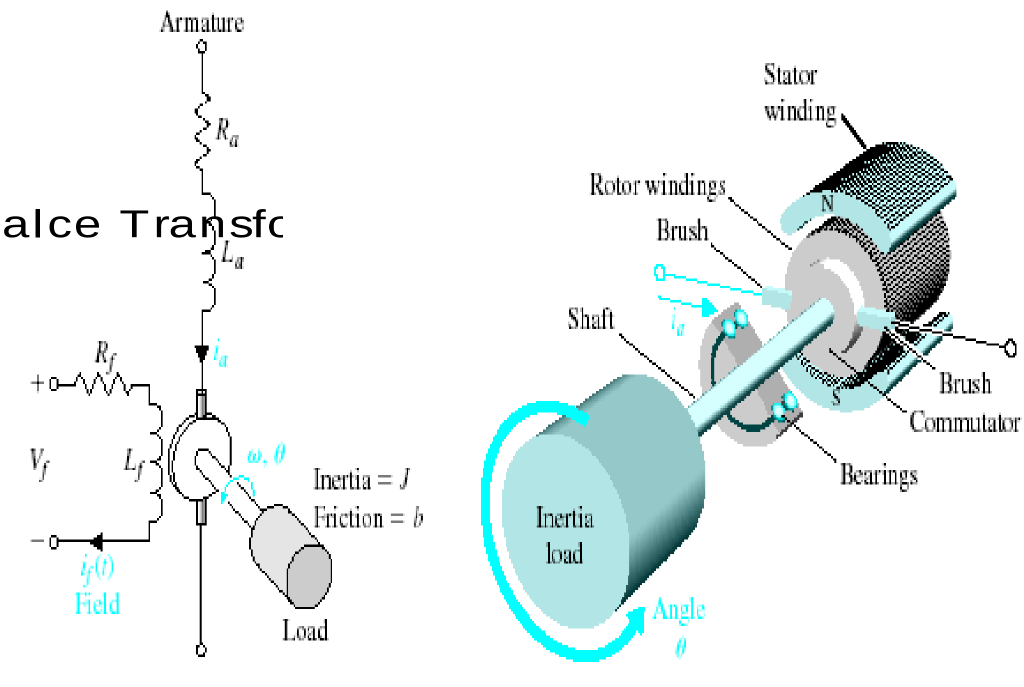
$T_L(s) = J \cdot s^2 \cdot \theta(s) + b \cdot s \cdot \theta(s)$

rearranging equations

$T_L(s) = T_m(s) - T_d(s)$

$T_m(s) = K_m \cdot I_f(s)$

$I_f(s) = \dfrac{Vf(s)}{R_f + L_f \cdot s}$

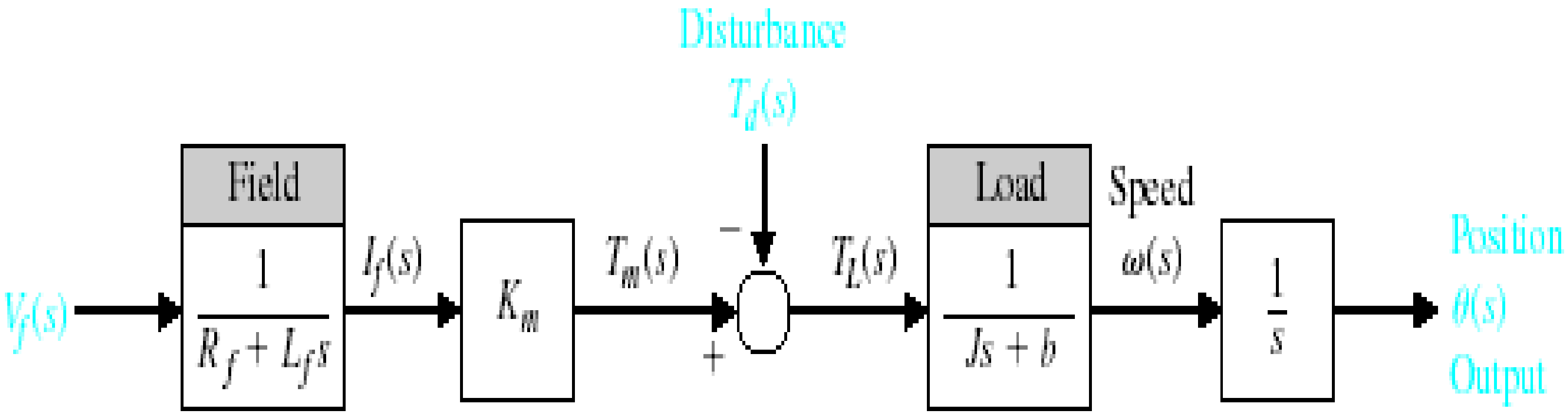


(a) (b)

$Td(s) = 0$

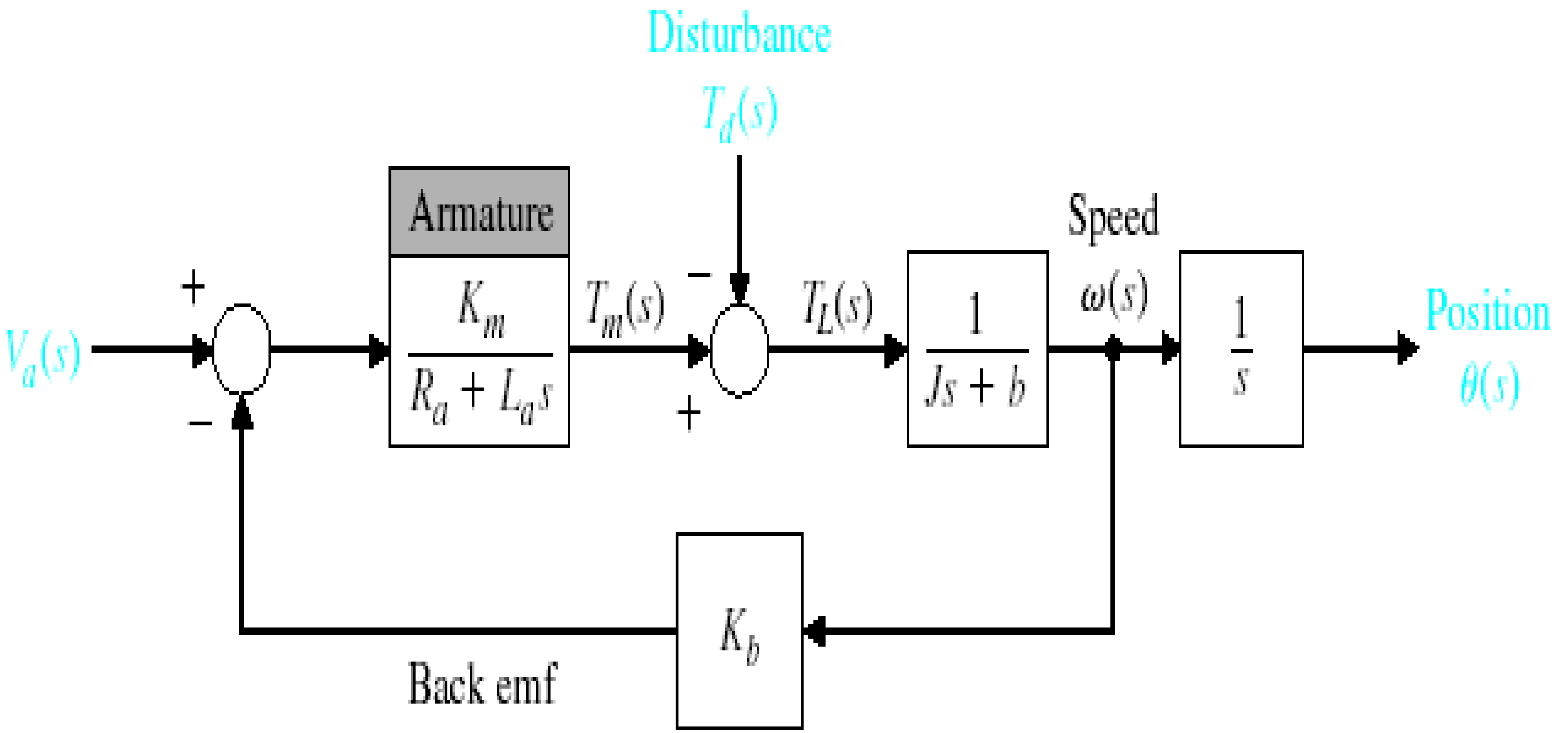
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s \cdot (J \cdot s + b) \cdot \left(L_f \cdot s + R_f\right)}$$

The Transfer Function of Linear Systems



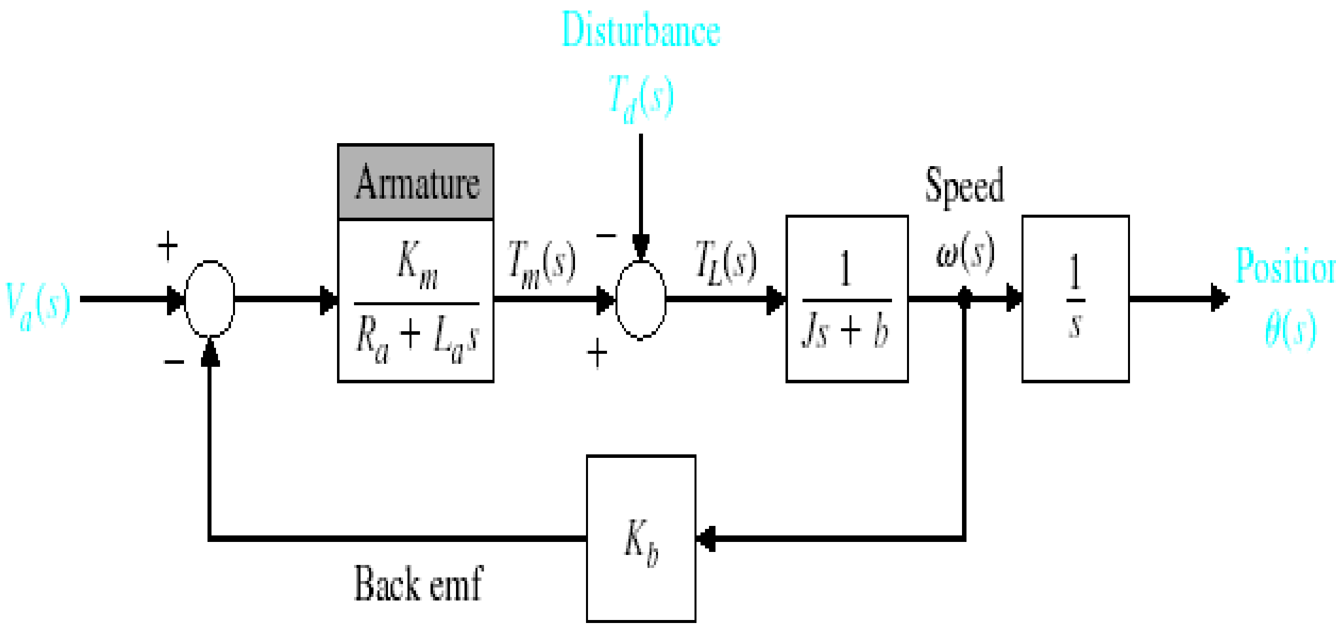
Block diagram model of field-controlled dc motor.

The Transfer Function of Linear Systems

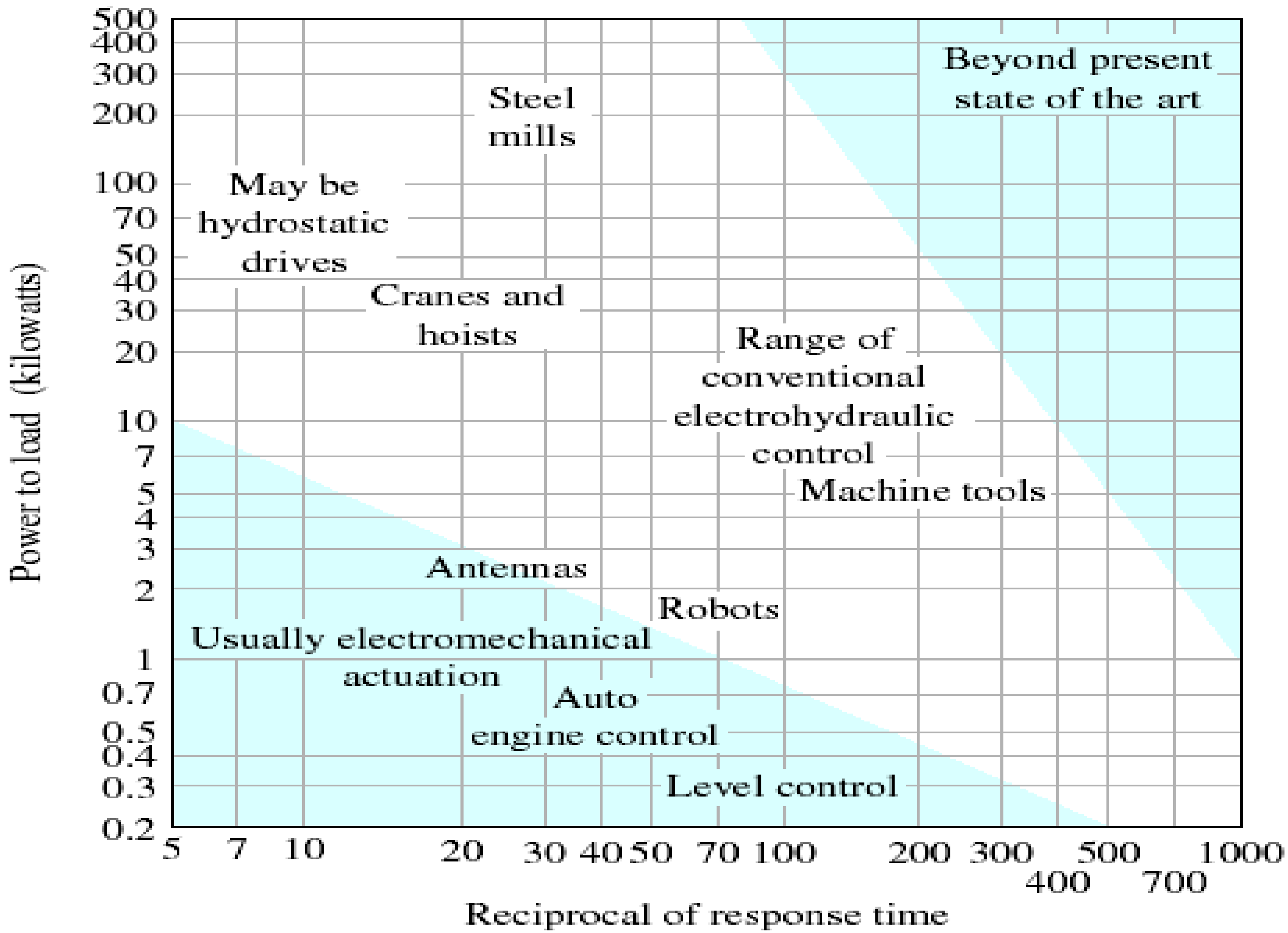


Armature-controlled dc motor.

The Transfer Function of Linear Systems

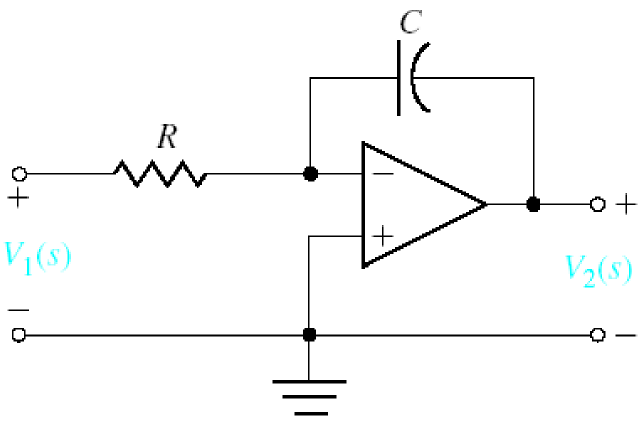


The Transfer Function of Linear Systems

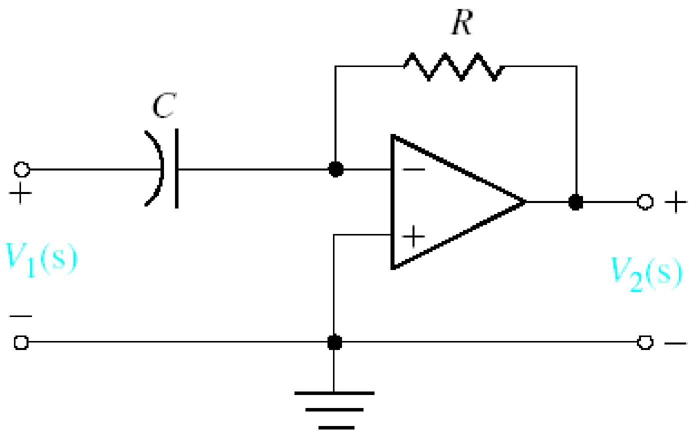


Range of control response time and power to load for electromechanical and electrohydraulic devices.

The Transfer Function of Linear Systems

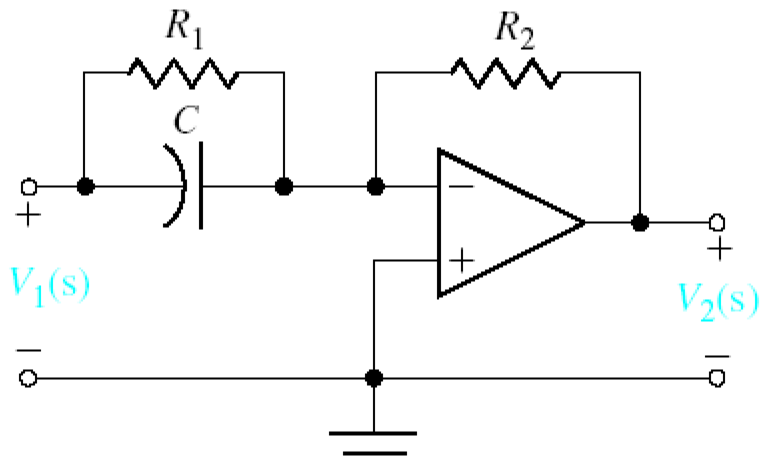


$$\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}$$

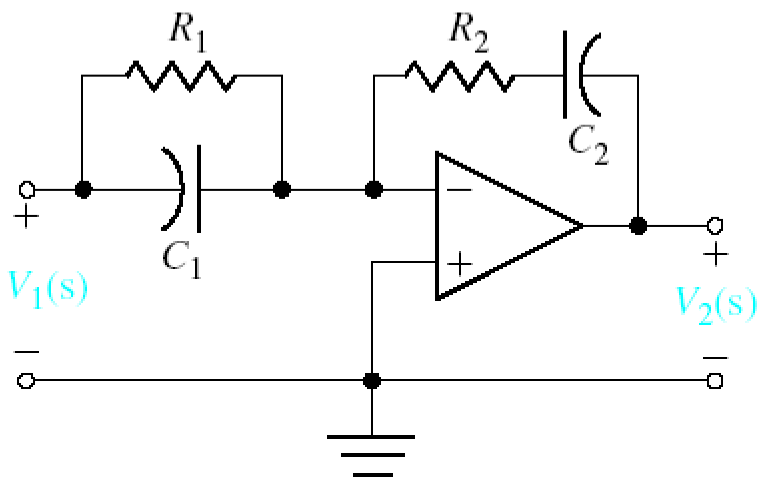


$$\frac{V_2(s)}{V_1(s)} = -RCs$$

The Transfer Function of Linear Systems

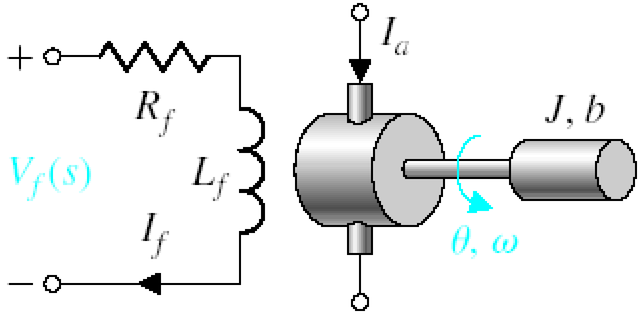


$$\frac{V_2(s)}{V_1(s)} = \frac{R_2\left(R_1 \cdot C \cdot s + 1\right)}{R_1}$$

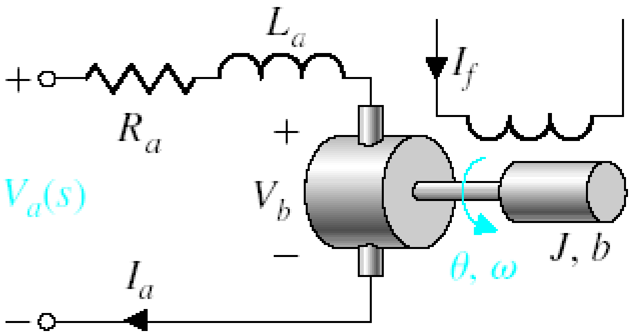


$$\frac{V_2(s)}{V_1(s)} = \frac{-\left(R_1 \cdot C_1 \cdot s + 1\right)\left(R_2 \cdot C_2 \cdot s + 1\right)}{R_1 \cdot C_2 \cdot s}$$

The Transfer Function of Linear Systems

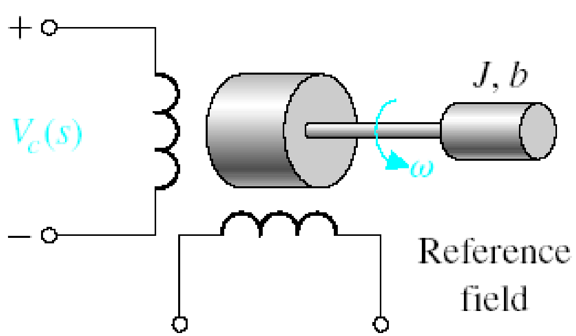


$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s\cdot(J\cdot s + b)\left(L_f\cdot s + R_f\right)}$$



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s\cdot\left[\left(R_a + L_a\cdot s\right)(J\cdot s + b) + K_b\cdot K_m\right]}$$

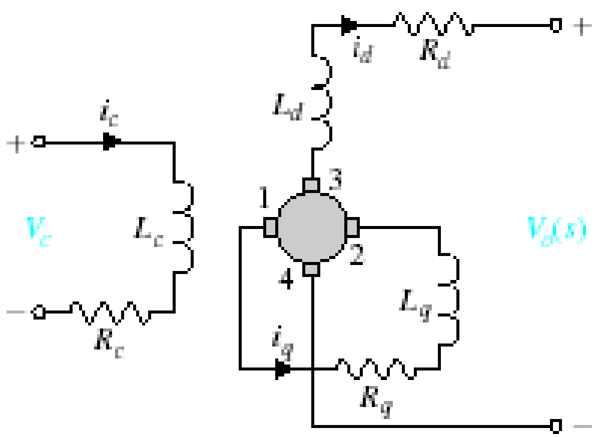
The Transfer Function of Linear Systems



$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau \cdot s + 1)}$$

$$\tau = \frac{J}{(b - m)}$$

m = slope of linearized torque-speed curve (normally negative)



$$\frac{V_o(s)}{V_c(s)} = \frac{\left(\frac{K}{R_c \cdot R_q}\right)}{(s \cdot \tau_c + 1) \cdot (s \cdot \tau_q + 1)}$$

$$\tau_c = \frac{L_c}{R_c} \qquad \tau_q = \frac{L_q}{R_q}$$

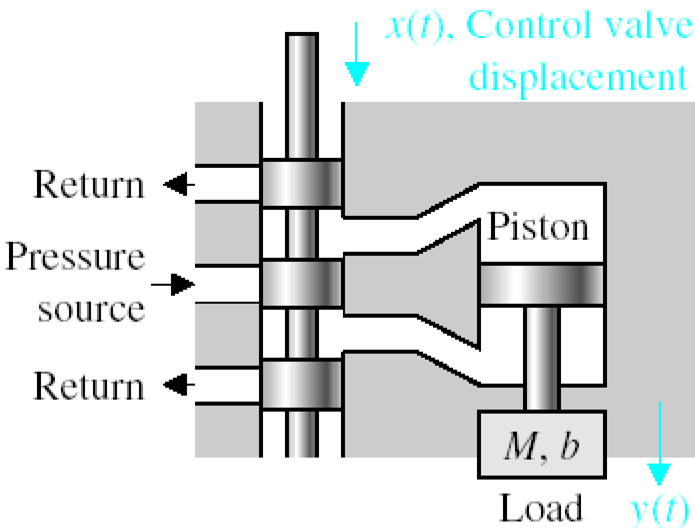
For the unloaded case:

$i_d = 0 \qquad \tau_c = \tau_q$

$0.05s < \tau_c < 0.5s$

$V_{12} = V_q \qquad V_{34} = V_d$

The Transfer Function of Linear Systems



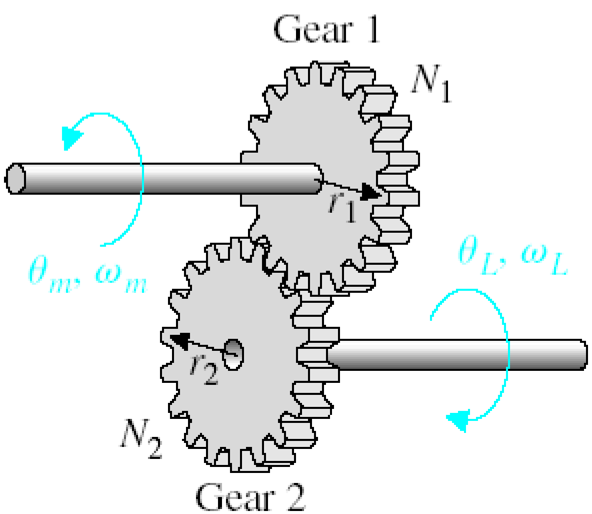
$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}$$

$$K = \frac{A \cdot k_x}{k_p} \qquad B = \left(b + \frac{A^2}{k_p}\right)$$

$$k_x = \frac{d}{dx} g \qquad k_p = \frac{d}{dP} g$$

$$g = g(x, P) = \text{flow}$$

A = area of piston



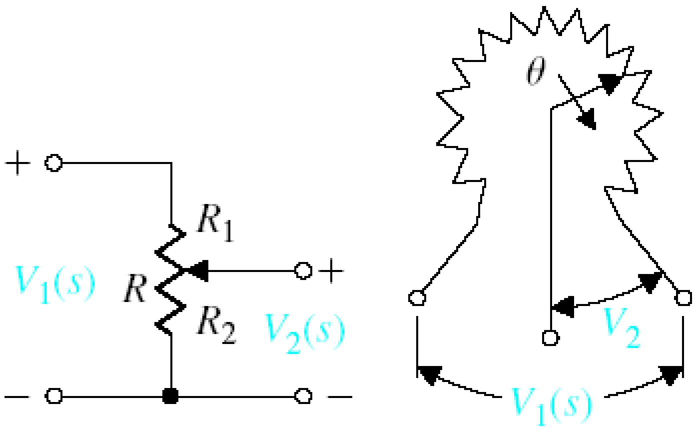
Gear Ratio = n = N1/N2

$$N_2 \cdot \theta_L = N_1 \cdot \theta_m$$

$$\theta_L = n \cdot \theta_m$$

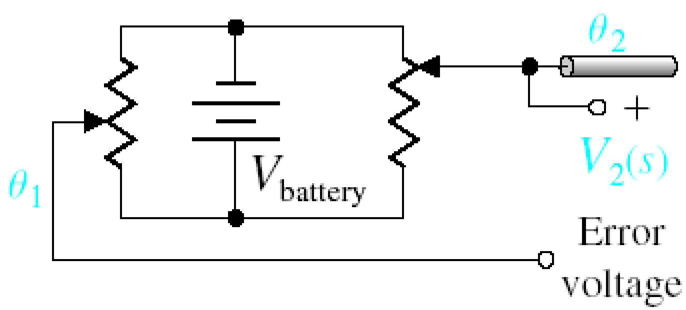
$$\omega_L = n \cdot \omega_m$$

The Transfer Function of Linear Systems



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{max}}$$

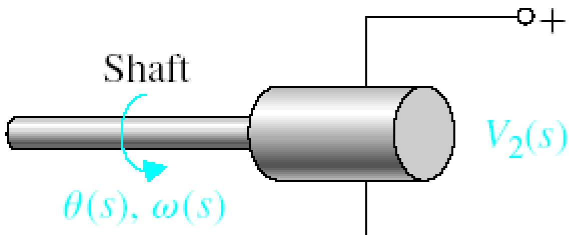


$$V_2(s) = k_s\left(\theta_1(s) - \theta_2(s)\right)$$

$$V_2(s) = k_s \cdot \theta_{error}(s)$$

$$k_s = \frac{V_{battery}}{\theta_{max}}$$

The Transfer Function of Linear Systems



$V_2(s) = K_t \cdot \omega(s) = K_t \cdot s \cdot \theta(s)$

K_t = constant

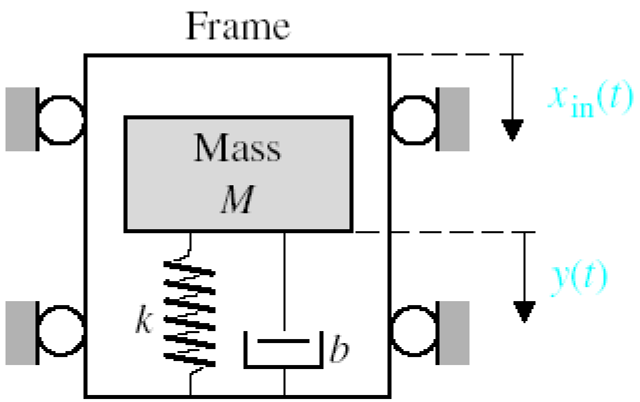
$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s \cdot \tau + 1}$$

Ro = output resistance
Co = output capacitance

$\tau = R_o \cdot C_o \qquad \tau < 1s$

and is often negligibl
for controller amplifie

The Transfer Function of Linear Systems

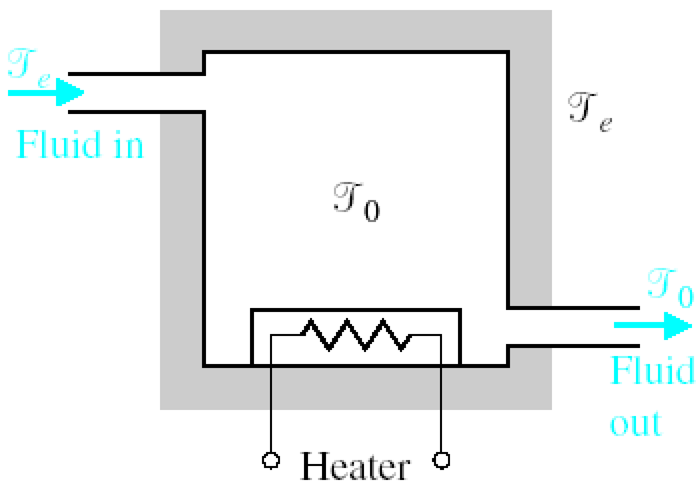


$$x_o(t) = y(t) - x_{in}(t)$$

$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + \left(\frac{b}{M}\right)\cdot s + \frac{k}{M}}$$

For low frequency oscillations, where $\omega < \omega_n$

$$\frac{X_o(j\cdot\omega)}{X_{in}(j\cdot\omega)} = \frac{\omega^2}{\frac{k}{M}}$$

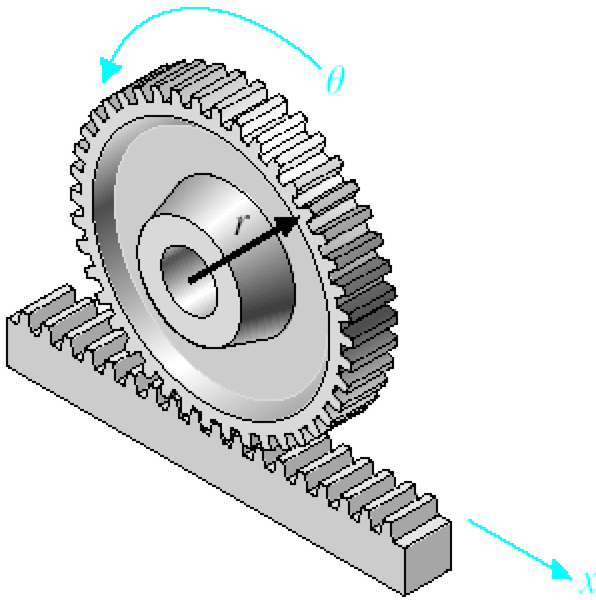


$$\frac{T(s)}{q(s)} = \frac{1}{C_t\cdot s + \left(Q\cdot S + \frac{1}{R}\right)}$$

$T = T_o - T_e$ = temperature difference due to thermal proc

C_t = thermal capacitance
Q = fluid flow rate = constant
S = specific heat of water
R_t = thermal resistance of insulation
$q(s)$ = rate of heat flow of heating element

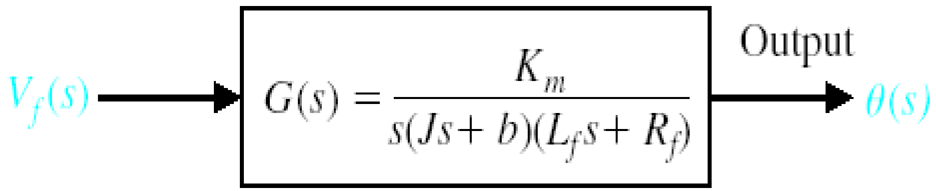
The Transfer Function of Linear Systems



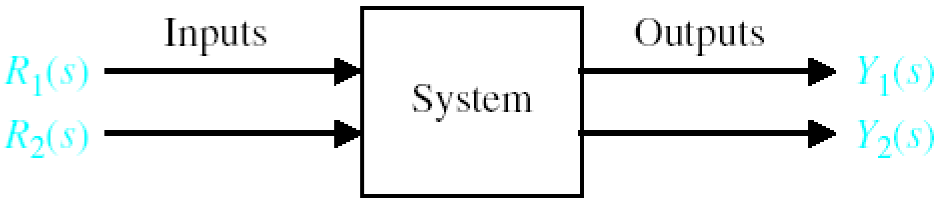
$x = r \cdot \theta$

converts radial motion to linear mo

Block Diagram Models

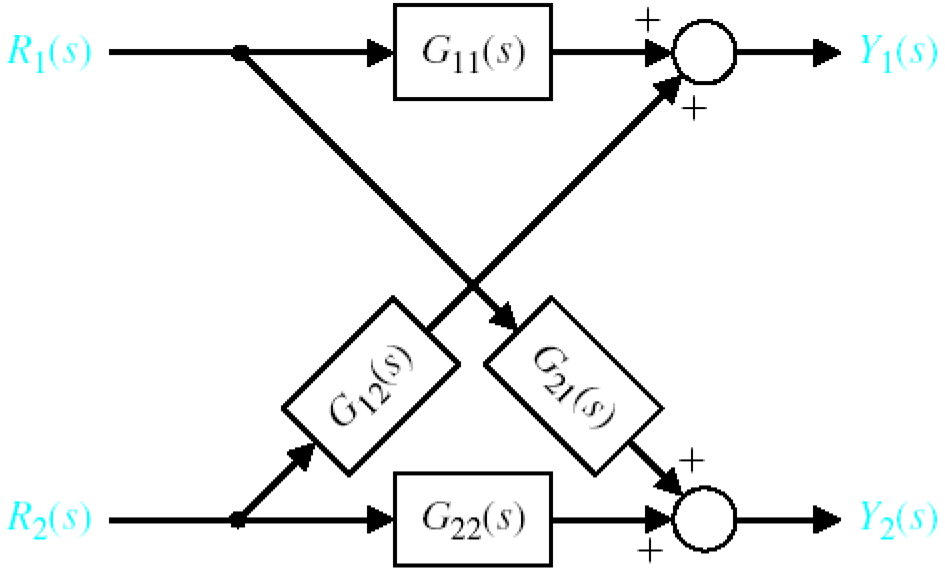


Block diagram of dc motor.



General block representation of two-input, two-output system.

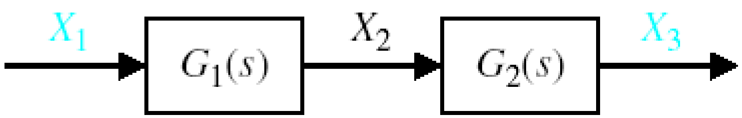
Block Diagram Models



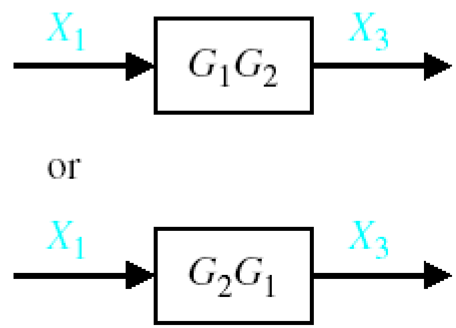
Block diagram of interconnected system.

Block Diagram Models

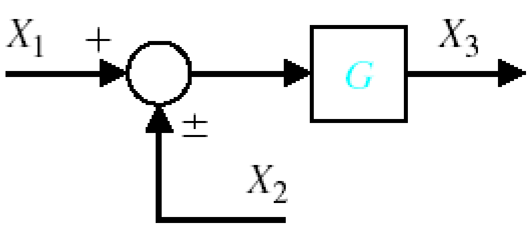
Original Diagram



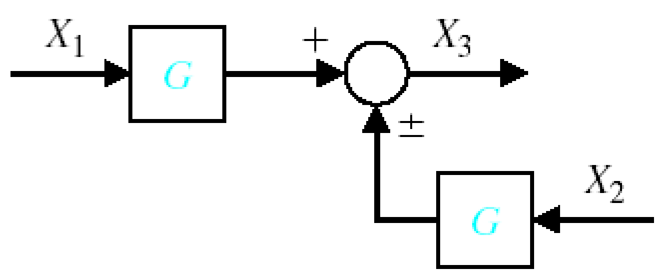
Equivalent Diagram



Original Diagram

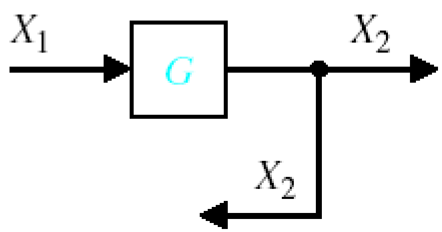


Equivalent Diagram

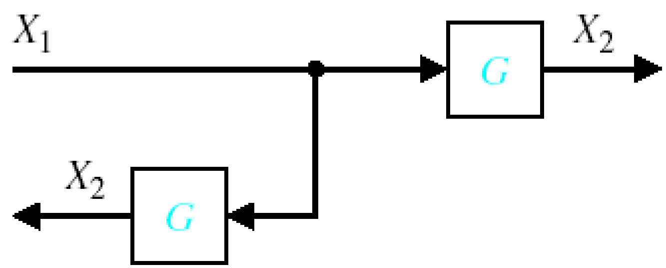


Block Diagram Models

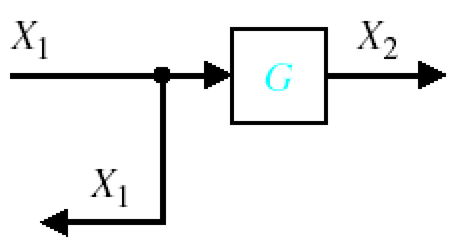
Original Diagram



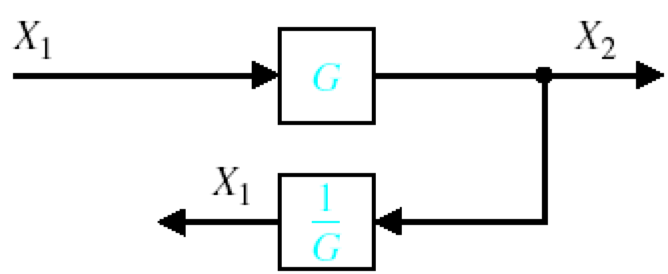
Equivalent Diagram



Original Diagram

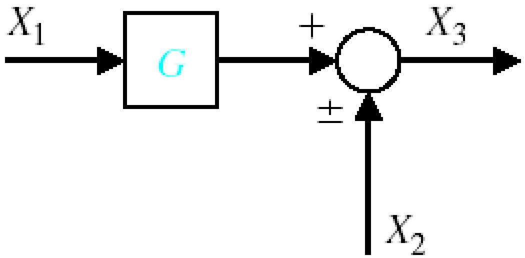


Equivalent Diagram

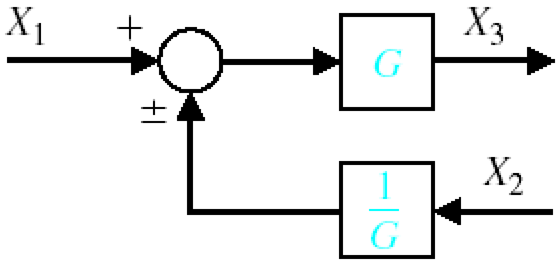


Block Diagram Models

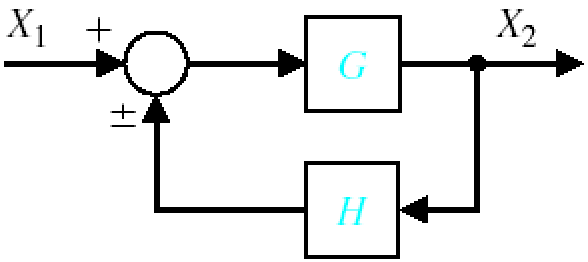
Original Diagram



Equivalent Diagram

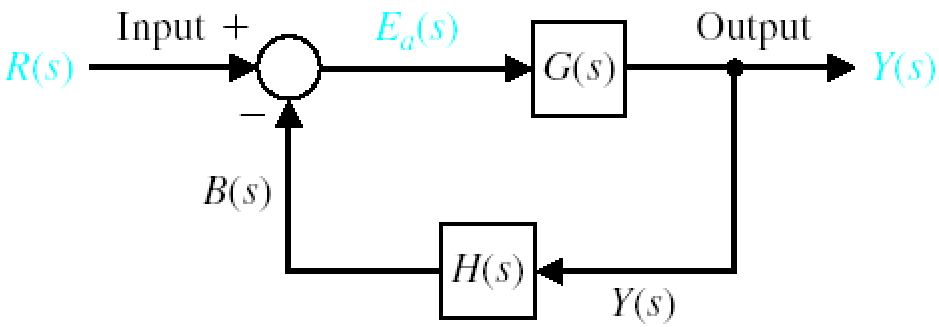


Original Diagram



Equivalent Diagram

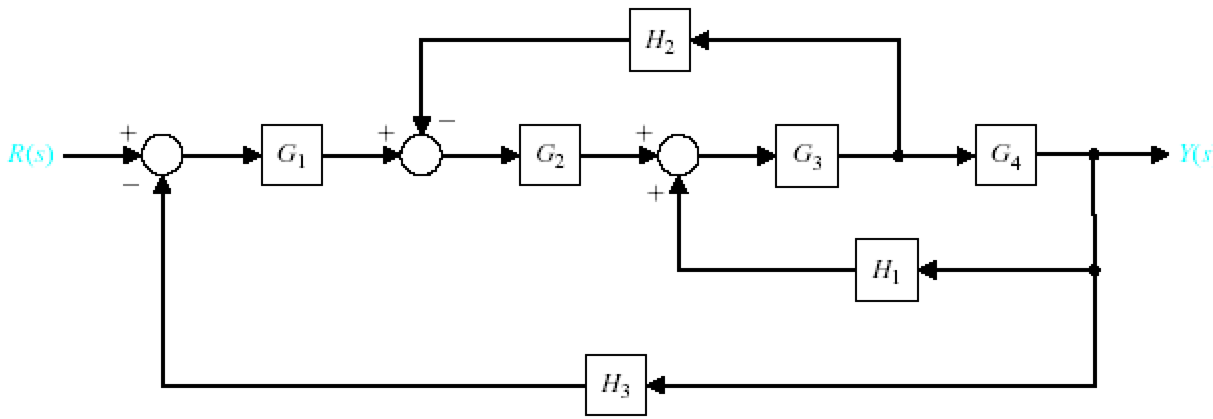
Block Diagram Models



Negative feedback control system.

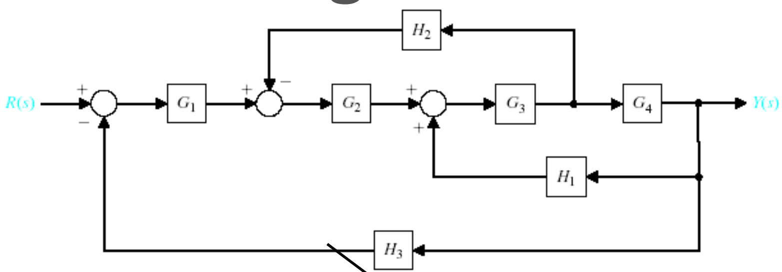
Block Diagram Models

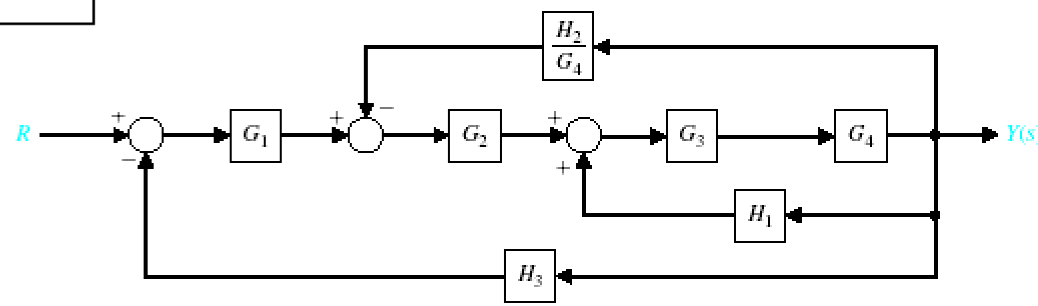
Example 2.7



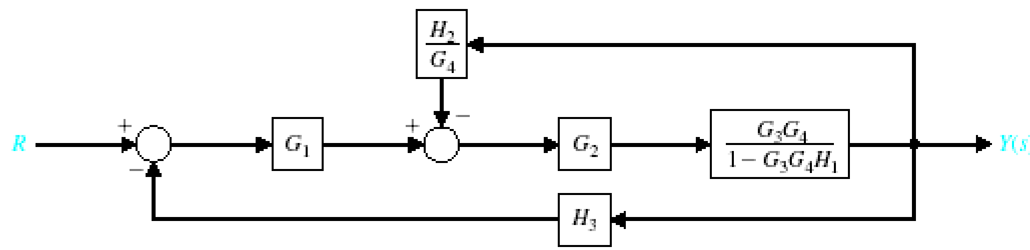
Block Diagram Models

Example 2.7

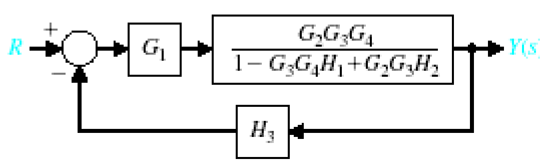




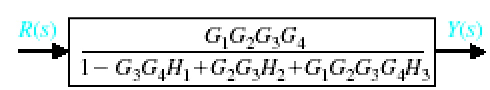
(a)



(b)

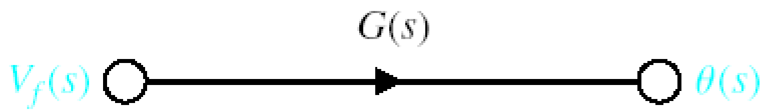


(c)



(d)

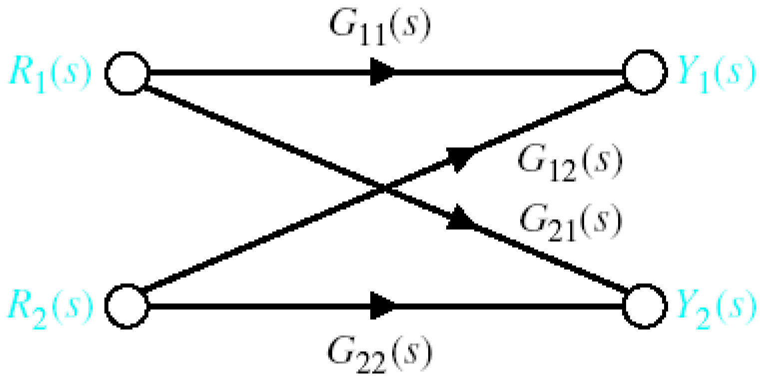
Signal-Flow Graph Models



Signal-flow graph of the dc motor.

For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.

Signal-Flow Graph Models

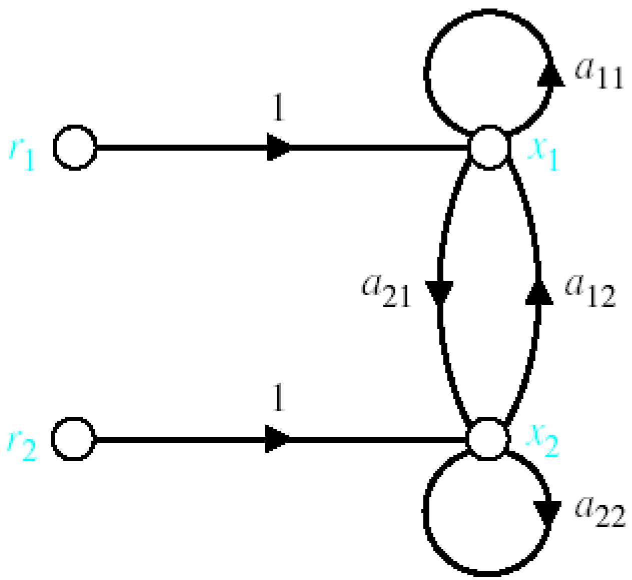


Signal-flow graph of interconnected system.

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

$$Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s)$$

Signal-Flow Graph Models



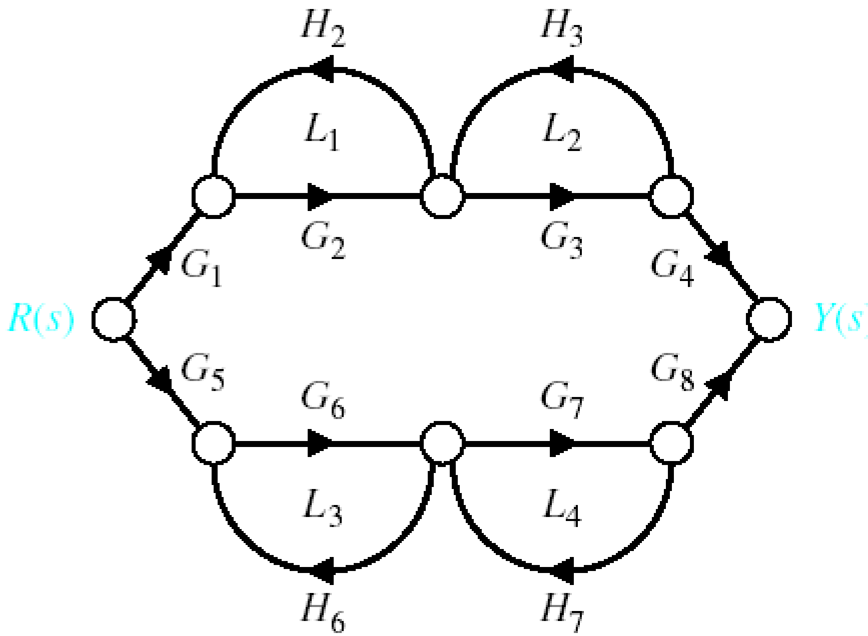
Signal-flow graph of two algebraic equations.

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$

Signal-Flow Graph Models

Example 2.8

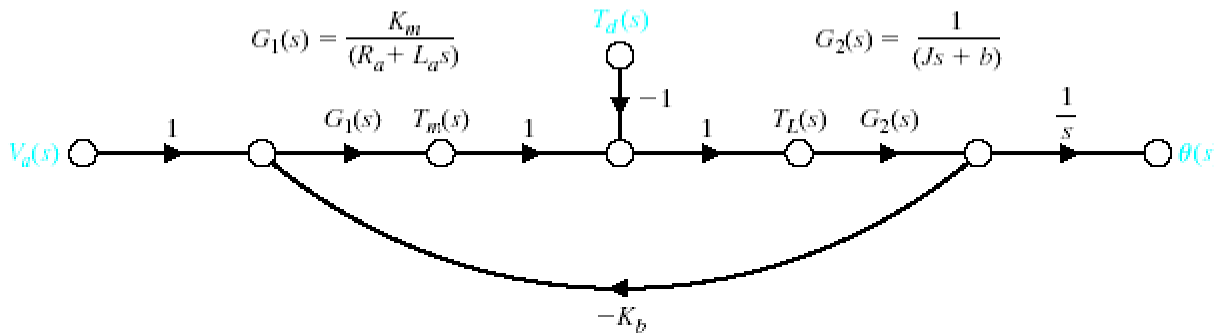


Two-path interacting system.

$$\frac{Y(s)}{R(s)} = \frac{\left[G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot \left(1 - L_3 - L_4\right)\right] + \left[G_5 \cdot G_6 \cdot G_7 \cdot G_8 \cdot \left(1 - L_1 - L_2\right)\right]}{1 - L_1 - L_2 - L_3 - L_4 + L_1 \cdot L_3 + L_1 \cdot L_4 + L_2 \cdot L_3 + L_2 \cdot L_4}$$

Signal-Flow Graph Models

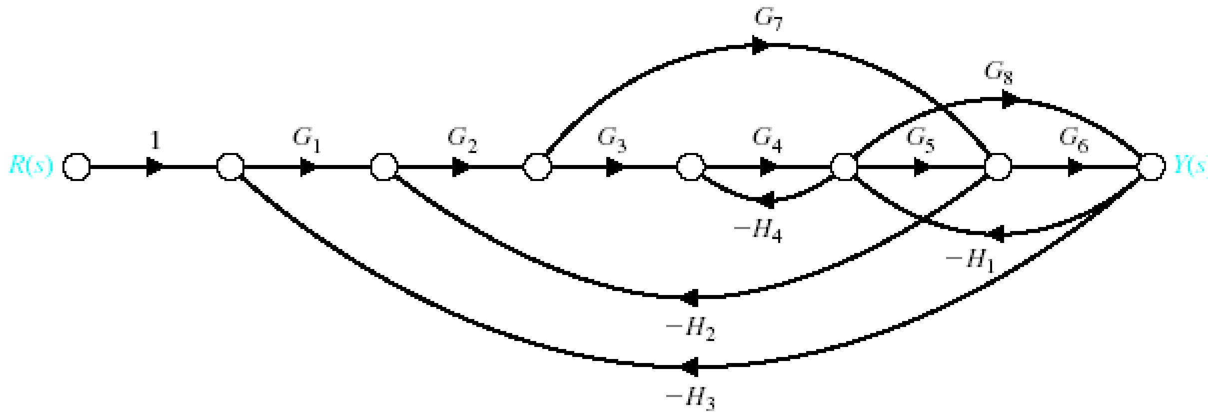
Example 2.10



The signal-flow graph of the armature-controlled dc motor.

$$\frac{Y(s)}{R(s)} = \frac{G_1 \cdot G_2 \cdot G_3 \cdot G_4}{1 + G_2 \cdot G_3 \cdot H_2 - G_3 \cdot G_4 \cdot H_1 + G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot H_3}$$

Signal-Flow Graph Models



Multiple-loop system.

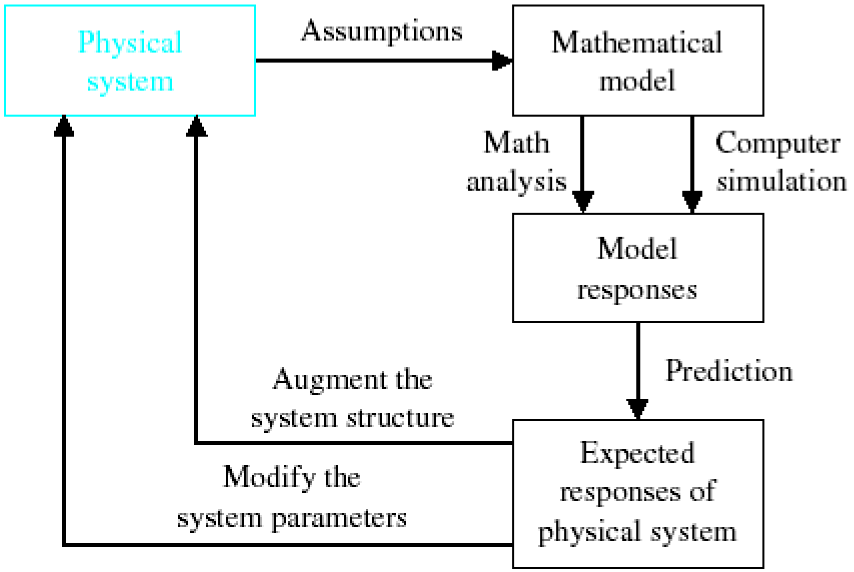
$$\frac{Y(s)}{R(s)} = \frac{P_1 + P_2 \cdot \Delta_2 + P_3}{\Delta}$$

$P_1 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6$ $\quad P_2 = G_1 \cdot G_2 \cdot G_7 \cdot G_6$ $\quad P_3 = G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_8$

$$\Delta = 1 - \left(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8\right) + \left(L_5 \cdot L_7 + L_5 \cdot L_4 + L_3 \cdot L_4\right)$$

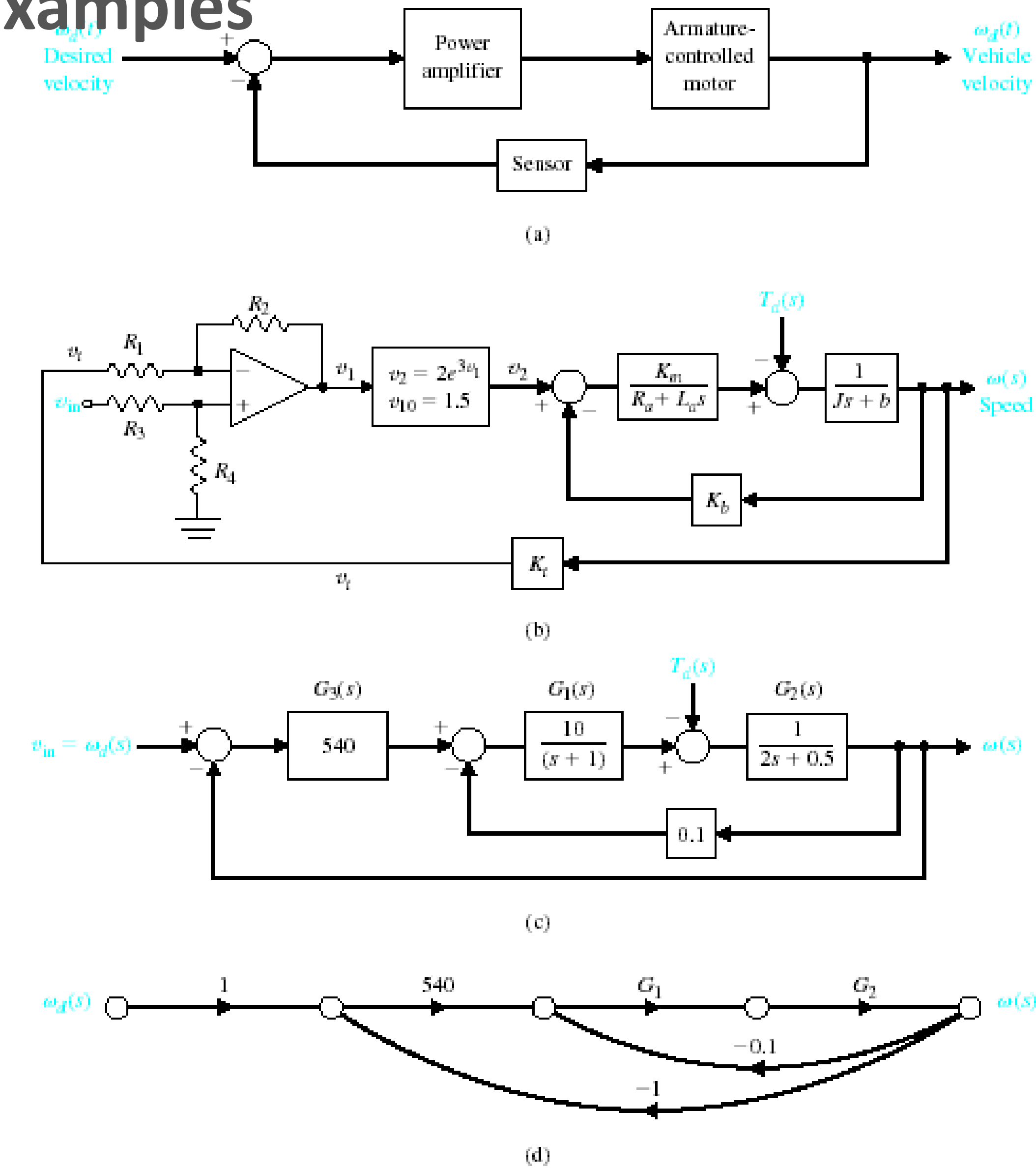
$\Delta_1 = \Delta_3 = 1$ $\quad \Delta_2 = 1 - L_5 = 1 + G_4 \cdot H_4$

Design Examples



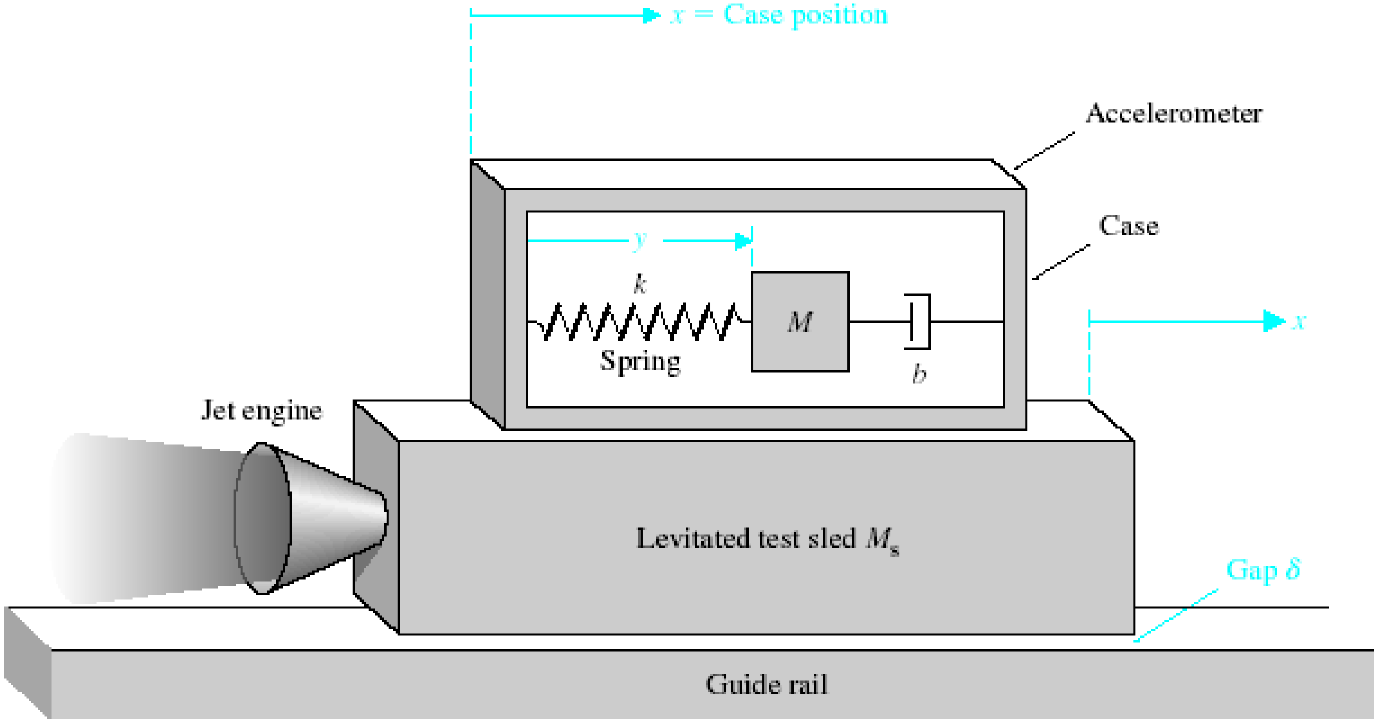
Analysis and design using a system model.

Design Examples



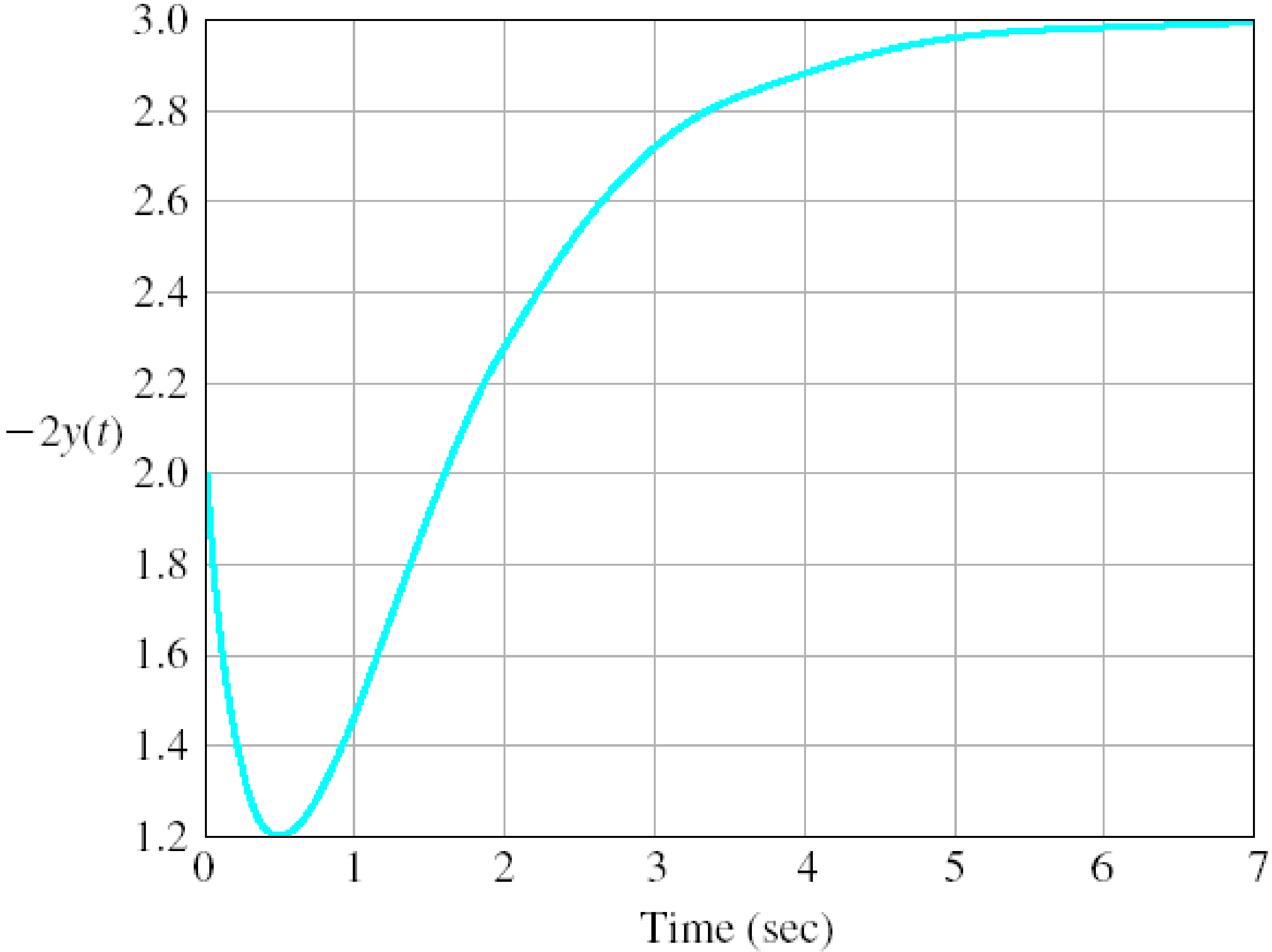
Speed control of an electric traction motor.

Design Examples



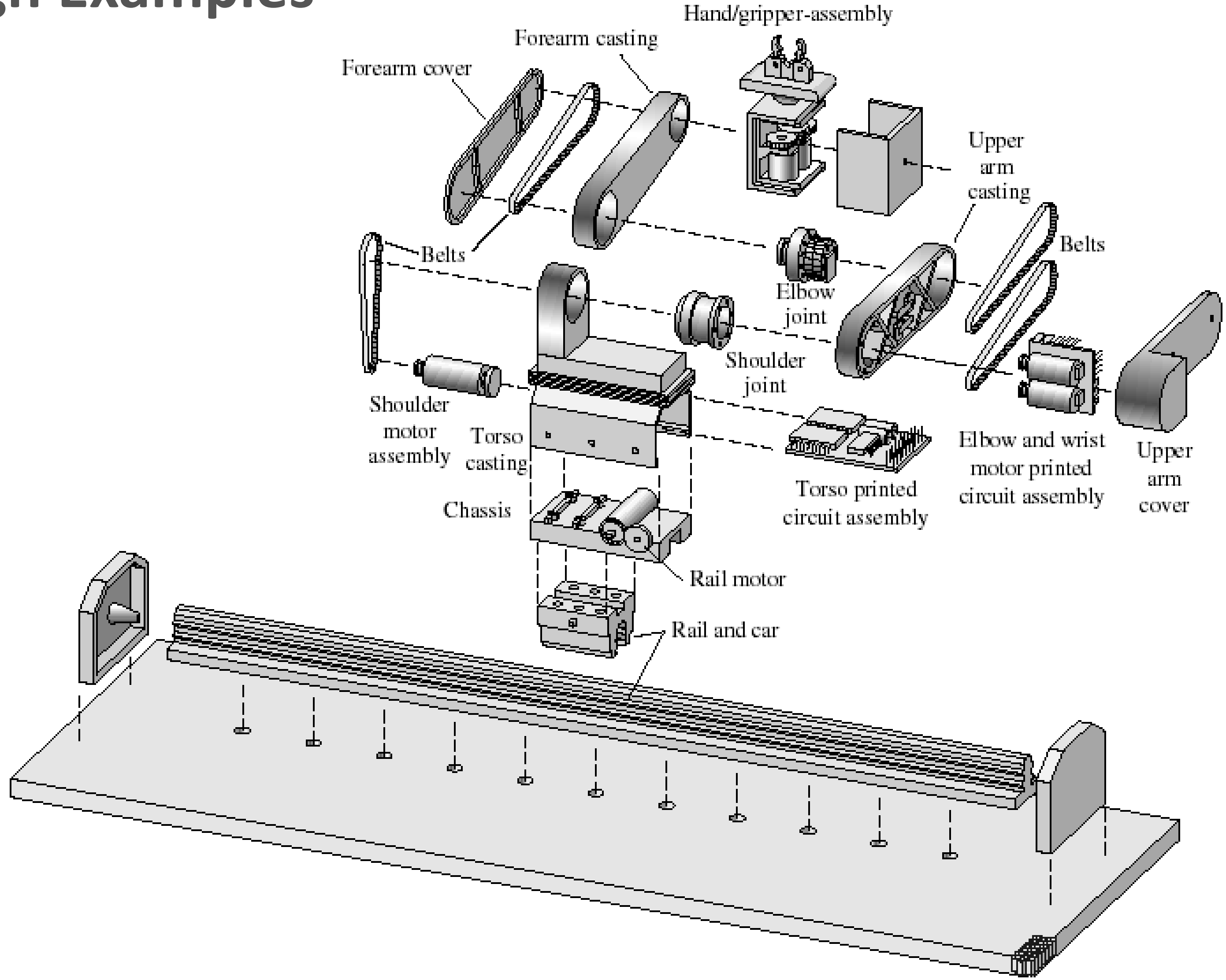
An accelerometer mounted on a jet-engine test sled.

Design Examples



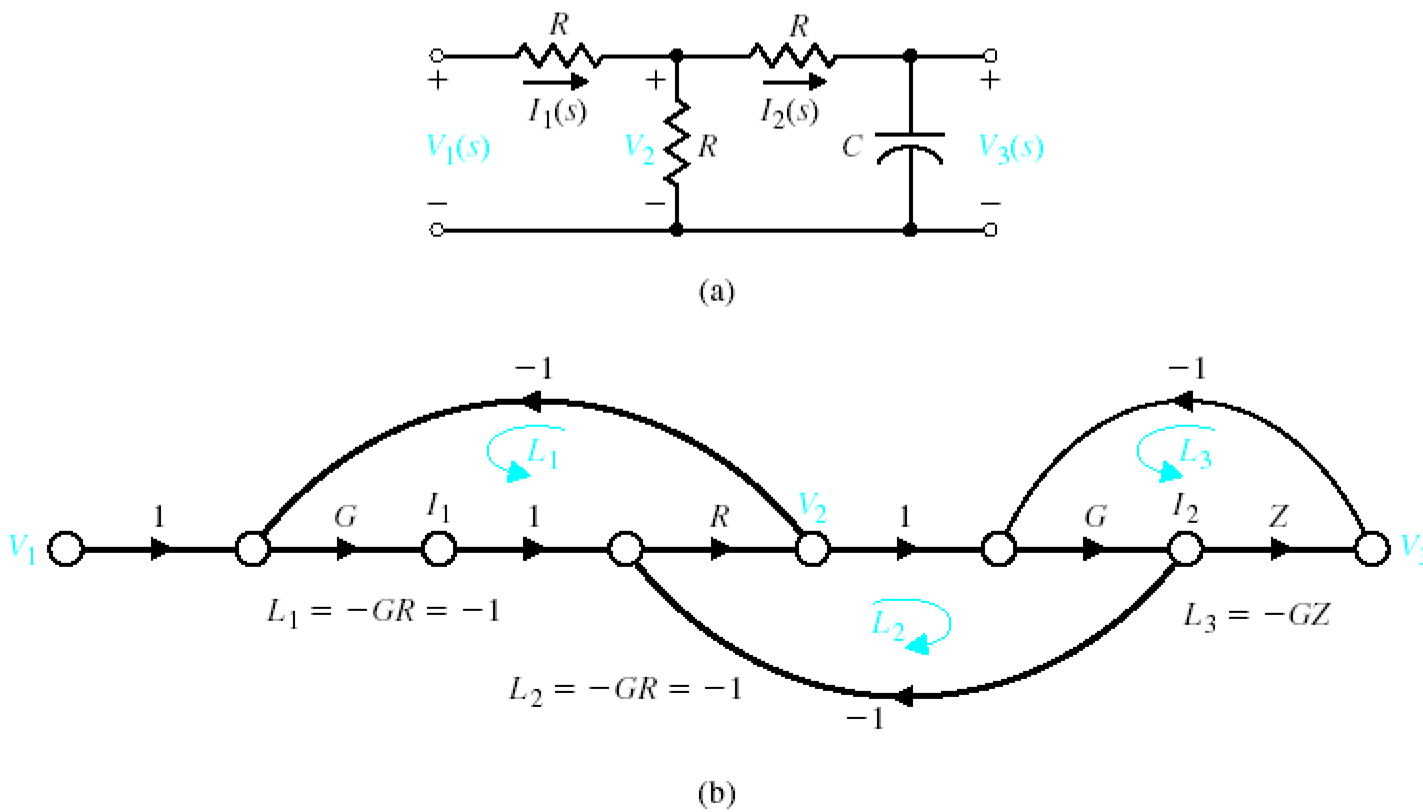
Accelerometer response.

Design Examples



Exploded view of the ORCA robot showing the components [15].
(Source: © Copyright 1993 Hewlett-Packard Company. Reproduced with permission.)

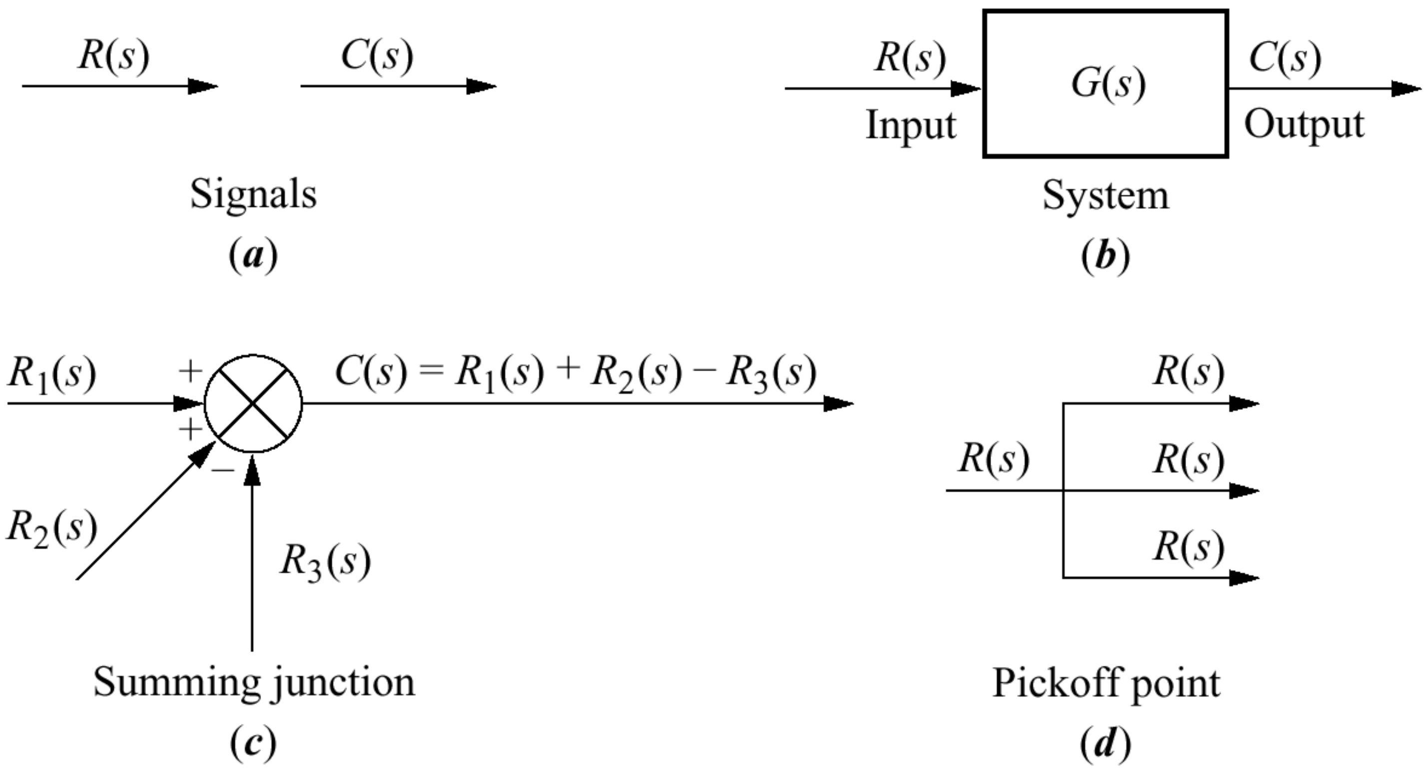
Design Examples



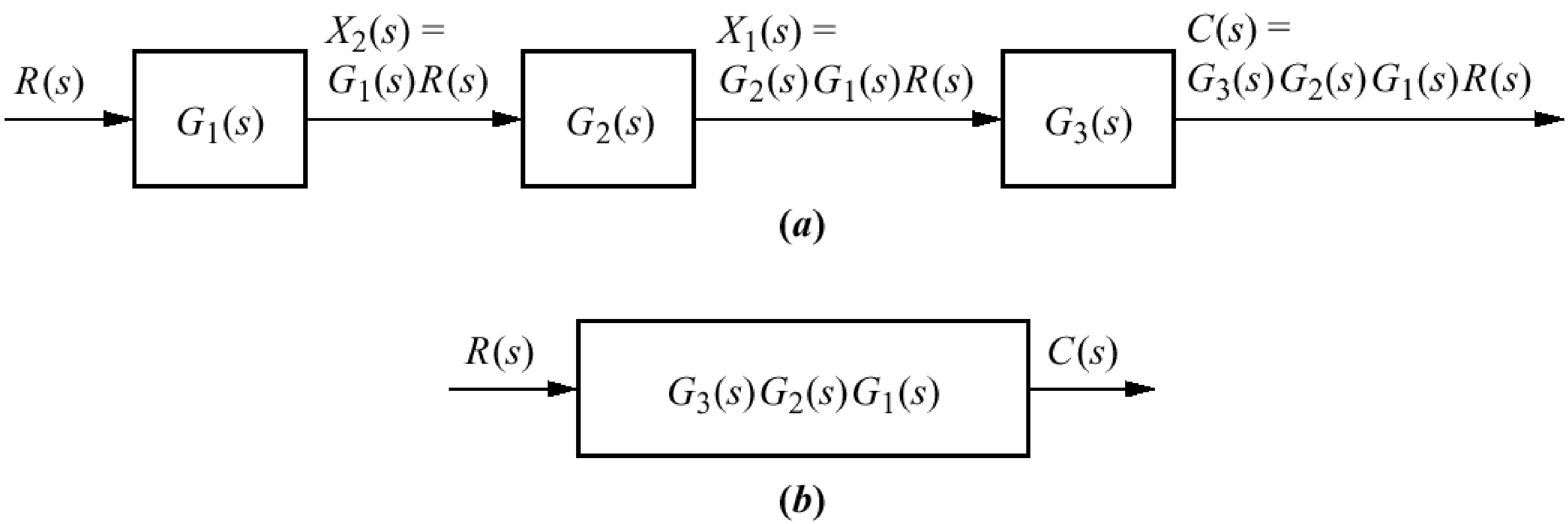
(a) Ladder network
and (b) its signal-flow graph.

BLOCK DIAGRAM REDUCTION OF MULTIPLE SYSTEMS

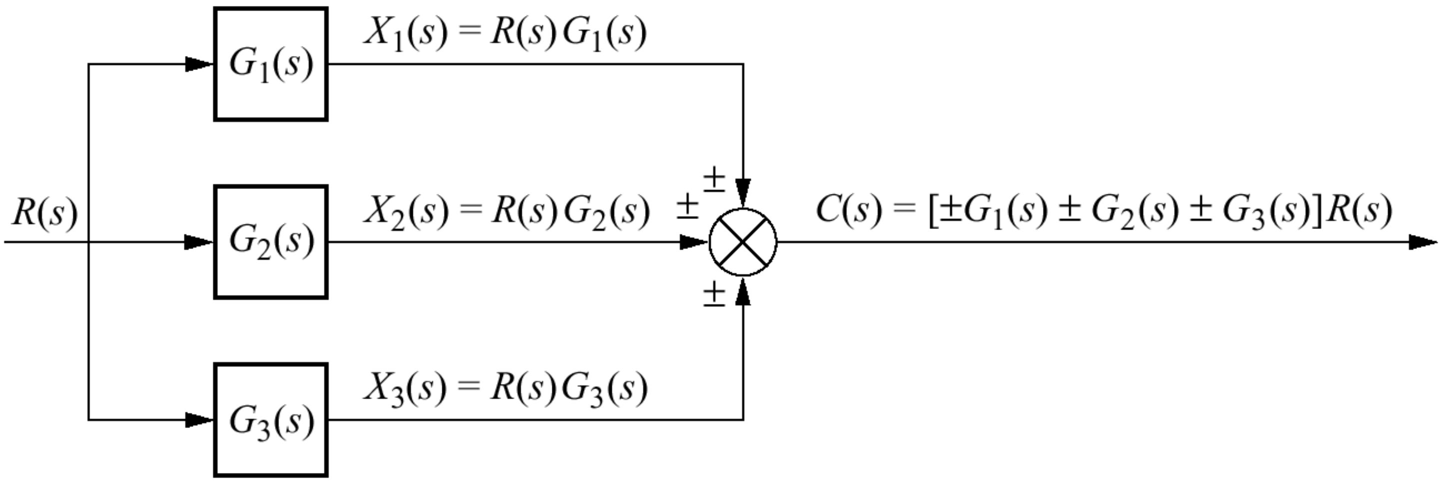
Components of a block diagram for a linear, time-invariant system



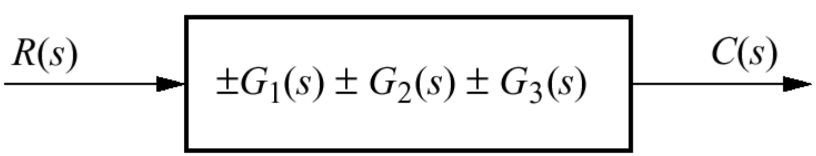
a. Cascaded subsystems;
b. equivalent transfer function



a. Parallel subsystems;
b. equivalent transfer function

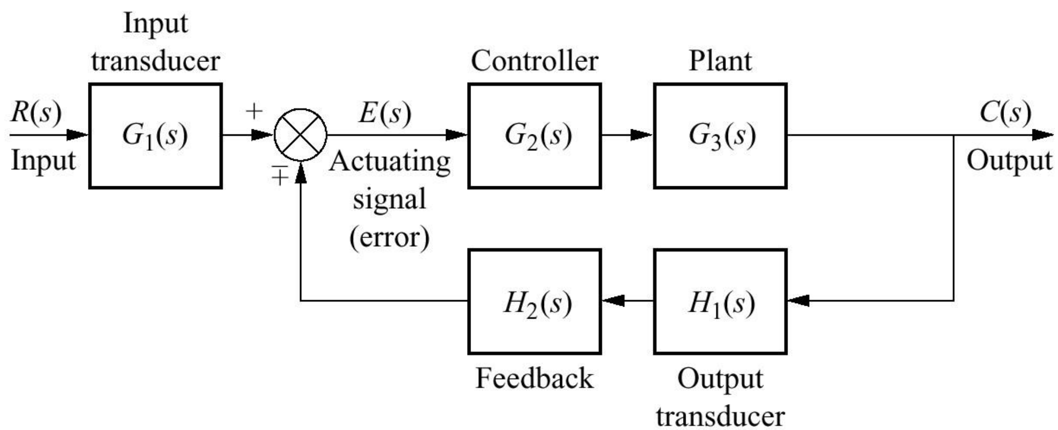


(*a*)

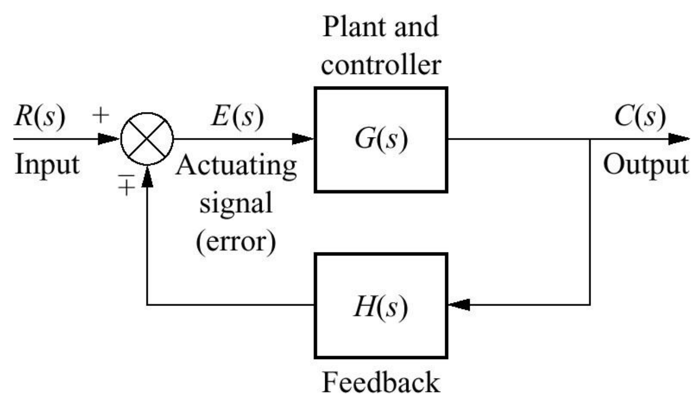


(*b*)

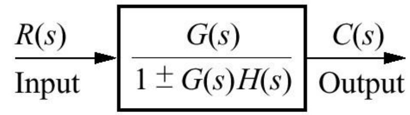
a. Feedback control system;
b. simplified model;
c. equivalent transfer function



(*a*)



(*b*)



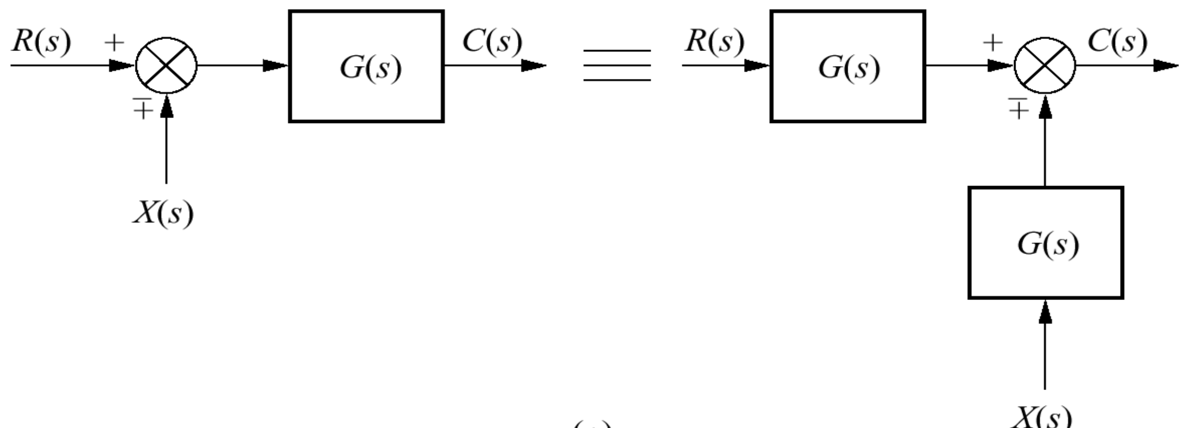
(*c*)

Block diagram algebra for summing junctions

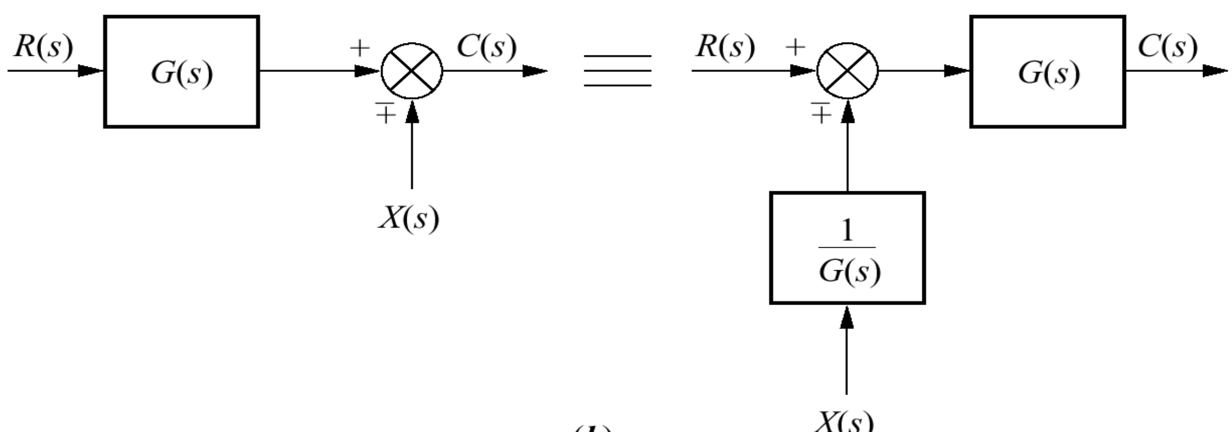
equivalent forms for moving a block

a. to the left past a summing junction;

b. to the right past a summing junction



(*a*)



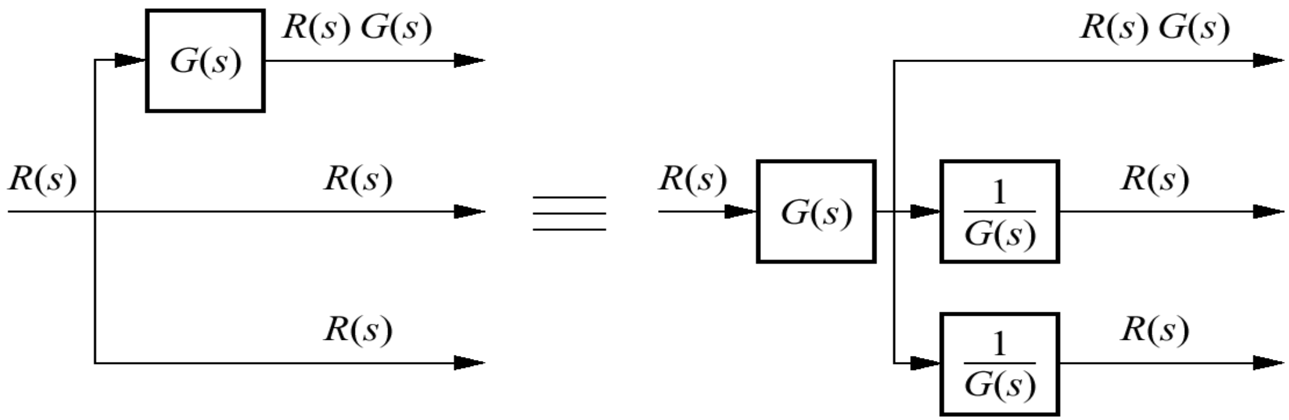
(*b*)

Block diagram algebra for pickoff points

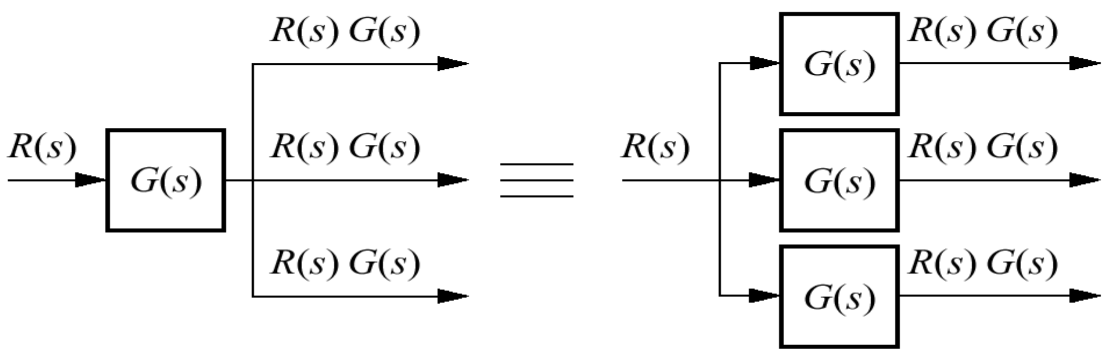
equivalent forms for moving a block

a. to the left past a pickoff point;

b. to the right past a pickoff point



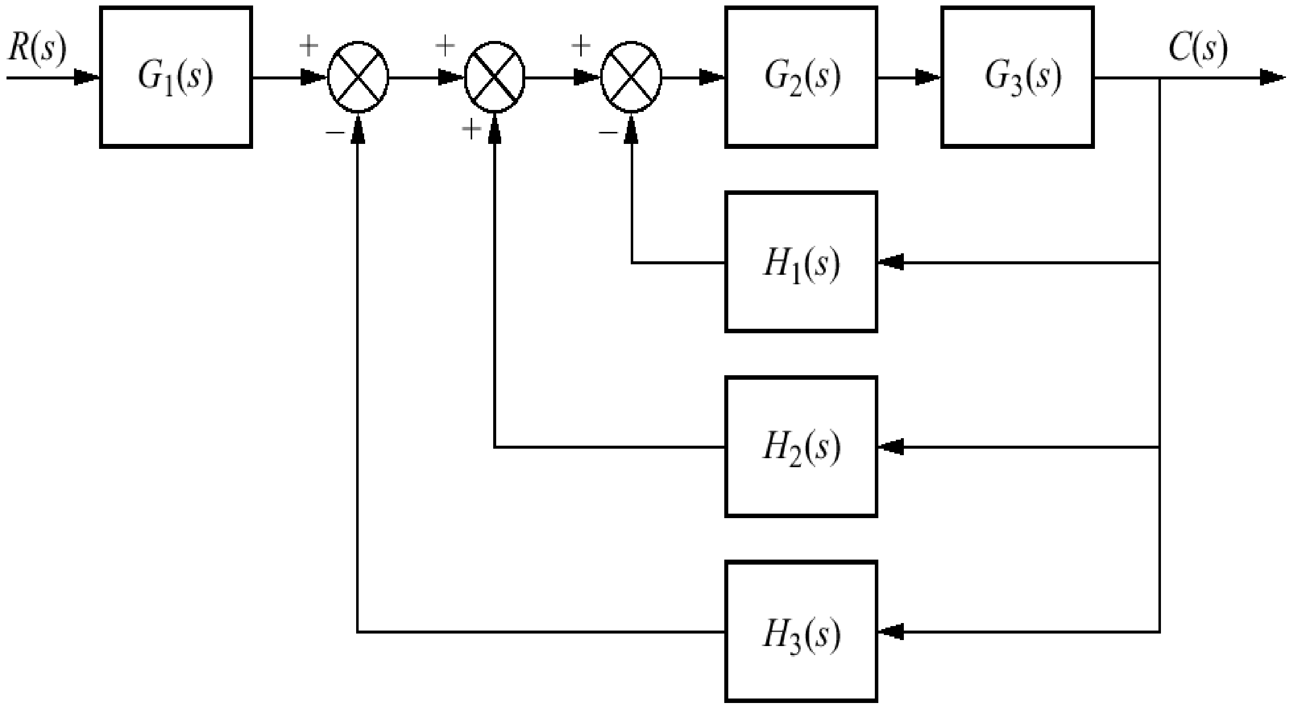
(a)



(b)

Block diagram reduction via familiar forms for Example

Problem: Reduce the block diagram shown in figure to a single transfer function



Block diagram reduction via familiar forms for Example Cont.

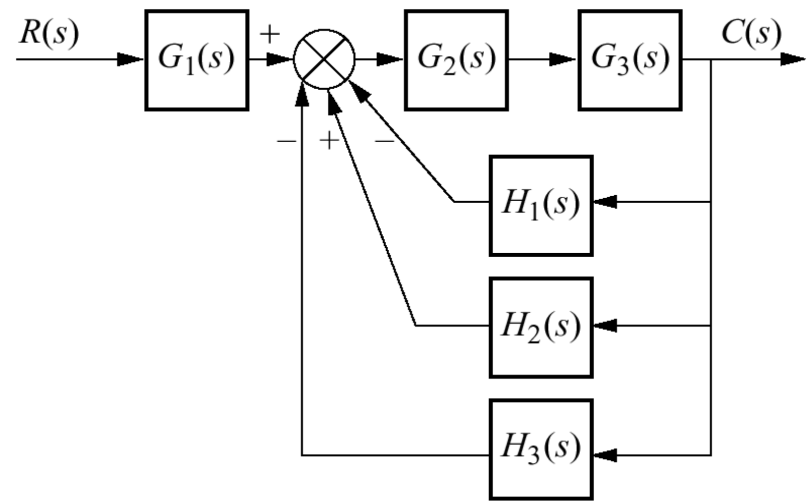
Steps in solving Example

a. collapse summing junctions;

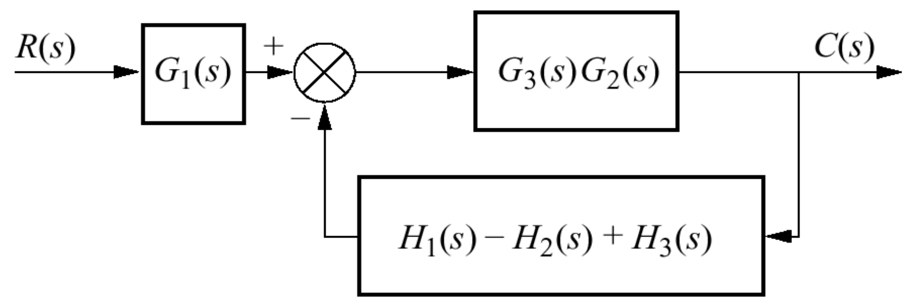
b. form equivalent cascaded system in the forward path

c. form equivalent parallel system in the feedback path;

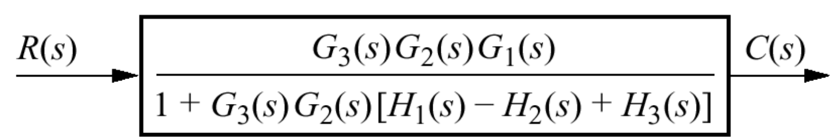
d. form equivalent feedback system and multiply by cascaded$G_1(s)$



(a)



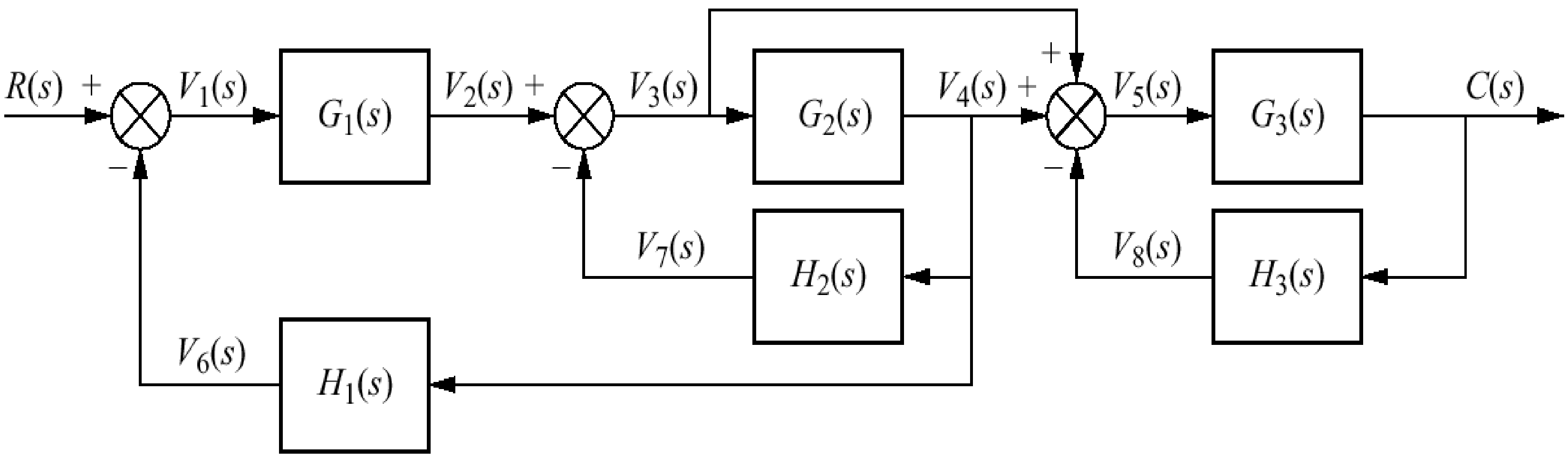
(b)



(c)

Block diagram reduction by moving blocks Example

Problem: Reduce the block diagram shown in figure to a single transfer function



Steps in the block diagram reduction for Example

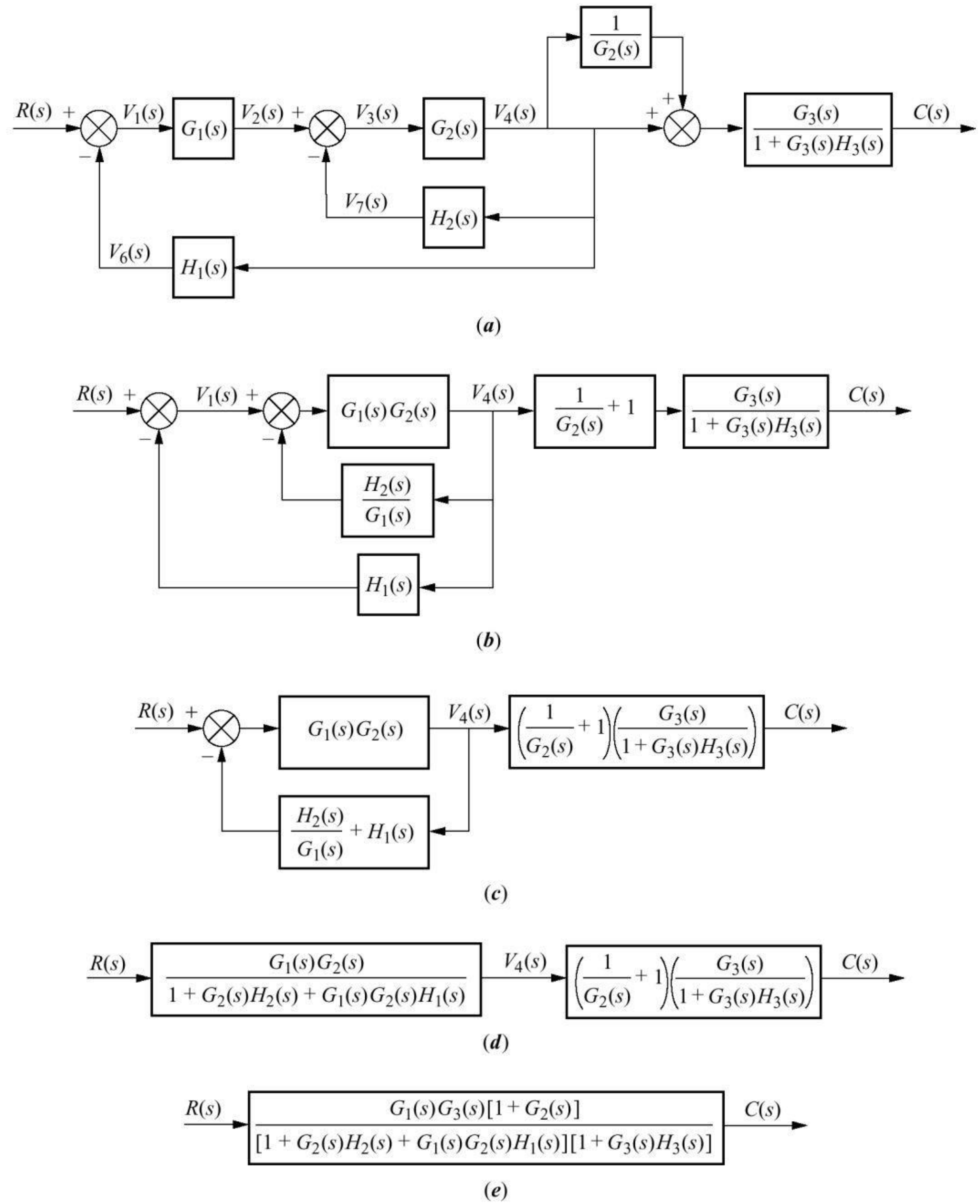
a) Move $G_2(s)$ to the left past of pickoff point to create parallel subsystems, and reduce the feedback system of $G_3(s)$ and $H_3(s)$

b) Reduce parallel pair of $1/G_2(s)$ and unity, and push $G_1(s)$ to the right past summing junction

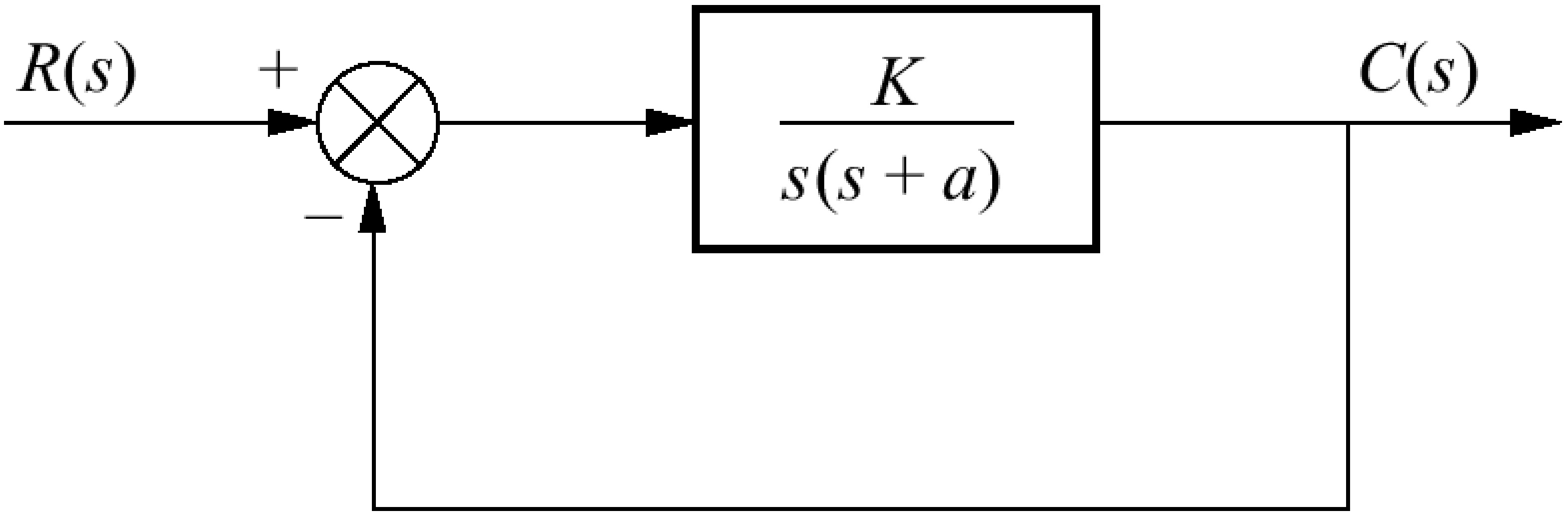
c) Collapse the summing junctions, add the 2 feedback elements, and combine the last 2 cascade blocks

d) Reduce the feedback system to the left

e) finally, Multiple the 2 cascade blocks and obtain final result.



Second-order feedback control system



The closed loop transfer function is $T(s) = \dfrac{K}{s^2 + as + K}$

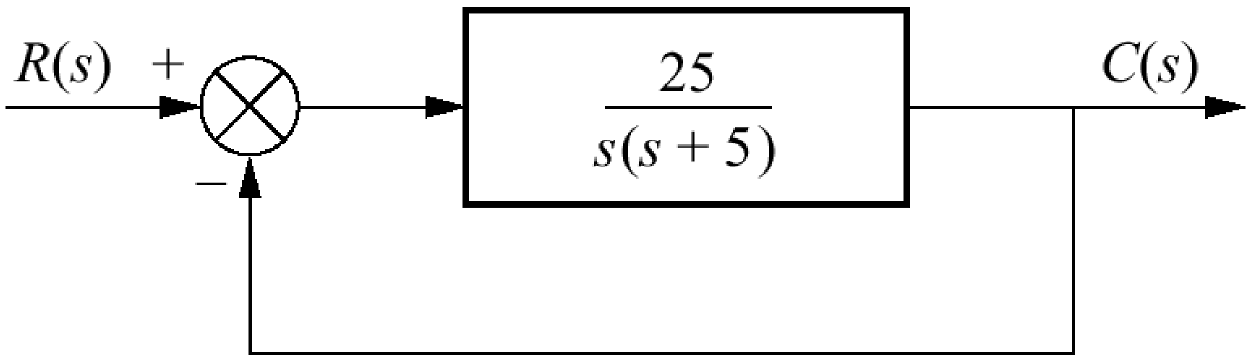
Note K is the amplifier gain, As K varies, the poles move through the three ranges of operations OD, CD, and UD

$0<K<a^2/4$ system is over damped

$K = a^2/4$ system is critically damped

$K > a^2/4$ system is under damped

Finding transient response Example



Problem: For the system shown, find peak time, percent overshot, and settling time.

Solution: The closed loop transfer function is $T(s) = \dfrac{25}{s^2 + 5s + 25}$

And
$$\omega_n = \sqrt{25} = 5$$
$$2\xi\omega_n = 5 \quad \text{so} \quad \xi = 0.5$$

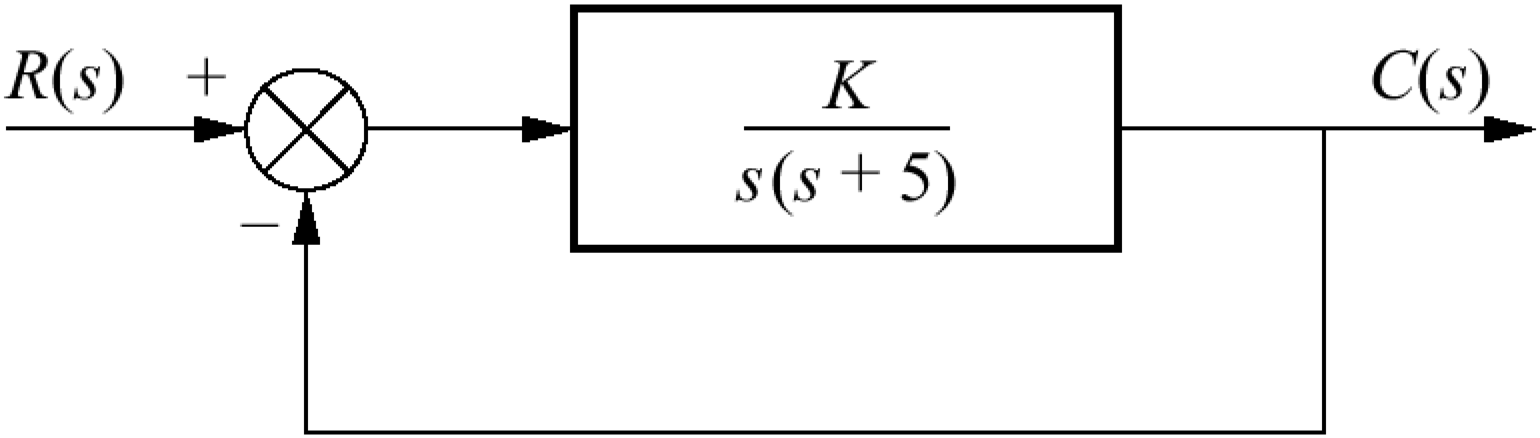
using values for ξ and ω_n and equation in chapter 4 we find

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.726 \quad \text{sec}$$

$$\%OS = e^{-\xi\pi/\sqrt{1-\xi^2}} X\ 100 = 16.303$$

$$T_s = \frac{4}{\xi\omega_n} = 1.6 \quad \text{sec}$$

Gain design for transient response Example



Problem: Design the value of gain K, so that the system will respond with a 10% overshot.

Solution: The closed loop transfer function is $T(s) = \dfrac{K}{s^2 + 5s + K}$

$$\omega_n = \sqrt{K} \quad \text{and} \quad 2\xi\omega_n = 5 \quad \text{thus} \quad \xi = \frac{5}{2\sqrt{K}}$$

For 10% OS we find $\xi = 0.591$

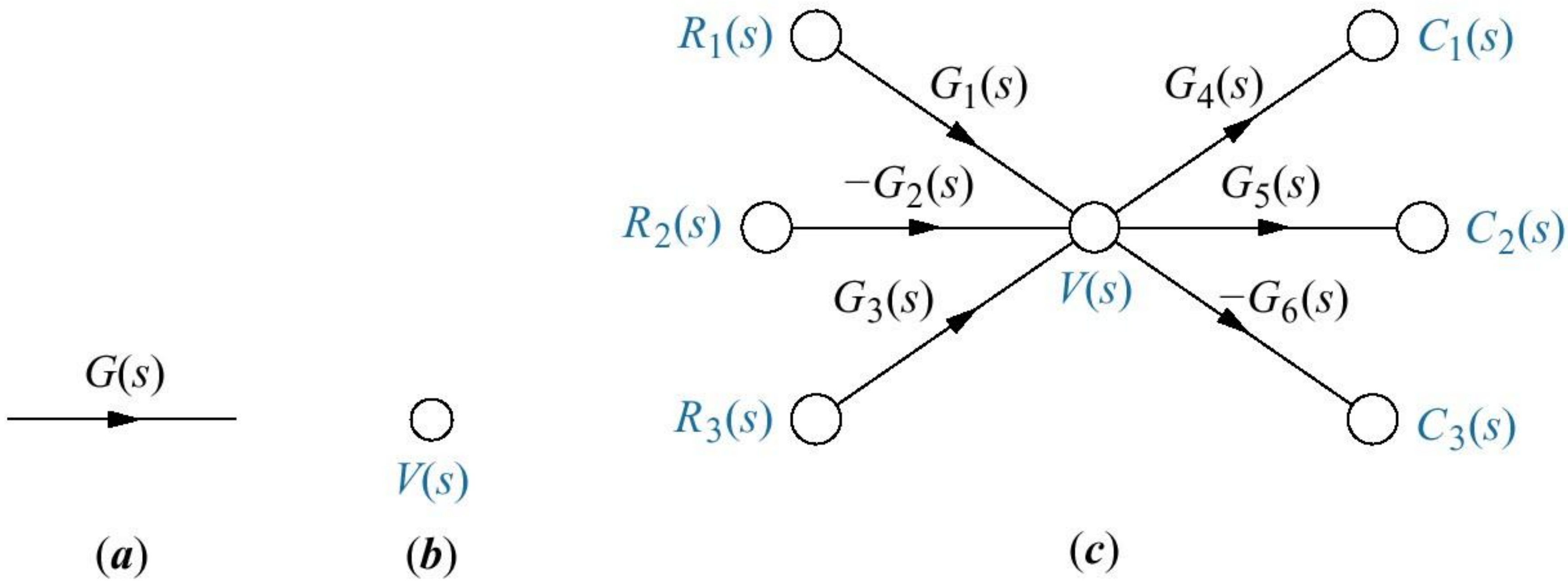
We substitute this value in previous equation to find K = 17.9

Signal-flow graph components:

a. system;

b. signal;

c. interconnection of systems and signals



Building signal-flow graphs

a. cascaded system nodes

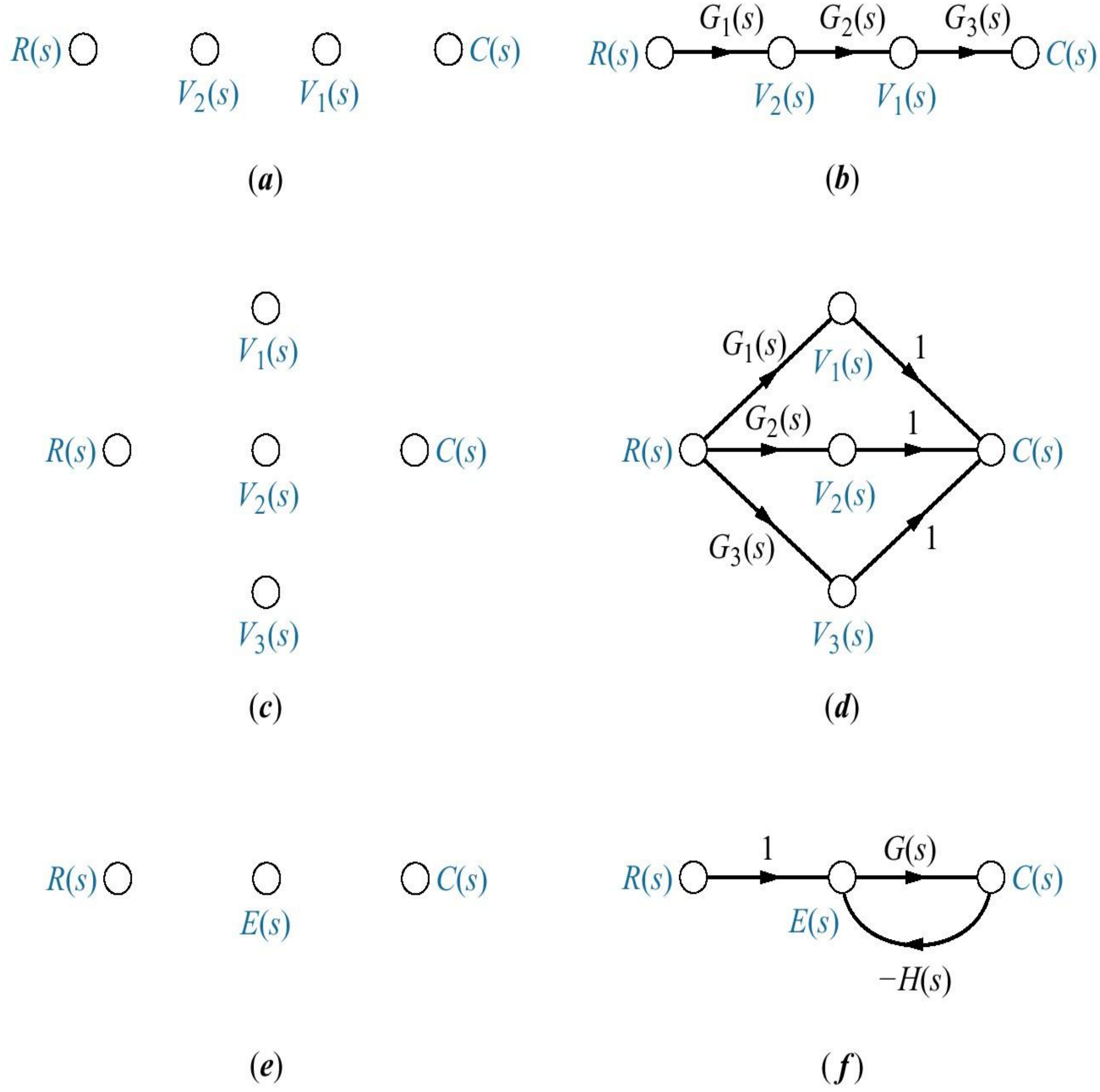
b. cascaded system signal-flow graph;

c. parallel system nodes

d. parallel system signal-flow graph;

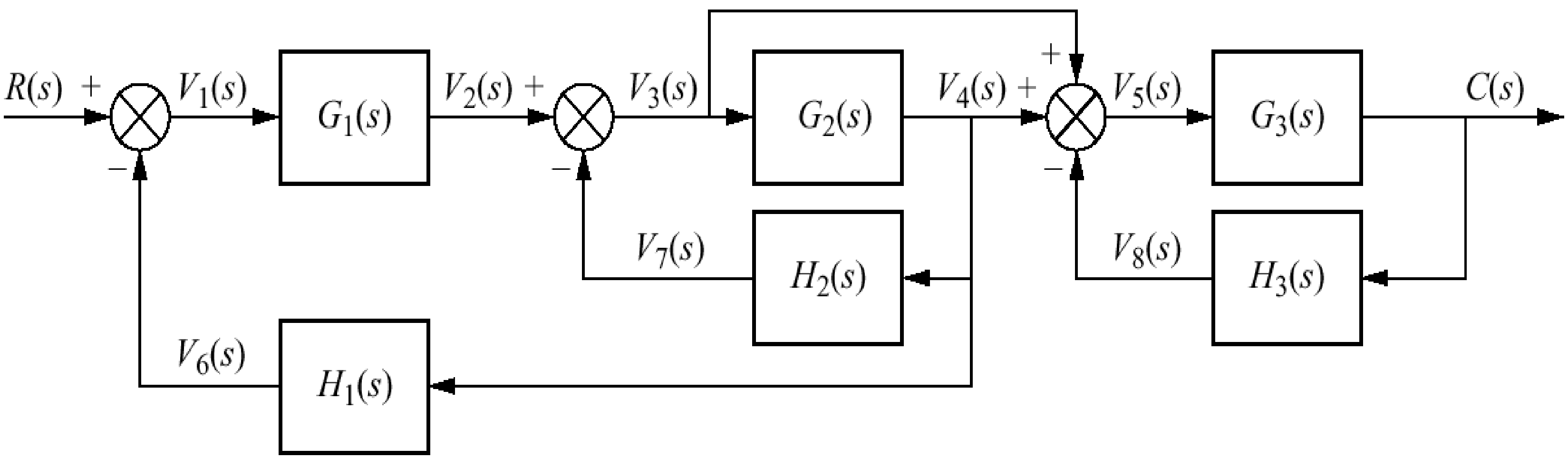
e. feedback system nodes

f. feedback system signal-flow graph



Converting a block diagram to a signal-flow graph

Problem: Convert the block diagram to a signal-flow graph.



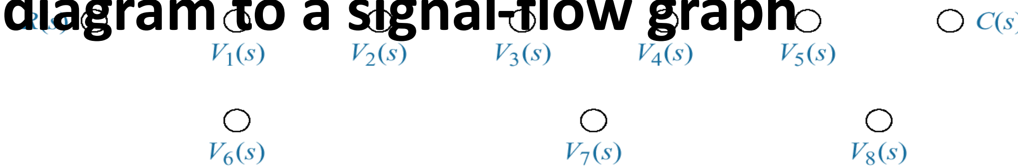
Converting a block diagram to a signal-flow graph

Signal-flow graph development:

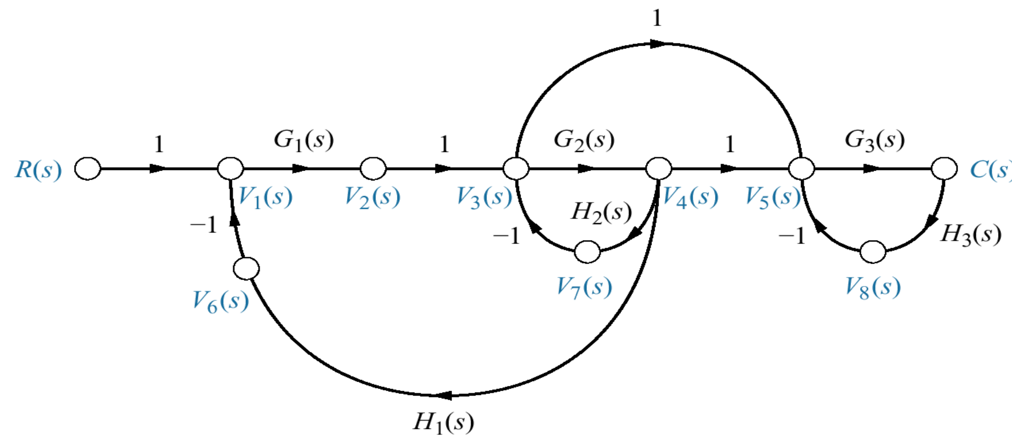
a. signal nodes;

b. signal-flow graph;

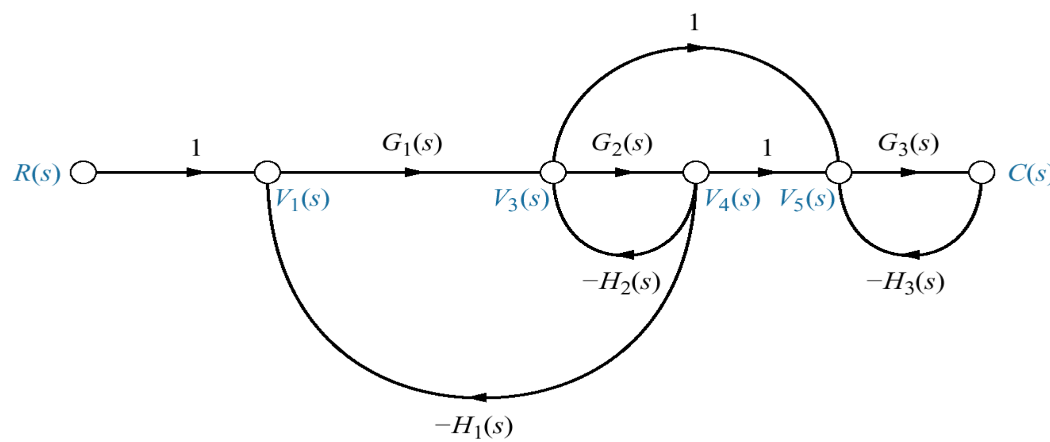
c. simplified signal-flow graph



(a)

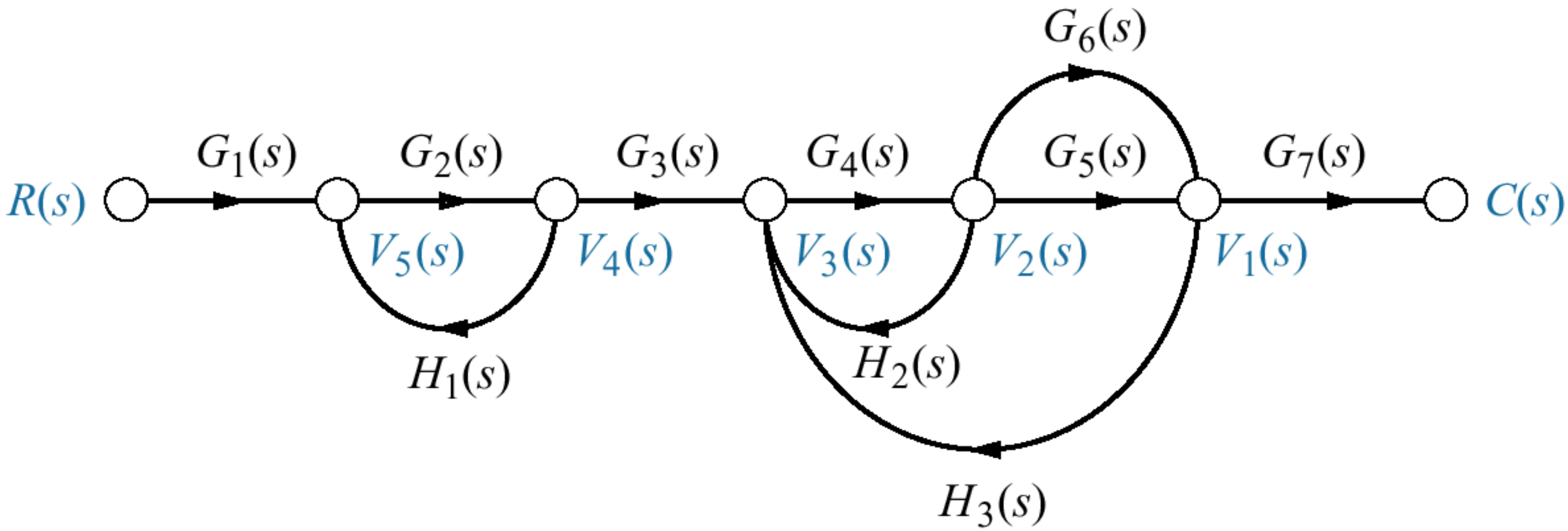


(b)



(c)

Mason’s Rule - Definitions



Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow, without passing through any other node more than once. **$G_2(s)H_2(s)$, $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, $G_4(s)G_6(s)H_3(s)$**

Forward-path gain: The product of gains found by traversing a path from input node to output node in the direction of signal flow. **$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$, $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$**

Nontouching loops: loops that do not have any nodes in common. **$G_2(s)H_1(s)$** does not touch **$G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$**

Nontouching-loop gain: The product of loop gains from nontouching loops taken 2, 3,4, or more at a time.

$[G_2(s)H_1(s)][G_4(s)H_2(s)]$, $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$, $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

Mason's Rule

The Transfer function. C(s)/ R(s), of a system represented by a signal-flow graph is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

Where

K = number of forward paths

T_k = the k*th* forward-path gain

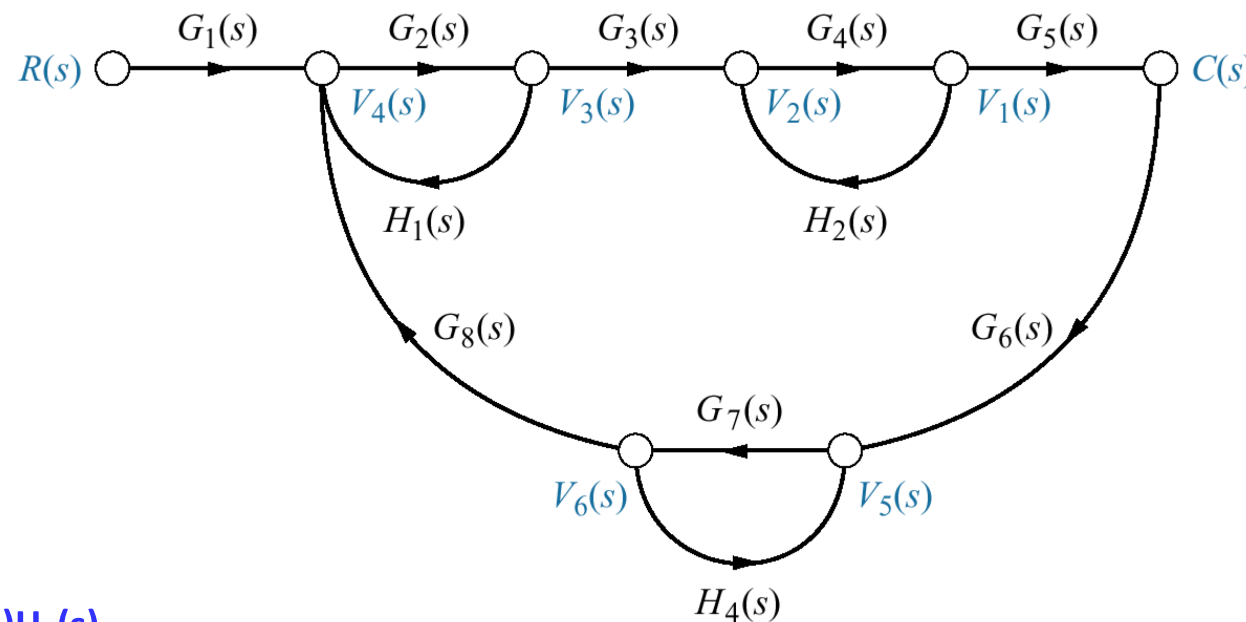
Δ = 1 - $\sum$ loop gains + $\sum$ nontouching-loop gains taken 2 at a time - $\sum$ nontouching-loop gains taken 3 at a time + $\sum$ nontouching-loop gains taken 4 at a time - ....

Δ_k = Δ - $\sum$ loop gain terms in Δ that touch the k*th* forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the k*th* forward path.

Transfer function via Mason's rule

Problem: Find the transfer function for the signal flow graph

Solution:



forward path

$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

Loop gains

$G_2(s)H_1(s)$, $G_4(s)H_2(s)$, $G_7(s)H_4(s)$,

$G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Nontouching loops

2at a time

$G_2(s)H_1(s)G_4(s)H_2(s)$

$G_2(s)H_1(s)G_7(s)H_4(s)$

$G_4(s)H_2(s)G_7(s)H_4(s)$

3at a time $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Now

Δ = 1-[G2(s)H1(s)+$G_4(s)H_2(s)$+$G_7(s)H_4(s)$+ $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$] + [$G_2(s)H_1(s)G_4(s)H_2(s)$ + $G_2(s)H_1(s)G_7(s)H_4(s)$ + $G_4(s)H_2(s)G_7(s)H_4(s)$] – [$G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$]

Δ_1 = 1 - G7(s)H4(s)

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1-G_7(s)H_4(s)]}{\Delta}$$

Signal-Flow Graphs of State Equations

- **Problem:** draw signal-flow graph for:

$x_1 = 2x_1 - 5x_2 + 3x_3 + 2r$

$x_2 = -6x_1 - 2x_2 + 2x_3 + 5r$

$x_3 = x_1 - 3x_2 - 4x_3 + 7r$

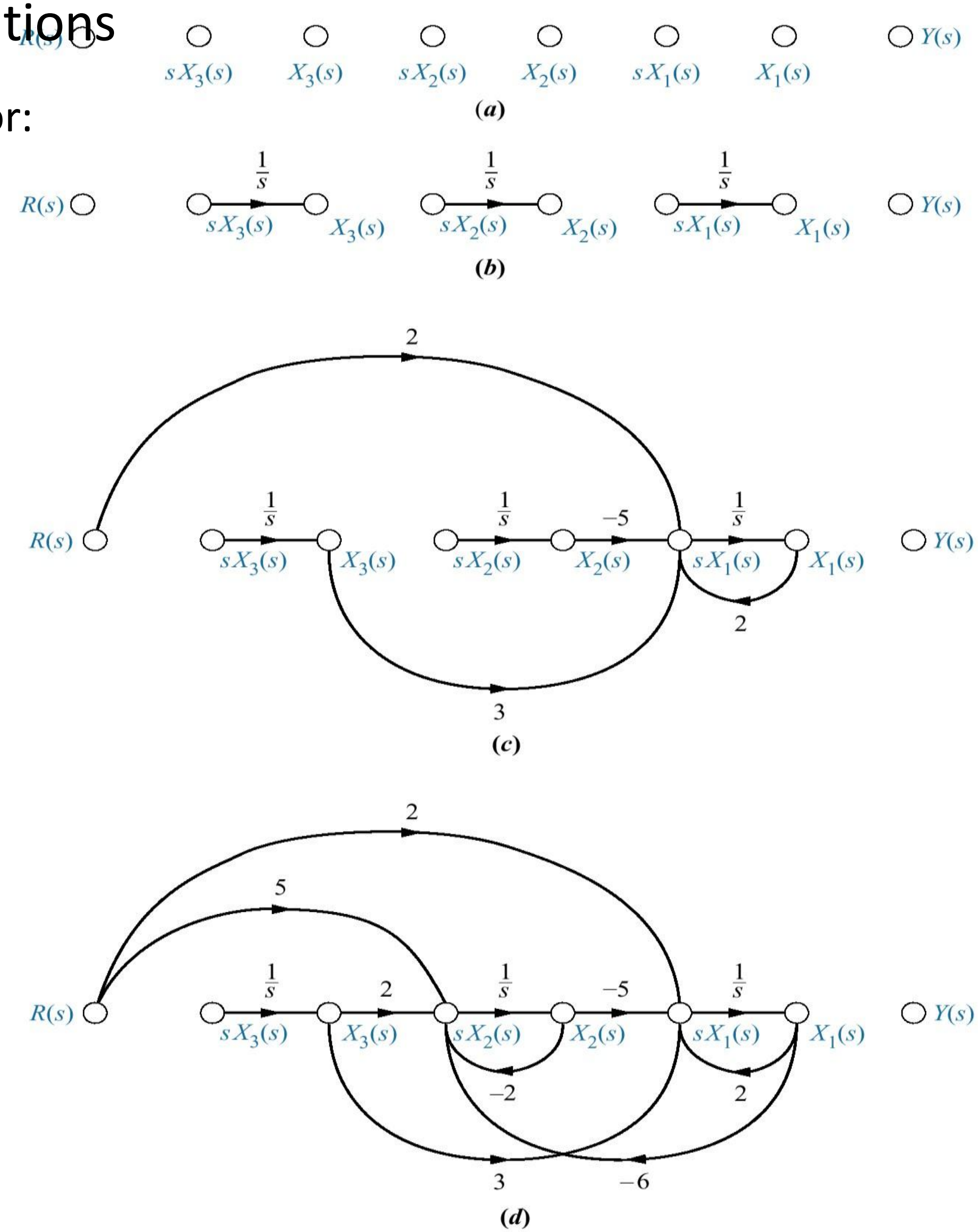
$y = -4x_1 + 6x_2 + 9x_3$

a. place nodes;

b. interconnect state variables and derivatives;

c. form dx1/dt ;

d. form dx2/dt

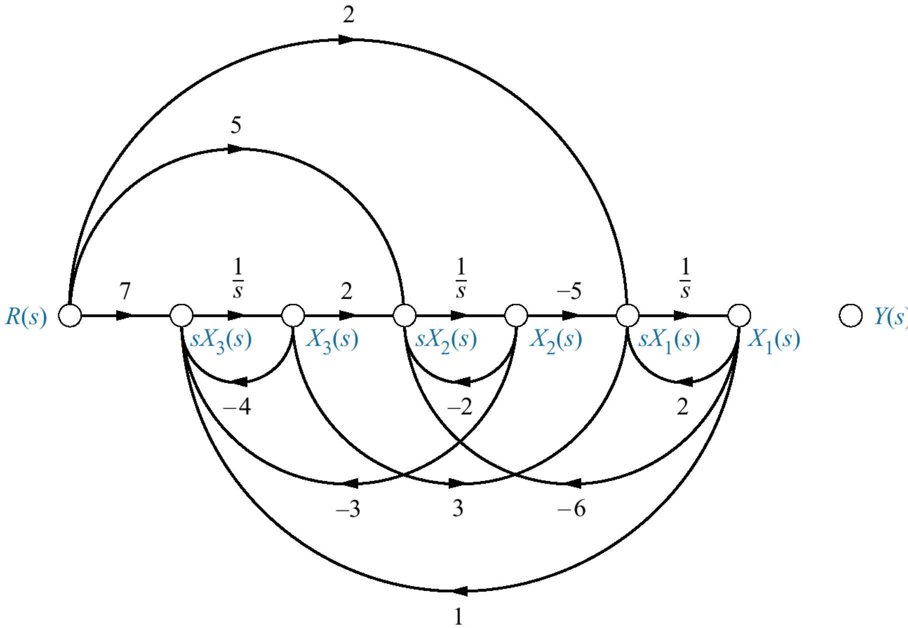


Signal-Flow Graphs of State Equations

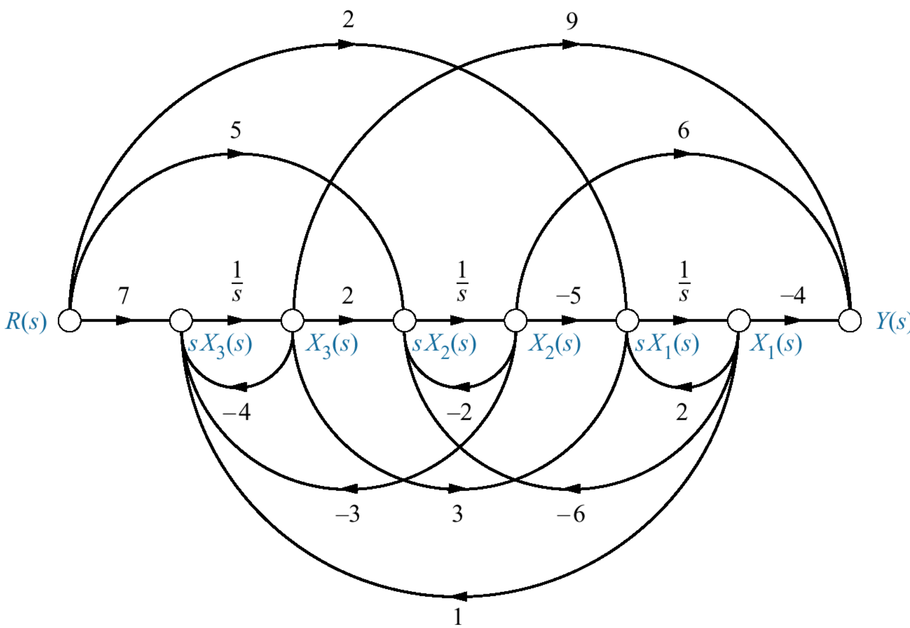
(continued)

e. form dx_3/dt;

f. form output

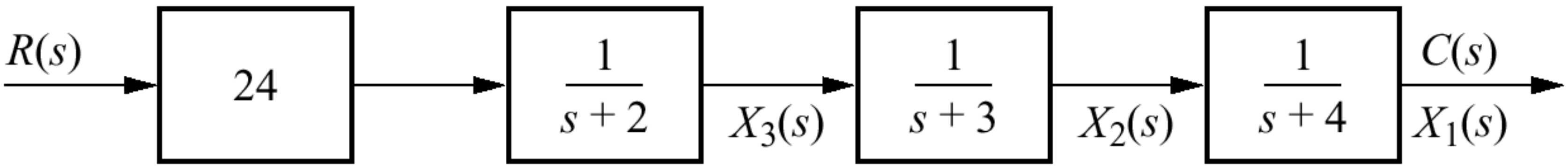


(*e*)

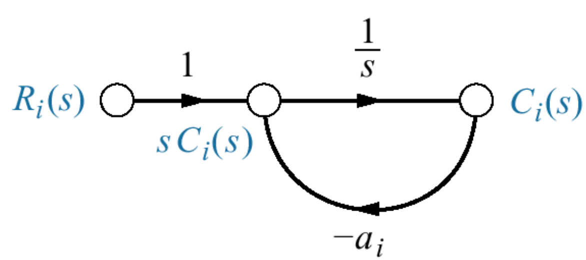


(*f*)

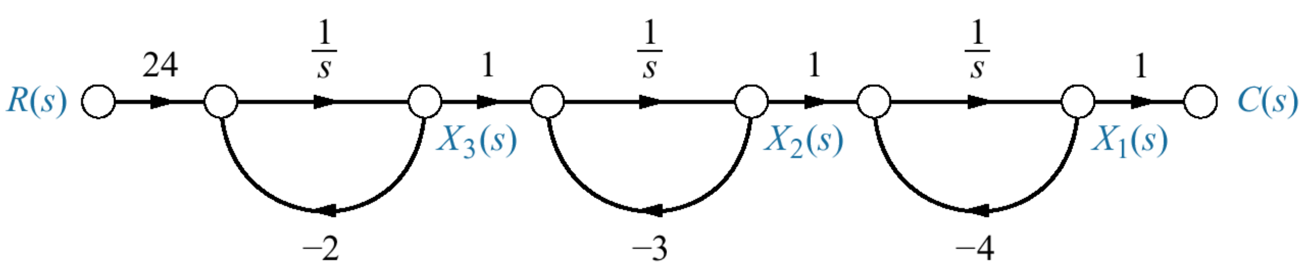
Alternate Representation: Cascade Form



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$

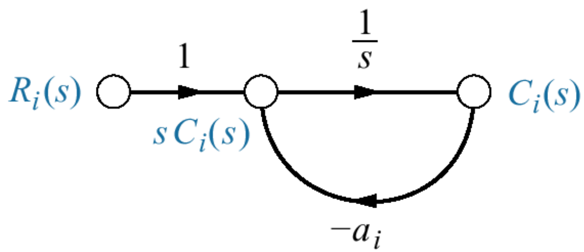


(*a*)

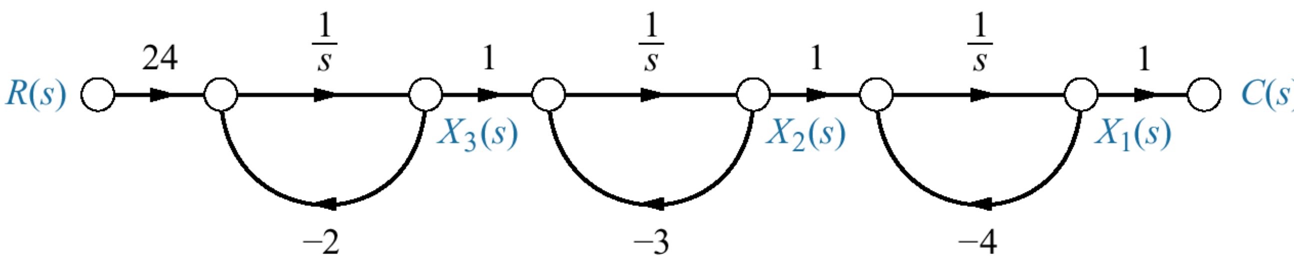


(*b*)

Alternate Representation: Cascade Form



(*a*)



(*b*)

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + 24r$$
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

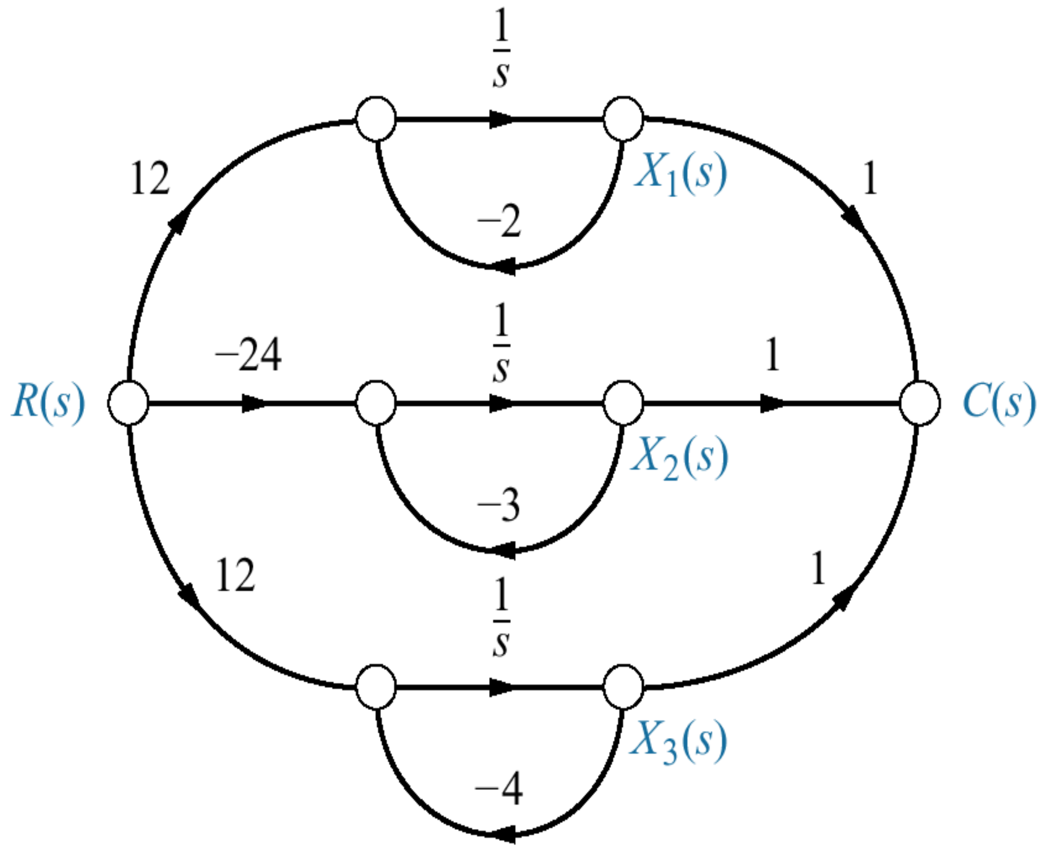
Alternate Representation: Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 12r \\
\dot{x}_2 &= -3x_2 - 24r \\
\dot{x}_3 &= -4x_3 + 12r \\
y &= c(t) = x_1 + x_2 + x_3
\end{aligned}$$

$$\dot{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$



Alternate Representation: Parallel Form Repeated roots

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} \quad \frac{1}{(s+1)} \quad \frac{1}{(s+2)}$$

$$\dot{x}_1 = -x_1 + x_2$$

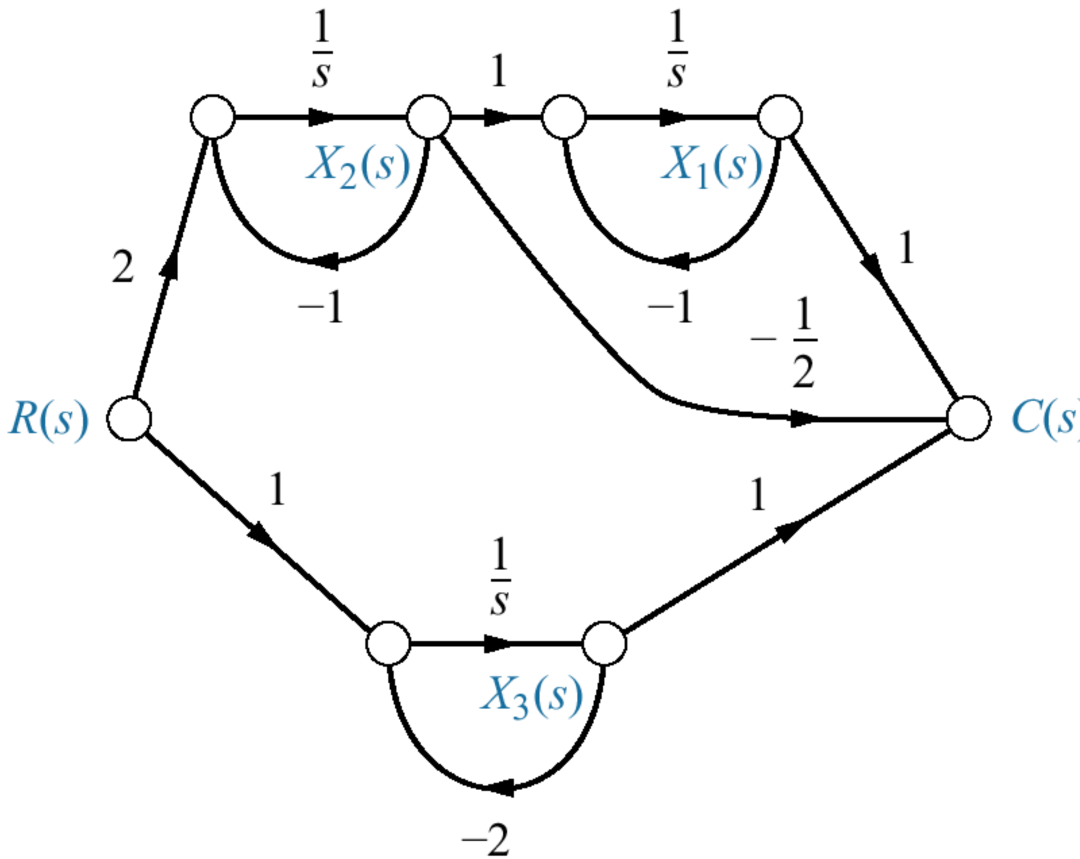
$$\dot{x}_2 = \qquad x_2 \qquad +2r$$

$$\dot{x}_3 = \qquad -2x_3 + r$$

$$y = c(t) = x_1 - 1/2x_2 + x_3$$

$$\dot{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

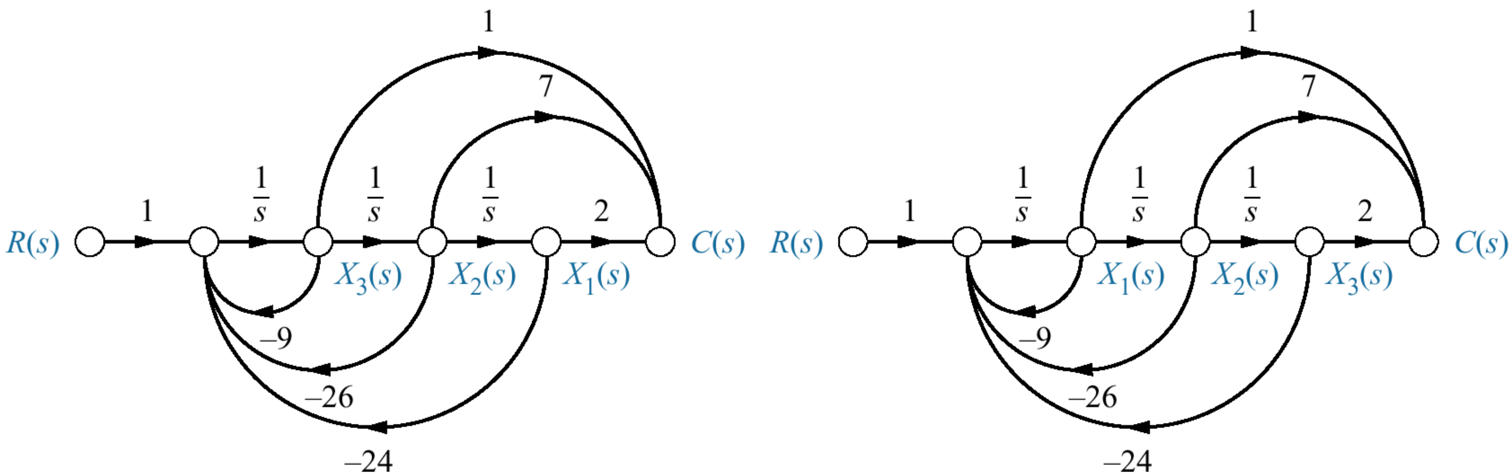
$$y = \begin{bmatrix} 1 & -1/2 & 1 \end{bmatrix} X$$



Alternate Representation: controller canonical form

G(s) = C(s)/R(s) = $(s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order



(a) (b)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r$$

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}r$$

$$y = \begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix

Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers

$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

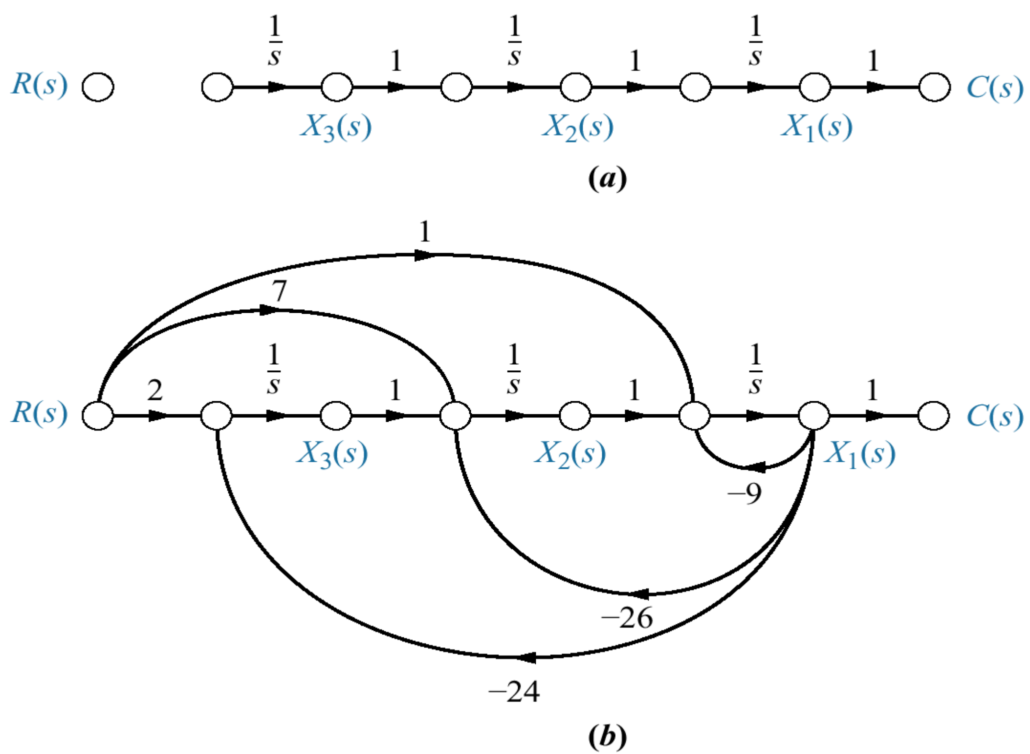
$= (1/s+7/s^2 +2/s^3)/(1+9/s+26/s^2 +24/s^3)$

Cross multiplying

$(1/s+7/s^2 +2/s^3)R(s) = (1+9/s+26/s^2 +24/s^3)\ C(s)$

And $C(s) = 1/s[R(s)-9C(s)] +1/s^2[7R(s)-26C(s)]+1/s^3[2R(s)-24C(s)]$

$= 1/s\{ [R(s)-9C(s)] + 1/s \{[7R(s)-26C(s)]+1/s [2R(s)-24C(s)]\}\}$



Alternate Representation: observer canonical form

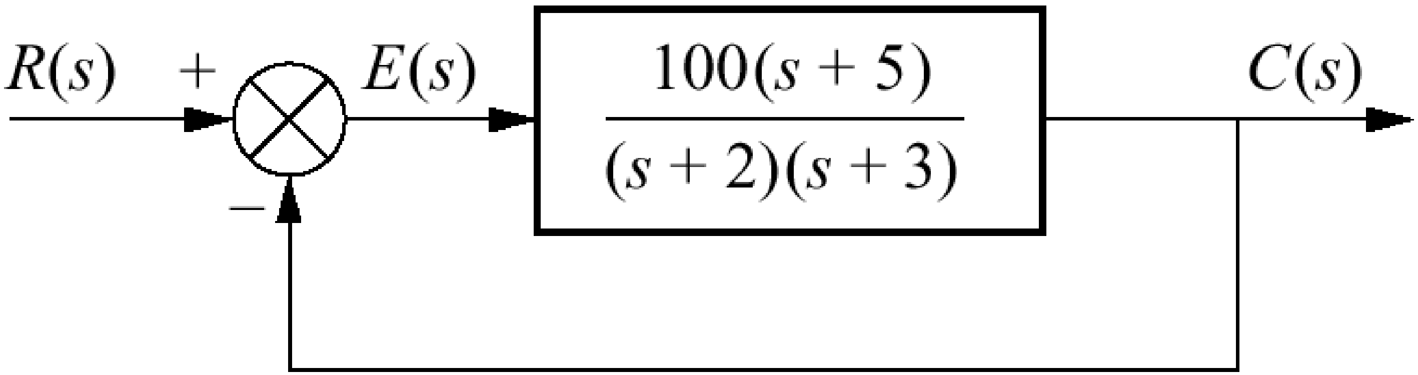
$$\dot{x}_1 = -9x_1 + x_2 + r$$
$$\dot{x}_2 = -26x_1 + x_3 + 7r$$
$$\dot{x}_3 = -24x_1 + 2r$$
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

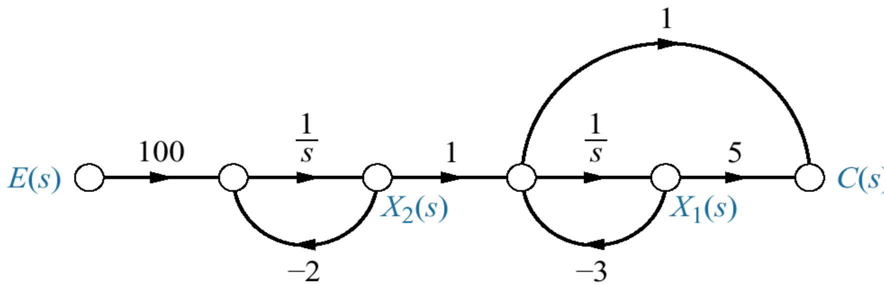
Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.

Feedback control system for Example



Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations

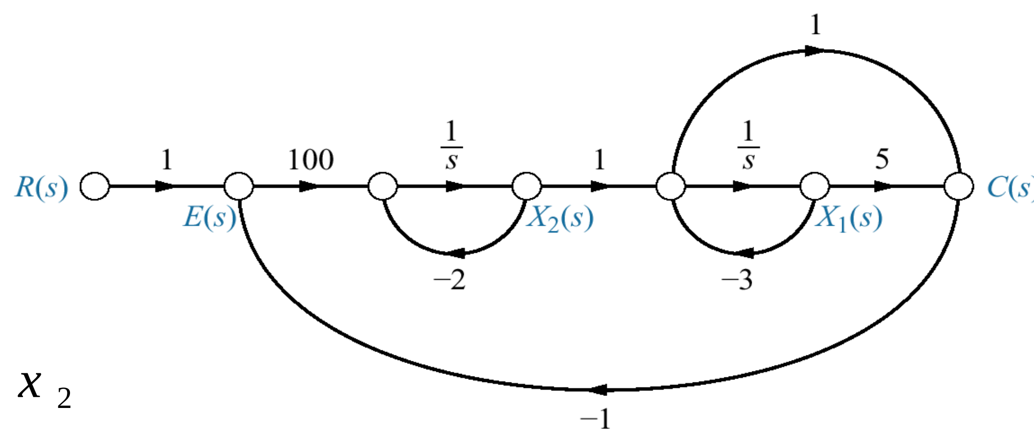


(a)

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -2x_2 + 100(r - c)$$

$$\text{but } c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$



(b)

Feedback control system for Example

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -200x_1 - 102x_2 + 100r$$

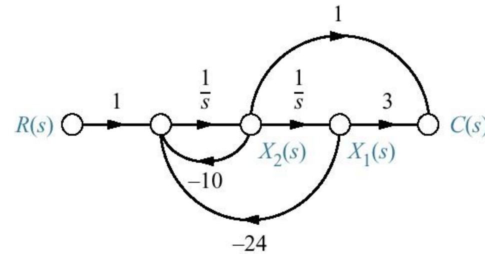
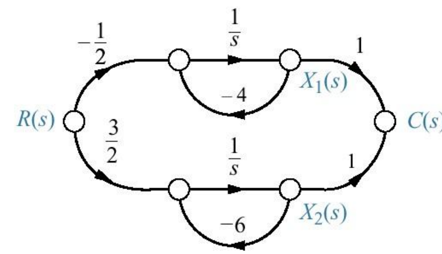
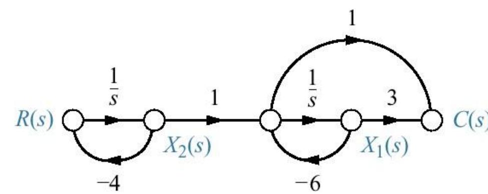
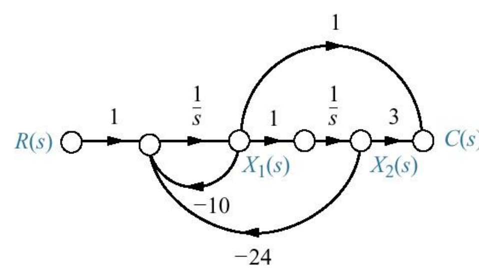
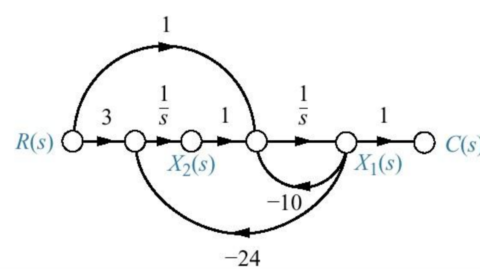
$$y = c(t) = 2x_1 + x_2$$

$$\dot{X} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} X$$

State-space forms for

C(s)/R(s) =(s+ 3)/[(s+ 4)(s+ 6)].
Note: y = c(t)

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

UNIT-II
TIME RESPONSE ANALYSIS

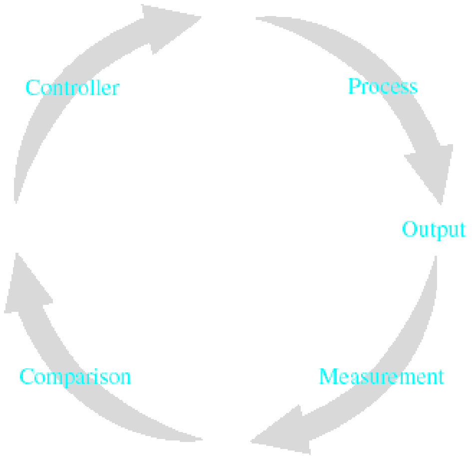
Feedback Control System Characteristics

Objectives

In this chapter we extend the ideas of modeling to include control system characteristics, such as sensitivity to model uncertainties, steady-state errors, transient response characteristics to input test signals, and disturbance rejection. We investigate the important role of the system error signal which we generally try to minimize.

We will also develop the concept of the sensitivity of a system to a parameter change, since it is desirable to minimize the effects of unwanted parameter variation. We then describe the transient performance of a feedback system and show how this performance can be readily improved. We will also investigate a design that reduces the impact of disturbance signals.

Open-And Closed-Loop Control Systems



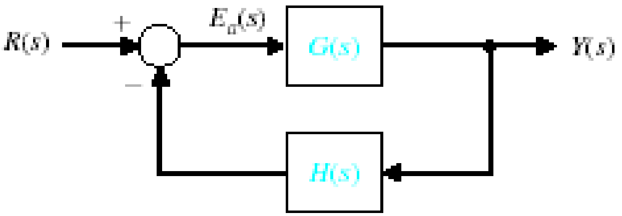
A closed-loop system.

An **open-loop** (direct) system operates without feedback and directly generates the output in response to an input signal.

A **closed-loop** system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is applied to the actuator.

Open-And Closed-Loop Control Systems

$$H(s) = 1$$



$$Y(s) = \frac{G(s)}{1 + G(s)} \cdot R(s)$$

$$E(s) = \frac{1}{1 + G(s)} \cdot R(s)$$ Error Signal

Thus, to reduce the error, the magnitude of $|1 + G(s)| \geq \blacksquare \geq 1$

$$H(s) \neq 1$$

$$Y(s) = \frac{G(s)}{1 + H(s)\,G(s)} \cdot R(s)$$

$$E(s) = \frac{1}{1 + H[(s) \cdot G(s)]} \cdot R(s)$$

Thus, to reduce the error, the magnitude of $|1 + G(s)\,H(s)| \geq \blacksquare \geq 1$

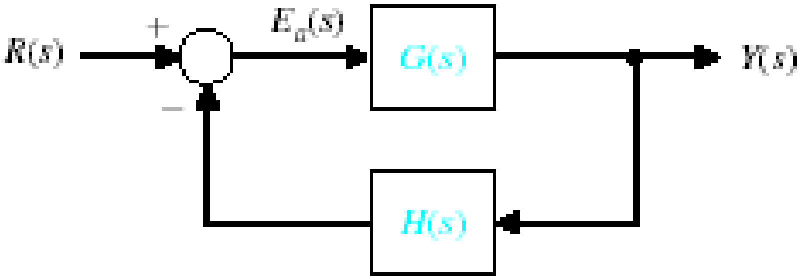
Sensitivity of Control Systems To Parameter Variations

For the closed-loop case if $GH(s) > 1$

$$Y(s) = \frac{1}{H(s)} \cdot R(s)$$

Output affected only by H(s)

$$G(s) + \Delta G(s)$$



Open Loop $\Delta Y(s) = \Delta G(s) \cdot R(s)$

Closed Loop

$$Y(s) + \Delta Y(s) = \frac{(G(s) + \Delta G(s))}{1 + (G(s) + \Delta G(s)) \cdot H(s)} \cdot R(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))} \cdot R(s)$$

$$GH(s) > \Delta GH(s)$$

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s))^2} \cdot R(s)$$

The change in the output of the closed system is reduced by a factor of 1+GH(s)

Sensitivity of Control Systems To Parameter Variations

$$T(s) = \frac{Y(s)}{R(s)}$$

$$S = \frac{\frac{\Delta T(s)}{T(s)}}{\frac{\Delta G(s)}{G(s)}} \qquad S = \frac{\frac{d\frac{d}{T}}{T}}{\left(\frac{d\frac{d}{G}}{G}\right)} = \frac{\left(\frac{d}{d\,T}T\right)}{\left(\frac{d}{d\,G}G\right)} \cdot \frac{G}{T}$$

$$T(s) = \frac{1}{1 + H[(s) \cdot G(s)]}$$

$$S_G^T = \frac{\left(\frac{d}{d\,T}T\right)}{\left(\frac{d}{d\,G}G\right)} \cdot \frac{G}{T} = \frac{\left(\frac{d}{d\,T}T\right)}{\left(\frac{d}{d\,G}G\right)} \cdot \frac{G}{T} = \frac{1}{(1 + GH)^2} \cdot \frac{G}{\frac{G}{(1 + GH)}}$$

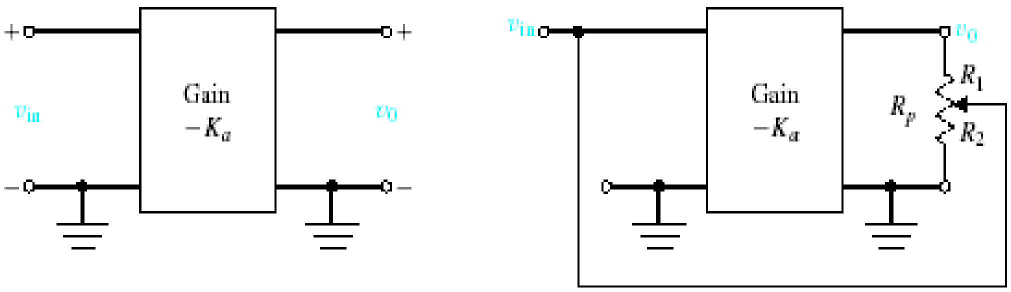
$$S_G^T = \frac{1}{(1 + GH)}$$

Sensitivity of the closed-loop to G variations reduced

$$S_H^T = \frac{-GH}{(1 + GH)}$$

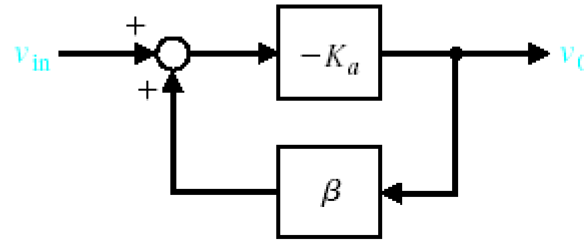
Sensitivity of the closed-loop to H variations
When GH is large sensitivity approaches 1
Changes in H directly affects the output response

Example 4.1



(a) (b)

(a) Open loop amplifier.
(b) Amplifier with feedback.



Block diagram model of feedback amplifier assuming $R_p \gg R_0$ of the amplifier.

Open loop

$$v_o = -K_a \cdot v_{in}$$

$$T = -k_a$$

$$S_{Ka}{}^{T} = 1$$

Closed loop

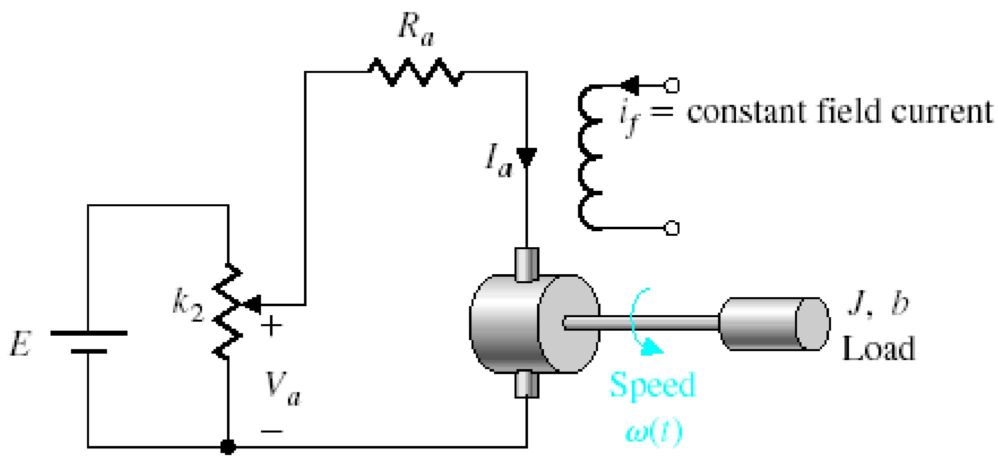
$$\beta = \frac{R2}{R1} \qquad R_p = R1 + R2$$

$$T = \frac{-K_a}{1 + K_a \cdot \beta} \qquad S_{Ka}{}^{T} = \frac{1}{1 + K_a \cdot \beta}$$

If Ka is large, the sensitivity is low.

$$K_a := 10^4 \qquad \beta := 0.1 \qquad S_{Ka}{}^{T} = \frac{1}{1 + 10^3} = 9.99 \times 10^{-4}$$

Control of the Transient Response of Control Systems



Open-loop speed control system (without feedback).

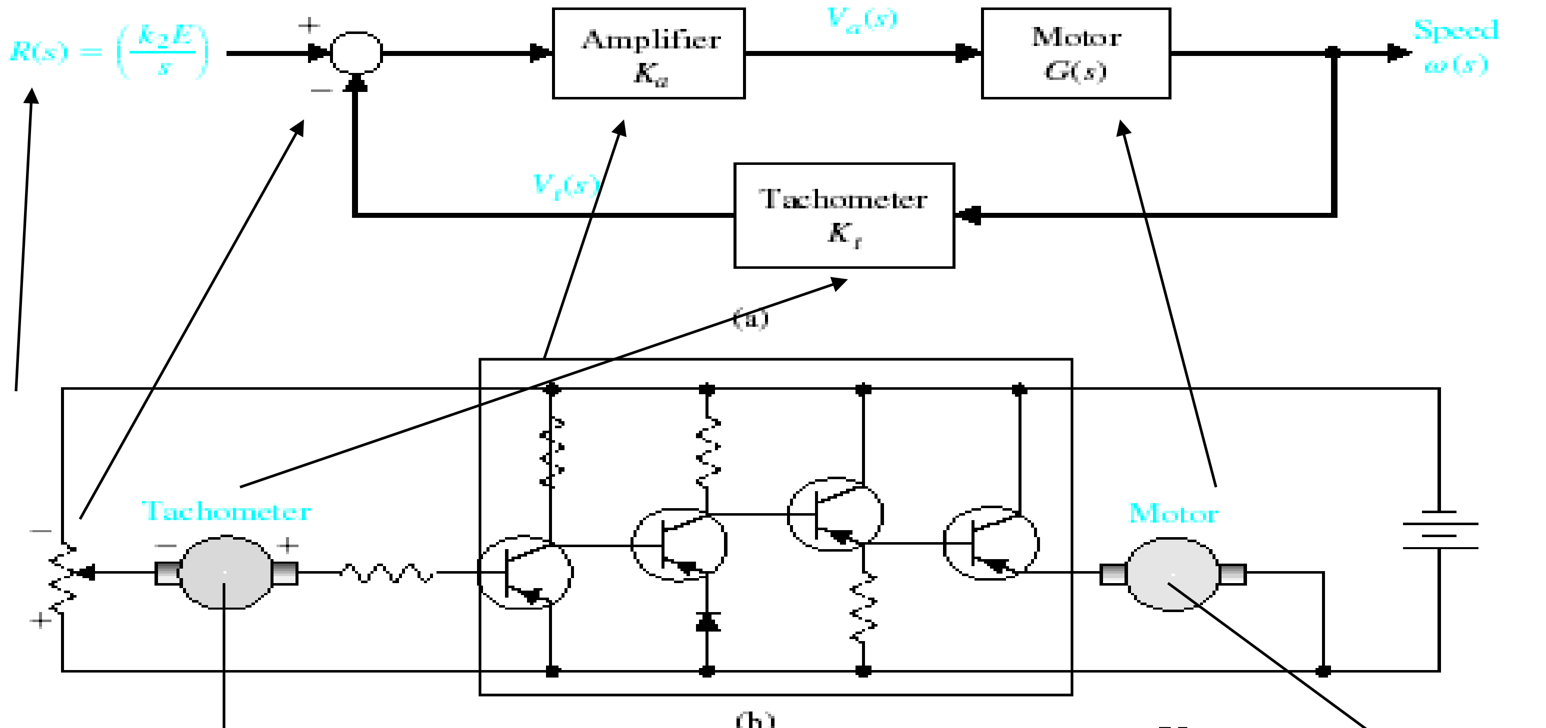
$$\frac{\omega(s)}{V_a(s)} = G(s) = \frac{K_1}{\tau_1 \cdot s + 1}$$

where,

$$K_1 = \frac{K_m}{R_a \cdot b + K_b \cdot K_m} \qquad \tau_1 = \frac{R_a \cdot J}{R_a \cdot b + K_b \cdot K_m}$$

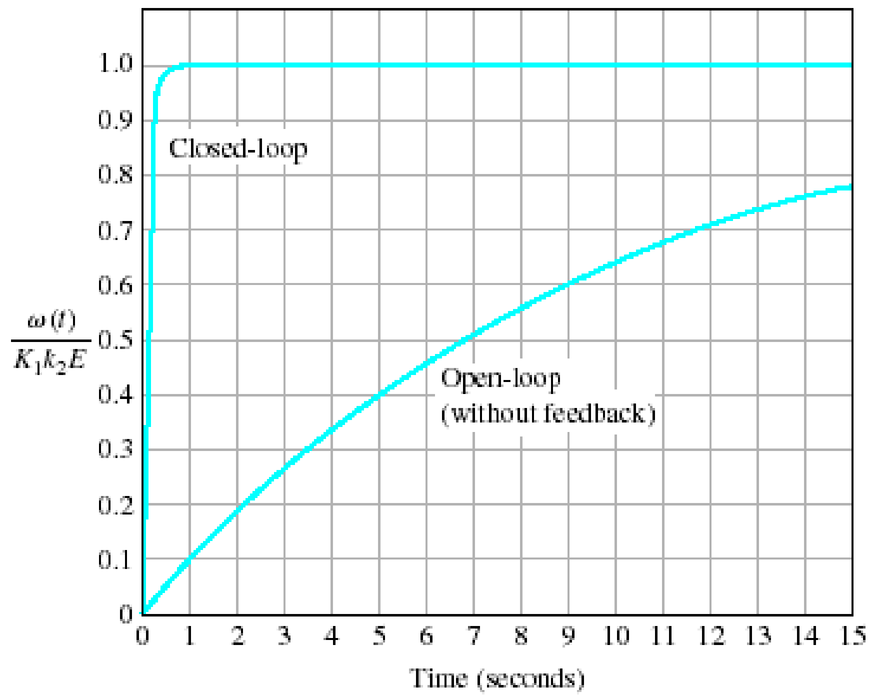
Control of the Transient Response of Control Systems

(a) Closed-loop speed control system.
(b) Transistorized closed-loop speed control system.



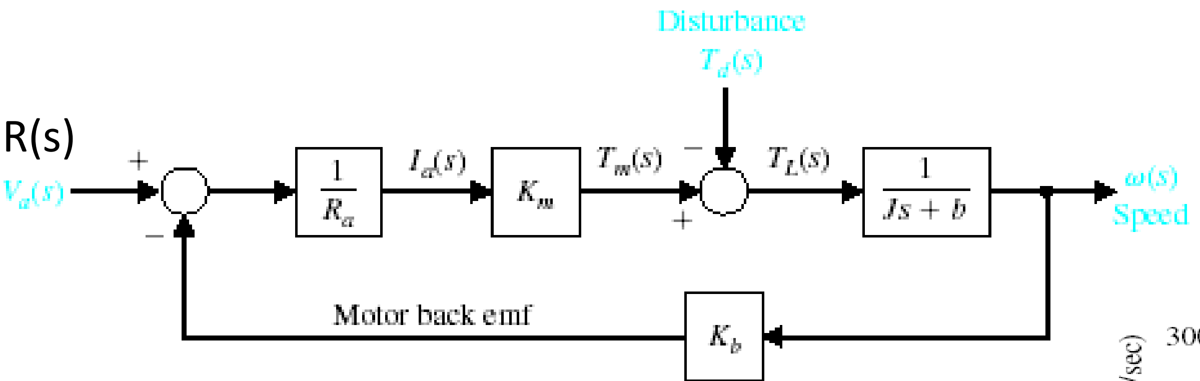
$$\frac{\omega(s)}{R(s)} = \frac{K_a \cdot G(s)}{1 + K_a \cdot K_t \cdot G(s)} = \frac{K_a \cdot K_1}{\tau_1 \cdot s + 1 + K_a \cdot K_t \cdot K_1} = \frac{K_a \cdot \frac{K_1}{\tau_1}}{s + \left[\frac{\left(1 + K_a \cdot K_t \cdot K_1\right)}{\tau_1}\right]}$$

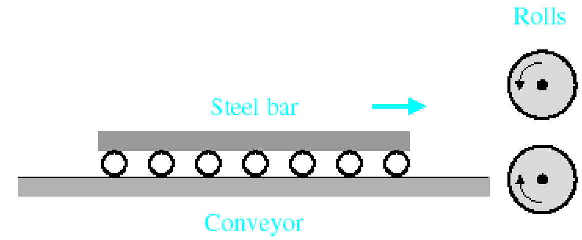
Control of the Transient Response of Control Systems

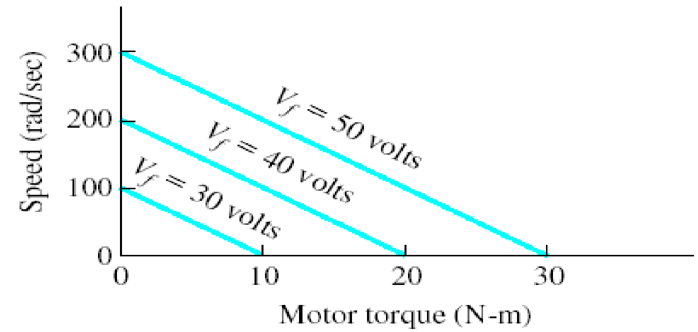


The response of the open-loop and closed-loop speed control system when τ 5 10 and $K_1 K_a K_t = 100$. The time to reach 98% of the final value for the open-loop and closed-loop system is 40 seconds and 0.4 second, respectively.

Disturbance Signals In a Feedback Control Systems







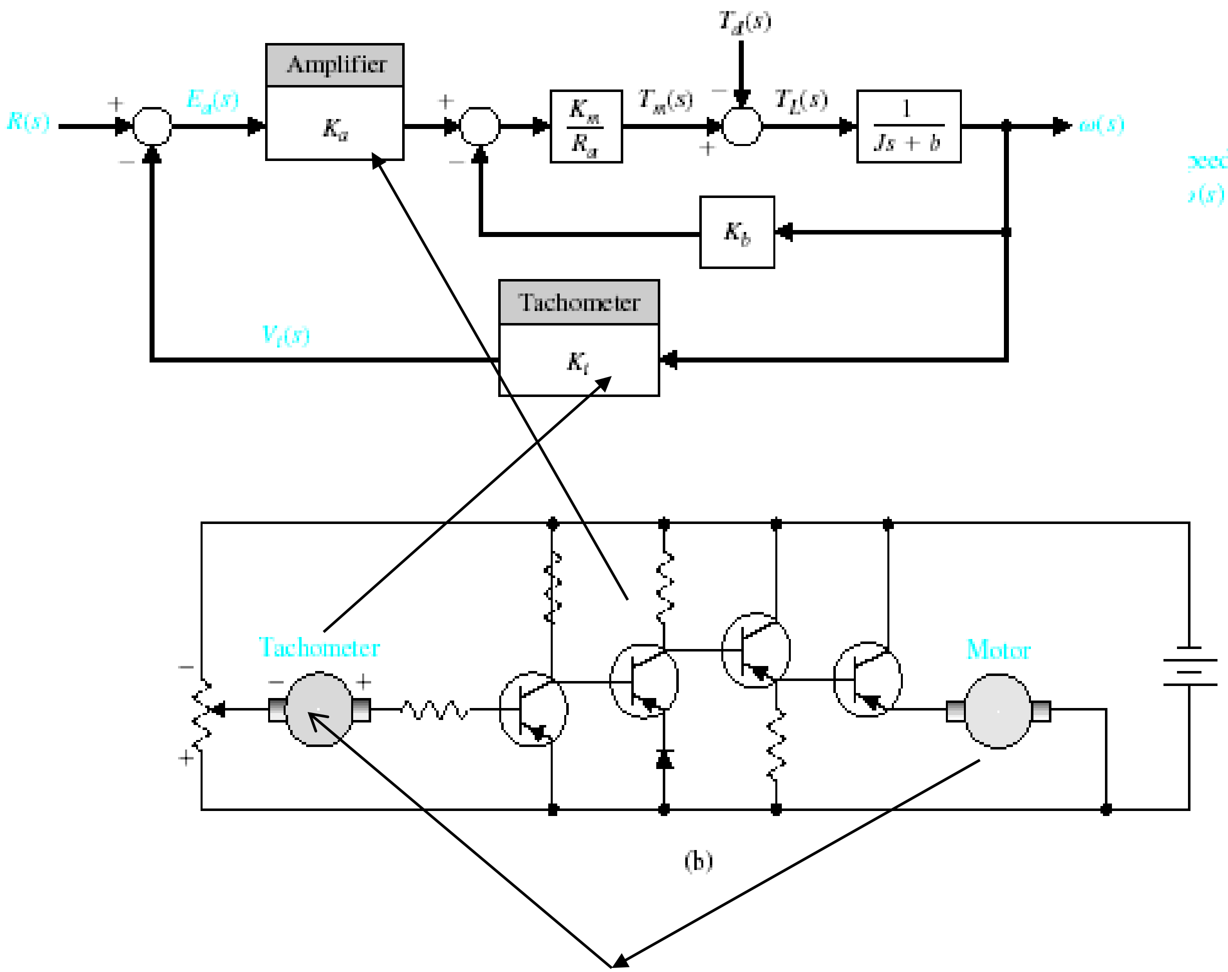
$$E(s) = R(s) - \omega(s) \qquad R(s) = 0$$

$$E(s) = -\omega(s)$$

$$E(s) = -\omega(s) = \frac{1}{\left[J\cdot s + b + \left(Km\cdot\frac{Kb}{Ra}\right)\right]}\cdot Td(s) \qquad Td(s) = \frac{D}{s}$$

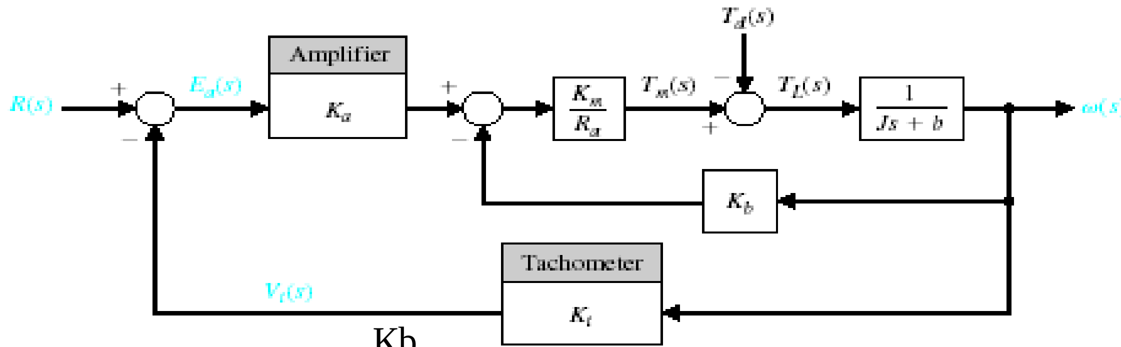
$$\lim_{t \to \text{infinite}} E(t) = \lim_{s \to 0} s\cdot\left[\frac{1}{J\cdot s + b + \left(Km\cdot\frac{Kb}{Ra}\right)}\right]\cdot\left(\frac{D}{s}\right) = \frac{D}{b + \left(Km\cdot\frac{Kb}{Ra}\right)} = -\omega(\text{infinite})$$

Disturbance Signals In a Feedback Control Systems



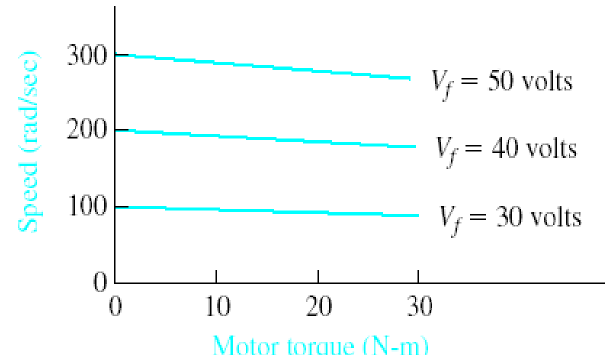
Closed-loop speed tachometer control system.

Disturbance Signals In a Feedback Control Systems



$$G1(s) = Ka\cdot\frac{Km}{Ra} \qquad G2(s) = \frac{1}{(J\cdot s + b)} \qquad H(s) = Kt + \frac{Kb}{Ka}$$

$$E(s) = -\omega(s) = \frac{G(s)}{1 + G1(s)\cdot G2(s)\cdot H(s)}\cdot Td(s)$$



$$G1G2H(s) > 1$$

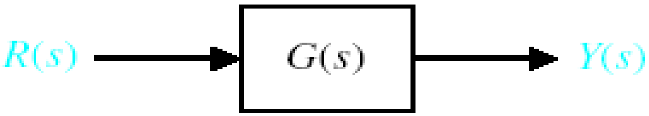
$$E(s) = \frac{1}{G1(s)\cdot H(s)}\cdot Td(s)$$

If G1(s)H(s) very large the effect of the disturbar can be minimized

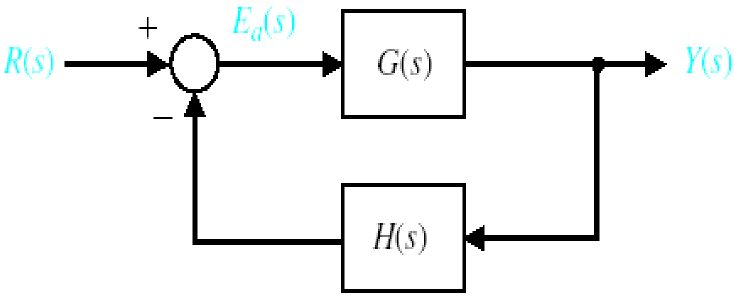
$$G1(s)\,H(s) = \frac{Ka\cdot Km}{Ra}\left(Kt + \frac{Kb}{Ka}\right)$$ approximately $$\frac{Ka\cdot Km\cdot Kt}{Ra}$$ since Ka >> Kb

Strive to maintain Ka large and Ra < 2 ohms

Steady-State Error



$$E_o(s) = R(s) - Y(s) = (1 - G(s))\cdot R(s)$$



$$E_c(s) = \frac{1}{1 + G(s)}\cdot R(s) \qquad H(s) = 1$$

Steady State Error

$$\lim_{t \to 0} e(t) = \lim_{s \to 0} s\cdot E(s)$$

For a step unit input

$$e_o(\text{infinite}) = \lim_{s \to 0} s\cdot(1 - G(s))\cdot\frac{1}{s} = \lim_{s \to 0} (1 - G(0))$$

$$e_c(\text{infinite}) = \lim_{s \to 0} s\cdot\left(\frac{1}{1 + G(s)}\right)\cdot\frac{1}{s} = \lim_{s \to 0} \left(\frac{1}{1 + G(0)}\right)$$

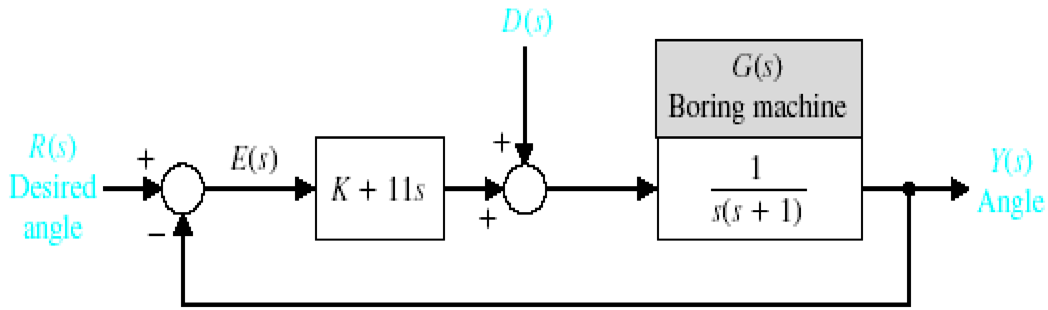
The Cost of Feedback

Increased Number of components and Complexity

Loss of Gain

Instability

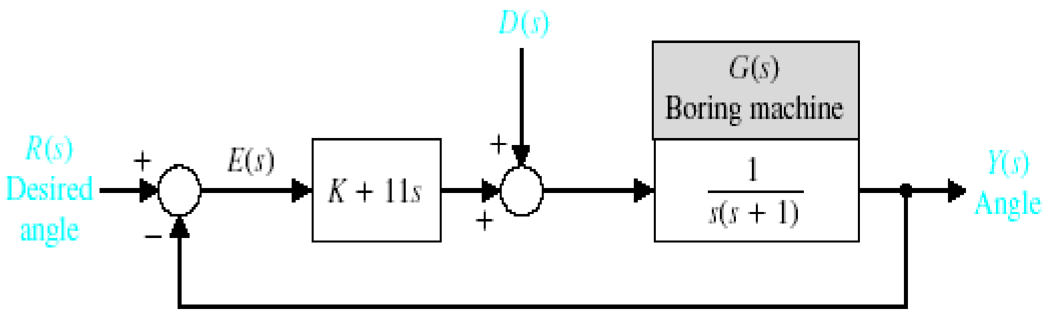
Design Example: English Channel Boring Machines



$$Y(s) = T(s) \cdot R(s) + Td(s) \cdot D(s)$$

$$Y(s) = \frac{K + 11 \cdot s}{s^2 + 12 \cdot s + K} \cdot R(s) + \frac{1}{s^2 + 12 \cdot s + K} \cdot D(s)$$

Design Example: English Channel Boring Machines



Study system for different
Values of gain K

Steady state error for R(s)=1/s and D(s)=0

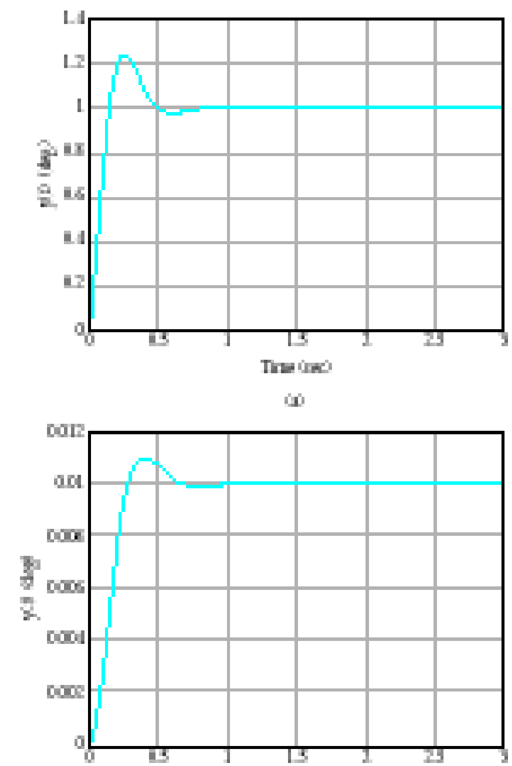
$$\lim_{t \to \text{infinite}} e(t) = \lim_{s \to 0} s \cdot \left(\frac{1}{1 + \frac{K + 11 \cdot s}{s^2 + s}} \right) \cdot \frac{1}{s} = 0$$

G

Steady state error for R(s)=0 D(s)=1/s

$$\lim_{t \to \text{infinite}} y(t) = \lim_{s \to 0} s \cdot \left(\frac{1}{s^2 + 12 \cdot s + K} \right) \cdot \frac{1}{s} = \frac{1}{K}$$

Td



The response y(t) to (a) a unit input step r(t) and (b) a unit disturbance step input D(s) 5 1/s for K = 100.

Transient vs Steady-State

The output of any differential equation can be broken up into two parts,

- a **transient part** (which decays to zero as t goes to infinity) and
- a **steady-state part** (which does not decay to zero as t goes to infinity).

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

$$\lim_{t \to 0} y_{tr}(t) = 0$$

Either part might be zero in any particular case.

Prototype systems

1st Order system

$$\dot{c}(t) + \frac{1}{\tau} c(t) = kr(t)$$

2nd order system

$$\ddot{c}(t) + 2\zeta\omega_n \dot{c}(t) + \omega_n^2 c(t) = kr(t)$$

Agenda:

- transfer function
- response to test signals
  - impulse
  - step ramp
  - parabolic
  - sinusoidal

1st order system

$$G(s)=\frac{C(s)}{R(s)}=\frac{1/T}{s+1/T}$$

Impulse response
Step response
Ramp response
Relationship between impulse, step and ramp
Relationship between impulse, step and ramp responses

$$r(t)=\delta(t), R(s)=1, \qquad c_{\delta}(t)=\frac{1}{T}e^{-t/T}1(t)$$

$$r(t)=1(t), R(s)=\frac{1}{s}, \qquad c_{step}(t)=\left[1-e^{-t/T}\right]1(t)$$

$$r(t)=t1(t), R(s)=\frac{1}{s^2}, \qquad c_{ramp}(t)=\left[t-T+Te^{-t/T}\right]1(t)$$

1^{st} Order system

Prototype parameter: Time constant

Relate problem specific parameter to prototype parameter.

Parameters: problem specific constants. Numbers that do not change with time, but do change from problem to problem.

We learn that the time constant defines a problem specific time scale that is more convenient than the arbitrary time scale of seconds, minutes, hours, days, etc, or fractions thereof.

Transient vs Steady state

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

1^{st} order system

Consider the impulse, step, ramp responses computed earlier. Identify the steady state and the transient parts.

Impulse response
Step response
Ramp response
Relationship between impulse, step and ramp
Relationship between impulse, step and ramp responses

$$G(s) = \frac{C(s)}{R(s)} = \frac{1/T}{s + 1/T}, \quad T > 0$$

$$r(t) = \delta(t), R(s) = 1, \quad c_{\delta}(t) = \frac{1}{T} e^{-t/T} 1(t)$$

$$r(t) = 1(t), R(s) = \frac{1}{s}, \quad c_{step}(t) = \left[1 - e^{-t/T}\right] 1(t)$$

$$r(t) = t1(t), R(s) = \frac{1}{s^2}, \quad c_{ramp}(t) = \left[t - T + Te^{-t/T}\right] 1(t)$$

Compare steady-state part to input function, transient part to TF.

2^{nd} order system $G(s)=\frac{C(s)}{R(s)}=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Over damped

- (two real distinct roots = two 1^{st} order systems with real poles)

Critically damped

- (a single pole of multiplicity two, highly unlikely, requires exact matching)

Underdamped

- (complex conjugate pair of poles, oscillatory behavior, most common)

step response

$$c_{step}(t)=K\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)\right)\right]1(t)$$

$$c_{\delta}(t)=K\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)\right]1(t)$$

2^{nd} Order System

Prototype parameters:

undamped natural frequency,

damping ratio

Relating problem specific parameters to prototype parameters

Transient vs Steady state

Consider the step, responses computed earlier. Identify the steady state and the transient parts.

2[nd] order system $G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Over damped

- (two real distinct roots = two 1st order systems with real poles)

Critically damped

- (a single pole of multiplicity two, highly unlikely, requires exact matching)

Underdamped

- (complex conjugate pair of poles, oscillatory behavior, most common)

step response

$$c_{step}(t) = K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)\right)\right]1(t)$$

$$c_\delta(t) = K\left[\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)\right]1(t)$$

Use of Prototypes

Too many examples to cover them all
We cover important prototypes
We develop intuition on the prototypes
We cover how to convert specific examples to prototypes
We transfer our insight, based on the study of the prototypes to the specific situations.

Transient-Response Spedifications

1. Delay time, t_d: The time required for the response to reach half the final value the very first time.
2. Rise time, t_r: the time required for the response to rise from
   10% to 90% (common for overdamped and 1^{st} order systems);
   5% to 95%;
   or 0% to 100% (common for underdamped systems);
   of its final value
1. Peak time, t_p:
2. Maximum (percent) overshoot, M_p:
3. Settling time, t_s

Derived relations for 2nd Order Systems

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\sigma = \zeta\omega_n$$

$$t_r = \frac{\pi - \beta}{\omega_d} \qquad t_p = \frac{\pi}{\omega_d} \qquad \beta = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right)$$

See book for details. (Pg. 232)

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad 2\%$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad 5\%$$

Allowable M_p determines damping ratio.

Settling time then determines undamped natural frequency.

Theory is used to derive relationships between design specifications and prototype parameters.

Which are related to problem parameters.

Higher order system

PFEs have linear denominators.

•each term with a real pole has a time constant

•each complex conjugate pair of poles has a damping ratio and an undamped natural frequency.

Proportional control of plant w integrator

$$G_C(s) = K_p, \quad G(s) = \frac{1}{s(Js+b)}$$

Integral control of Plant w disturbance

$$G_C(s) = \frac{K}{s}, \quad G(s) = \frac{1}{s(Js+b)}$$

Proportional Control of plant w/o integrator

$$G_C(s) = K, \quad G(s) = \frac{1}{Ts+1}$$

Integral control of plant w/o integrator

$$G_C(s) = \frac{K}{s}, \quad G(s) = \frac{1}{Ts+1}$$

UNIT-III
STABILITY ANALYSIS IN S- DOMAIN

Chapter 6 – The Stability of Linear Feedback Systems

The issue of ensuring the stability of a closed-loop feedback system is central to control system design. Knowing that an unstable closed-loop system is generally of no practical value, we seek methods to help us analyze and design stable systems. A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input, bounded-output stability and is one of the main topics of this chapter.

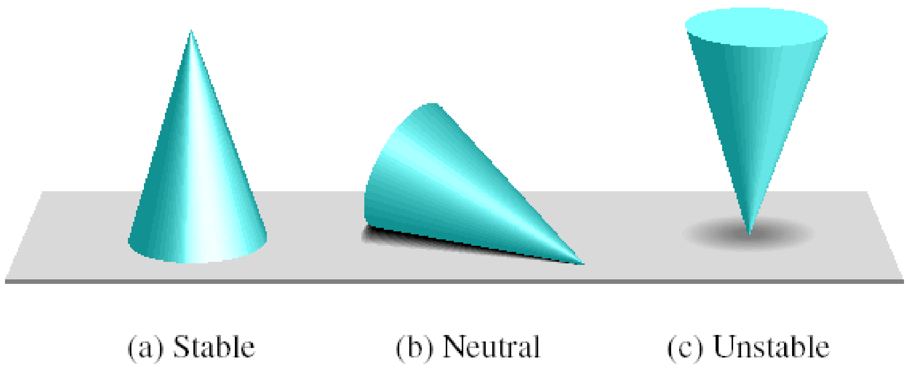
The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh–Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half-plane without actually computing the values of the roots. Thus we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems we will introduce the notion of relative stability, which allows us to characterize the degree of stability.

The Concept of Stability

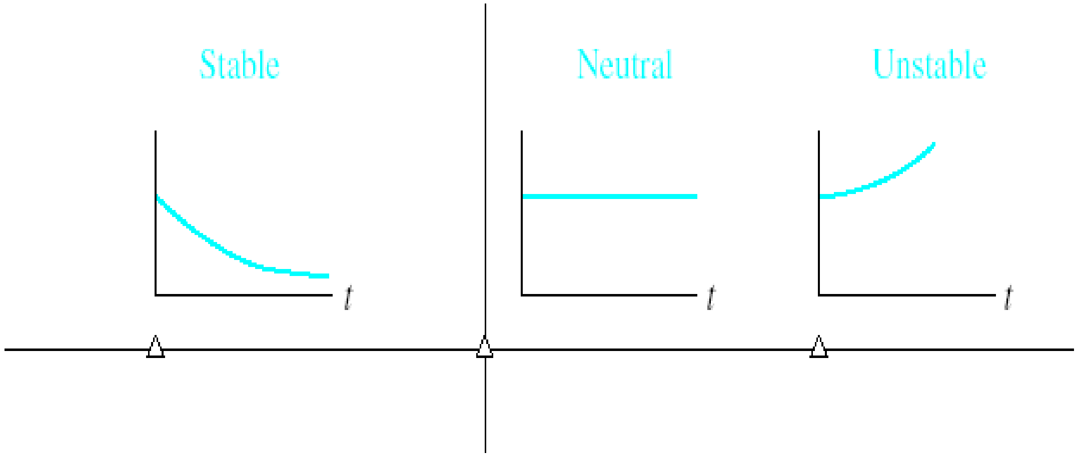
A stable system is a dynamic system with a bounded response to a bounded input.

Absolute stability is a stable/not stable characterization for a closed-loop feedback system. Given that a system is stable we can further characterize the degree of stability, or the relative stability.

The Concept of Stability



The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

The Routh-Hurwitz Stability Criterion

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are in the left-hand plane.

These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system.

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

The Routh-Hurwitz Stability Criterion

Characteristic equation, q(s) → $a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$

Routh array →

$$
\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
s^0 & h_{n-1} & &
\end{array}
$$

The Routh-Hurwitz criterion states that the number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

The Routh-Hurwitz Stability Criterion

Case One: No element in the first column is zero.

Example 6.1 Second-order system

The Characteristic polynomial of a second-order system is:

$$q(s) = a_2 \cdot s^2 + a_1 \cdot s + a_0$$

The Routh array is written as:

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

where:

$$b_1 = \frac{a_1 \cdot a_0 - (0) \cdot a_2}{a_1} = a_0$$

Therefore the requirement for a stable second-order system is simply that all coefficients be positive or all the coefficients be negative.

The Routh-Hurwitz Stability Criterion

Case Two: Zeros in the first column while some elements of the row containing a zero in the first column are nonzero.

If only one element in the array is zero, it may be replaced with a small positive number ε that is allowed to approach zero after completing the array.

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is then:

s^5	1	2	11
s^4	2	4	10
s^3	b_1	6	0
s^2	c_1	10	0
s^1	d_1	0	0
s^0	10	0	0

where:

$$b_1 = \frac{2 \cdot 2 - 1 \cdot 4}{2} = 0 = \varepsilon \qquad c_1 = \frac{4\varepsilon - 2 \cdot 6}{\varepsilon} = \frac{-12}{\varepsilon} \qquad d_1 = \frac{6 \cdot c_1 - 10\varepsilon}{c_1} = 6$$

There are two sign changes in the first column due to the large negative number calculated for c1. Thus, the system is unstable because two roots lie in the right half of the plane.

The Routh-Hurwitz Stability Criterion

Case Three: Zeros in the first column, and the other elements of the row containing the zero are also zero.

This case occurs when the polynomial q(s) has zeros located symetrically about the origin of the s-plane, such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$. This case is solved using the auxiliary polynomial, U(s), which is located in the row above the row containing the zero entry in the Routh array.

$$q(s) = s^3 + 2\cdot s^2 + 4s + K$$

Routh array:

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \frac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system we require that $0 < s < 8$

For the marginally stable case, K=8, the s^1 row of the Routh array contains all zeros. The auxiliary plynomial comes from the s^2 row.

$$U(s) = 2s^2 + Ks^0 = 2\cdot s^2 + 8 = 2\left(s^2 + 4\right) = 2(s + j\cdot 2)(s - j\cdot 2)$$

It can be proven that U(s) is a factor of the characteristic polynomial:

$$\frac{q(s)}{U(s)} = \frac{s + 2}{2}$$

Thus, when K=8, the factors of the characteristic polynomial are:

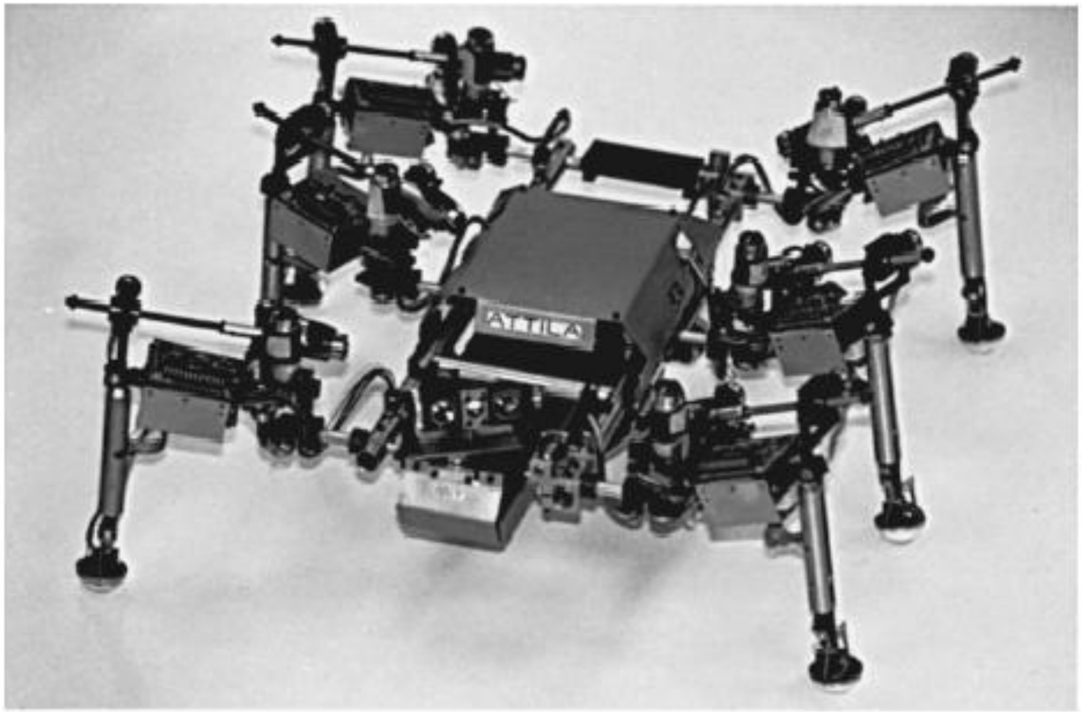
$$q(s) = (s + 2)(s + j\cdot 2)(s - j\cdot 2)$$

The Routh-Hurwitz Stability Criterion

Case Four: Repeated roots of the characteristic equation on the jw-axis.

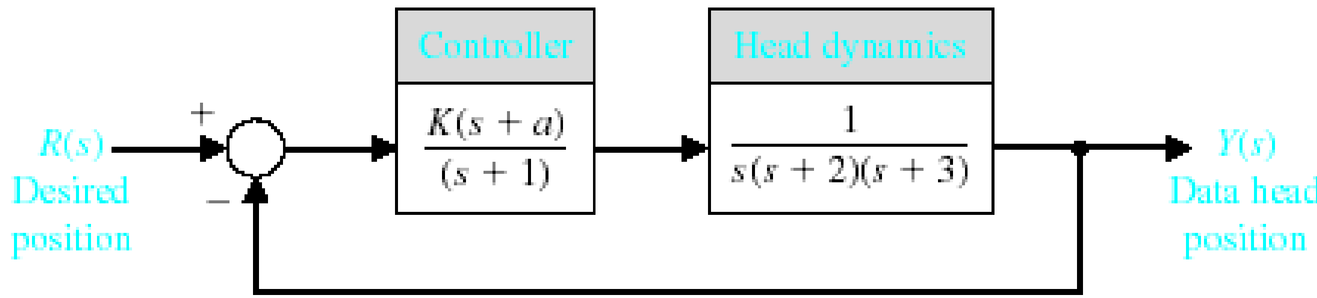
With simple roots on the jw-axis, the system will have a marginally stable behavior. This is not the case if the roots are repeated. Repeated roots on the jw-axis will cause the system to be unstable. Unfortunately, the routh-array will fail to reveal this instability.

Example 6.4



A completely integrated, six-legged, micro robot system. The six-legged design provides maximum dexterity. Legs also provide a unique sensory system for environmental interaction. It is equipped with a sensor network that includes 150 sensors of 12 different types. The legs are instrumented so that the robot can determine the lay of the terrain, the surface texture, hardness, and even color. The gyro-stabilized camera and range finder can be used for gathering data beyond the robot's immediate reach. This high-performance system is able to walk quickly, climb over obstacles, and perform dynamic motions. (Courtesy of IS Robotics Corporation.)

Example 6.5 Welding control



Welding head position control.

Using block diagram reduction we find that: $q(s) = s^4 + 6s^3 + 11s^2 + (K + 6)s + Ka$

The Routh array is then:

s^4	1	11	Ka
s^3	6	$(K+6)$	
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

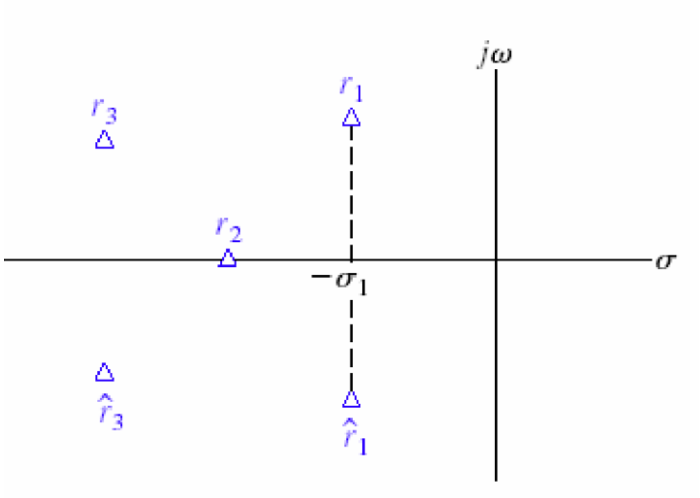
where: $b_3 = \dfrac{60 - K}{6}$ and $c_3 = \dfrac{b_3(K + 6) - 6 \cdot Ka}{b_3}$

For the system to be stable both b_3 and c_3 must be positive.

Using these equations a relationship can be determined for K a

The Relative Stability of Feedback Control Systems

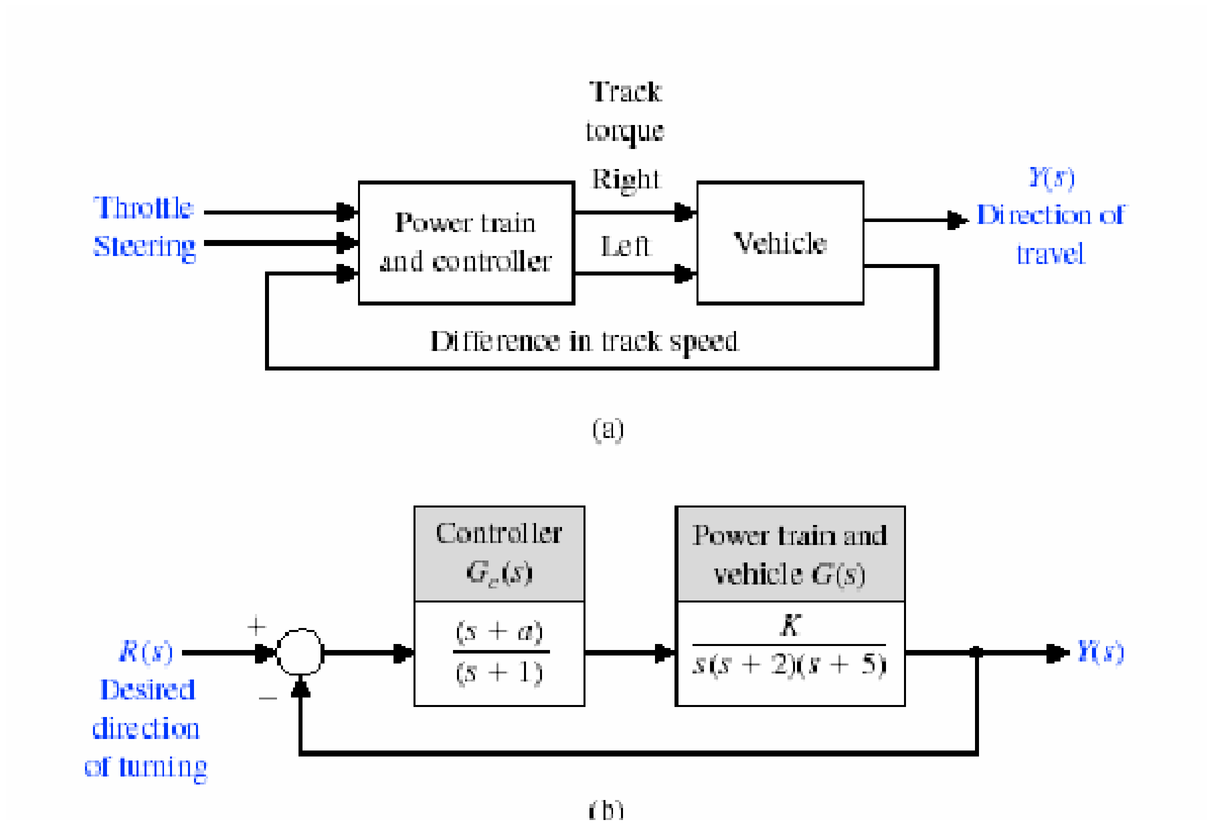
It is often necessary to know the relative damping of each root to the characteristic equation. Relative system stability can be measured by observing the relative real part of each root. In this diagram r2 is relatively more stable than the pair of roots labeled r1.



One method of determining the relative stability of each root is to use an axis shift in the s-domain and then use the Routh array as shown in Example 6.6 of the text.

Design Example: Tracked Vehicle Turning Control

Problem statement: Design the turning control for a tracked vehicle. Select K and a so that the system is stable. The system is modeled below.



Design Example: Tracked Vehicle Turning Control

The characteristic equation of this system is:

$$1 + G_c \cdot G(s) = 0$$

or

$$1 + \frac{K(s+a)}{s(s+1)(s+2)(s+5)} = 0$$

Thus,

$$s(s+1)(s+2)(s+5) + K(s+a) = 0$$

or

$$s^4 + 8s^3 + 17s^2 + (K+10)s + Ka = 0$$

To determine a stable region for the system, we establish the Routh arra

s^4	1	17	Ka
s^3	8	$(K+10)$	0
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

where

$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K+10) - 8Ka}{b_3}$$

Design Example: Tracked Vehicle Turning Control

s^4	1	17	Ka
s^3	8	$(K+10)$	0
s^2	b_3	Ka	
s^1	c_3		
s^0	Ka		

where

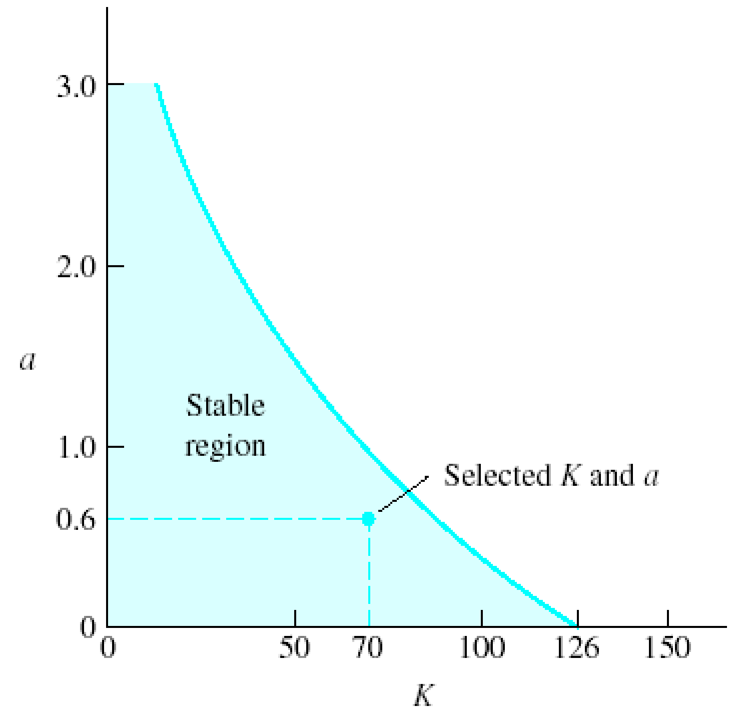
$$b_3 = \frac{126 - K}{8} \quad \text{and} \quad c_3 = \frac{b_3(K + 10) - 8Ka}{b_3}$$

Therefore,

$$K < 126$$

$$K \cdot a > 0$$

$$(K + 10)(126 - K) - 64Ka > 0$$



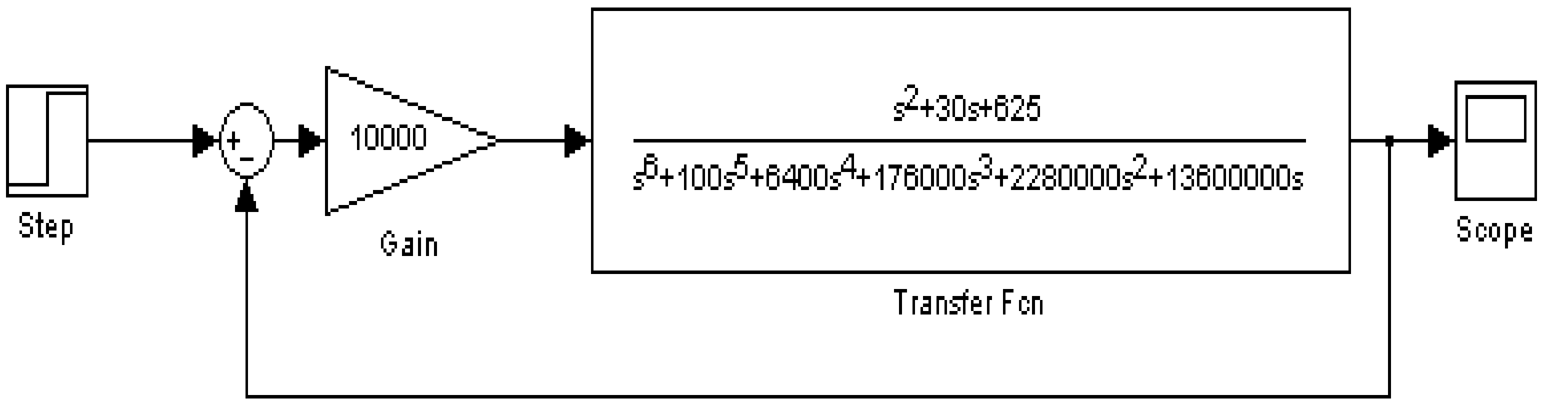
The Root Locus Method

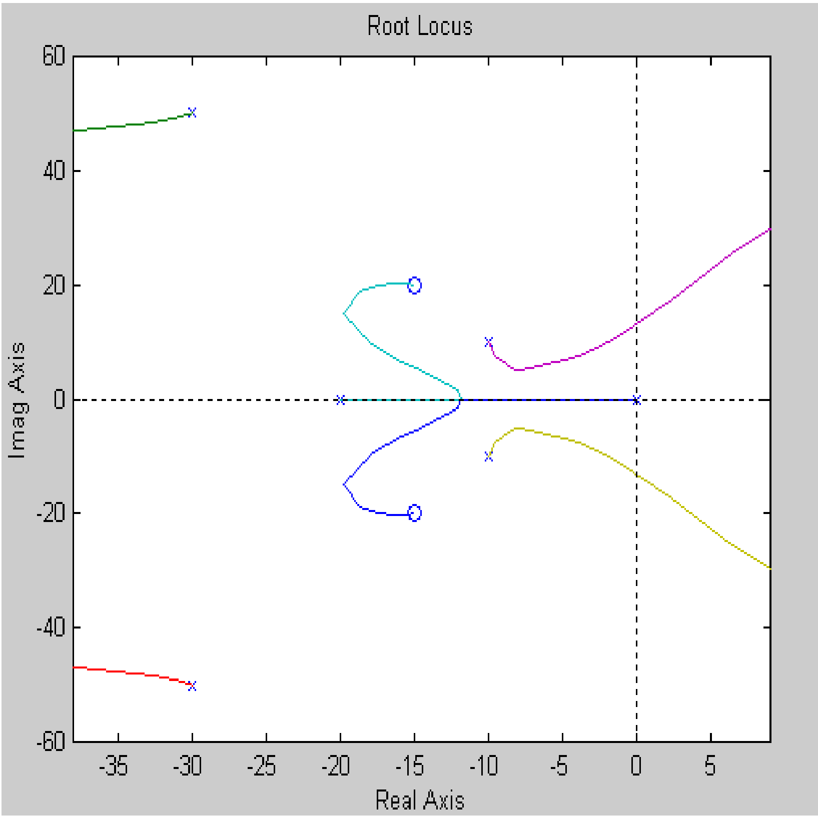
In the preceding chapters we discussed how the performance of a feedback system can be described in terms of the location of the roots of the characteristic equation in the s-plane. We know that the response of a closed-loop feedback system can be adjusted to achieve the desired performance by judicious selection of one or more system parameters. It is very useful to determine how the roots of the characteristic equation move around the s-plane as we change one parameter.

The locus of roots in the s-plane can be determined by a graphical method. A graph of the locus of roots as one system parameter varies is known as a root locus plot. The root locus is a powerful tool for designing and analyzing feedback control systems and is the main topic of this chapter. We will discuss practical techniques for obtaining a sketch of a root locus plot by hand. We also consider computer-generated root locus plots and illustrate their effectiveness in the design process. The popular PID controller is introduced as a practical controller structure.

We will show that it is possible to use root locus methods for design when two or three parameters vary. This provides us with the opportunity to design feedback systems with two or three adjustable parameters. For example the PID controller has three adjustable parameters. We will also define a measure of sensitivity of a specified root to a small incremental change in a system parameter.

The Root Locus Method



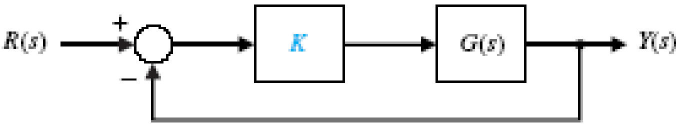


The Root Locus Method

The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity.

In mathematical terms, given a forward-loop transfer function, K**G**(s)
where K is the root locus gain, and the corresponding closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)}$$

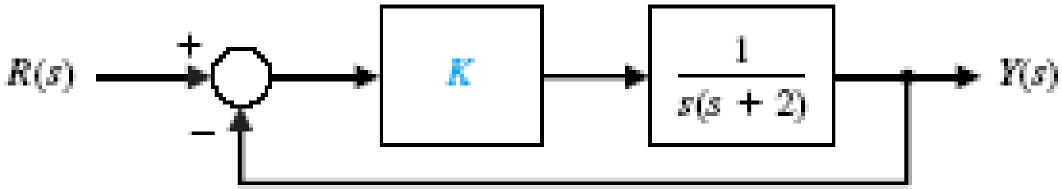


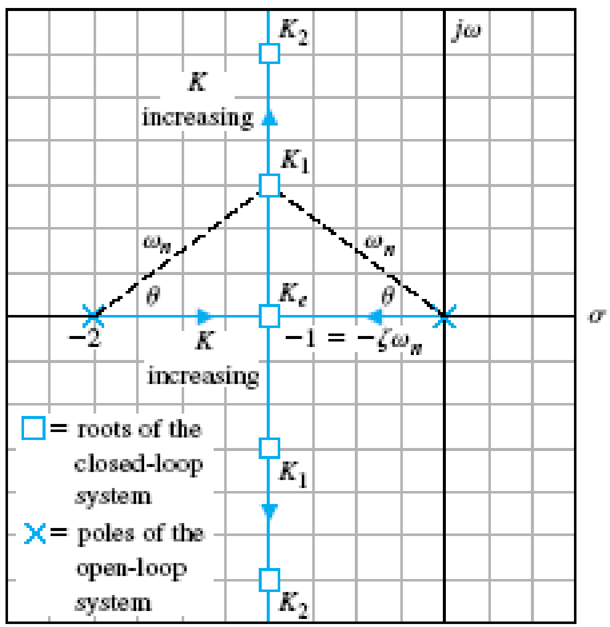
the root locus is the set of paths traced by the roots of

$$1 + KG(s) = 0$$

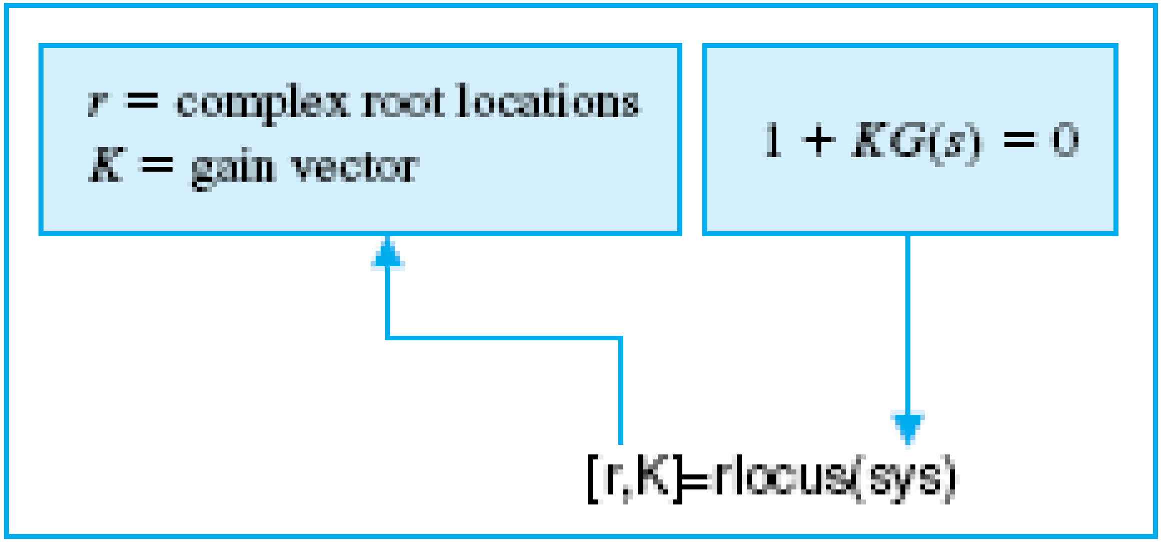
as K varies from zero to infinity. As K changes, the solution to this equation changes. This equation is called the characteristic equation. This equation defines where the poles will be located for any value of the root locus gain, K. In other words, it defines the characteristics of the system behavior for various values of controller gain.

The Root Locus Method

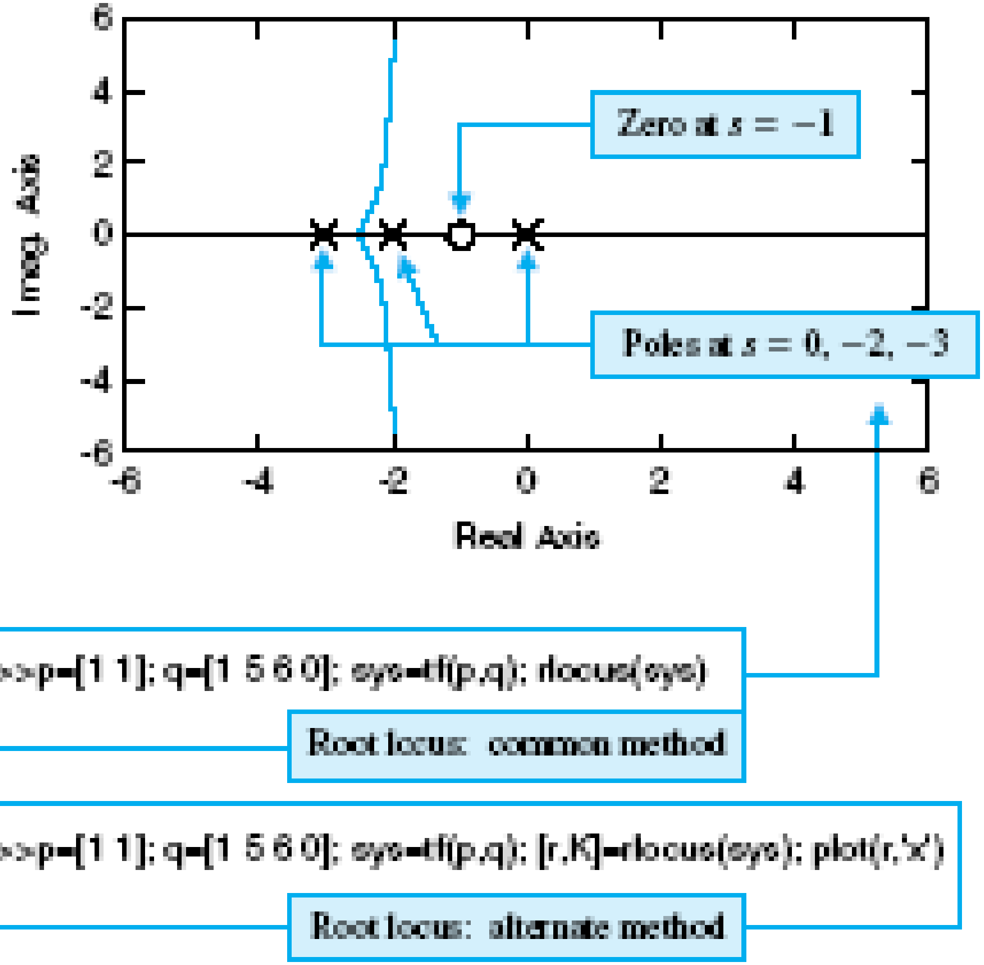


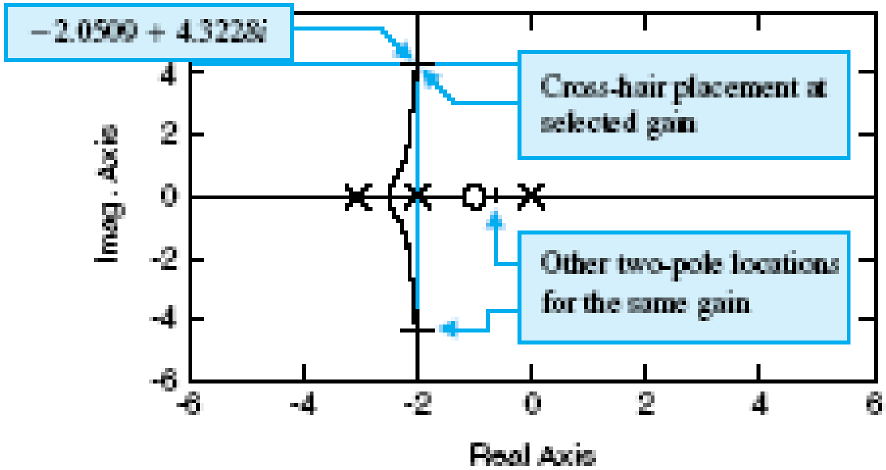


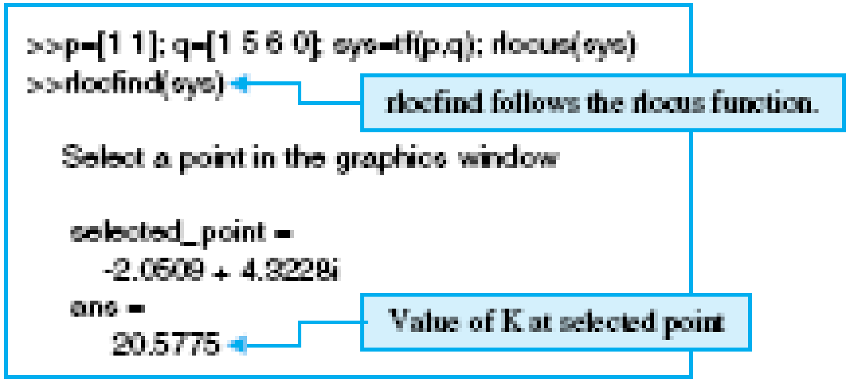
The Root Locus Method



The Root Locus Method




−2.0509 + 4.3228i
Cross-hair placement at selected gain
Other two-pole locations for the same gain
Imag. Axis
Real Axis
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6


>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
>>rlocfind(sys)
rlocfind follows the rlocus function.
Select a point in the graphics window
selected_point =
-2.0509 + 4.3228i
ans =
20.5775
Value of K at selected point

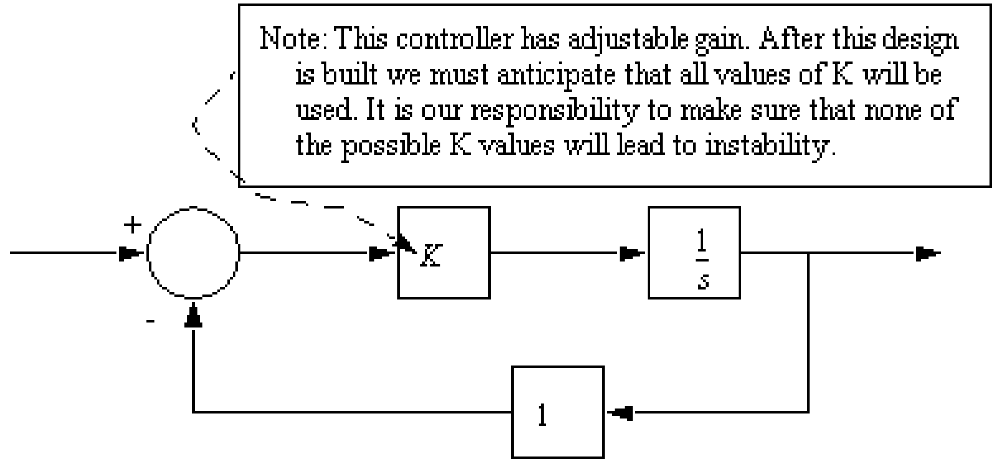
The Root Locus Method

No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of G(s).
The root locus must have n branches, each branch starts at a pole of G(s) and goes to a zero of G(s).
If G(s) has more poles than zeros (as is often the case), $m < n$ and we say that G(s) has zeros at infinity. In this case, the limit of G(s) as s -> infinity is zero.

The number of zeros at infinity is n-m, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example



$$G(s) = \frac{K}{s} \qquad H(s) = 1$$

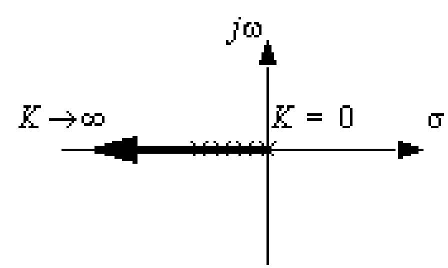
First, we must develop a transfer function for the entire control system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$$s + K = 0$$

K	root
0	
1	
2	
3	
etc..	



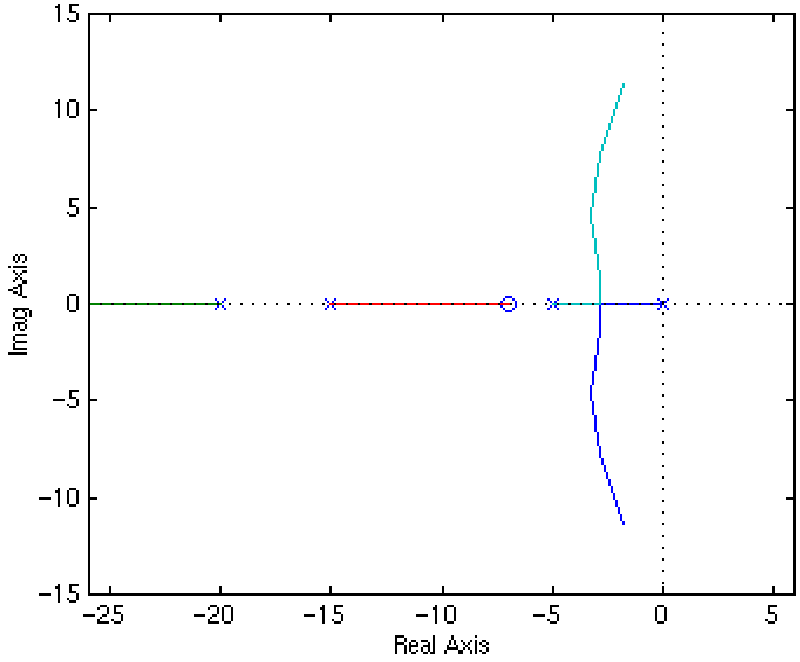
Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes.

Example

MATLAB Example - Plotting the root locus of a transfer function

Consider an open loop system which has a transfer function of
G(s) = (s+7)/s(s+5)(s+15)(s+20)
How do we design a feedback controller for the system by using the root locus method?
Enter the transfer function, and the command to plot the root locus:

```
num=[1 7];
den=conv(conv([1 0],[1 5]),conv([1 15],[1 20]));
rlocus(num,den)
axis([-22 3 -15 15])
```



Graphical Method

Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2+3s+2} \qquad H(s) = 1$$

Step 1: (put equation in standard form)

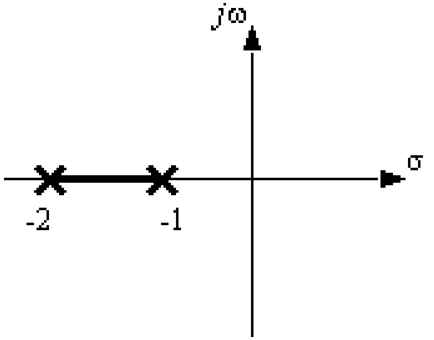
$$1 + G(s)H(s) = 1 + \left(\frac{K}{s^2+3s+2}\right)(1) = 1 + K\frac{1}{(s+1)(s+2)}$$

Step 2: (find loci ending at infinity)

$m = 0 \quad n = 2$ (from the poles and zeros of the previous step)

$n - m = 2$ (loci end at infinity)

Step 3: (plot roots)



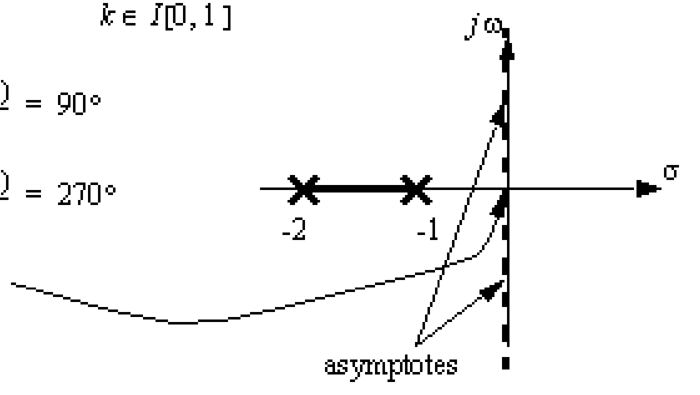
Step 4: (find asymptotes angles and real axis intersection)

$$\beta(k) = \frac{180°(2k+1)}{2} \qquad k \in I[0,1]$$

$$\beta(0) = \frac{180°(2(0)+1)}{2} = 90°$$

$$\beta(1) = \frac{180°(2(1)+1)}{2} = 270°$$

$$\sigma = \frac{(0)(-1-2)}{2} = 0$$



Graphical Method

Step 5: (find the breakout points for the roots)

$$A = 1 \qquad B = s^2 + 3s + 2$$

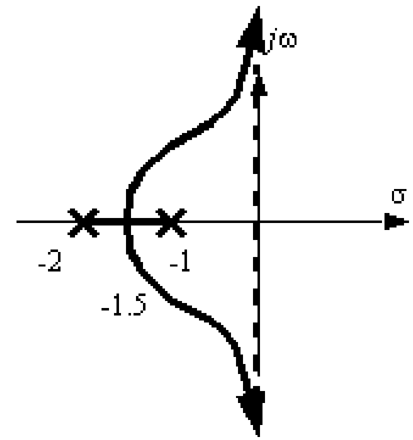
$$\frac{d}{ds}A = 0 \qquad \frac{d}{ds}B = 2s + 3$$

$$A\left(\frac{d}{ds}B\right) - B\left(\frac{d}{ds}A\right) = 0$$

$$1(2s+3) - (s^2+3s+2)(0) = 0$$

$$2s + 3 = 0$$

$$s = -1.5$$



Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

$$1 + G(s)H(s) = 0$$

$$1 + K\frac{1}{s^2+3s+2} = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$(j\omega)^2 + 3(j\omega) + 2 + K = 0$$

$$-\omega^2 + 3j\omega + 2 + K = 0$$

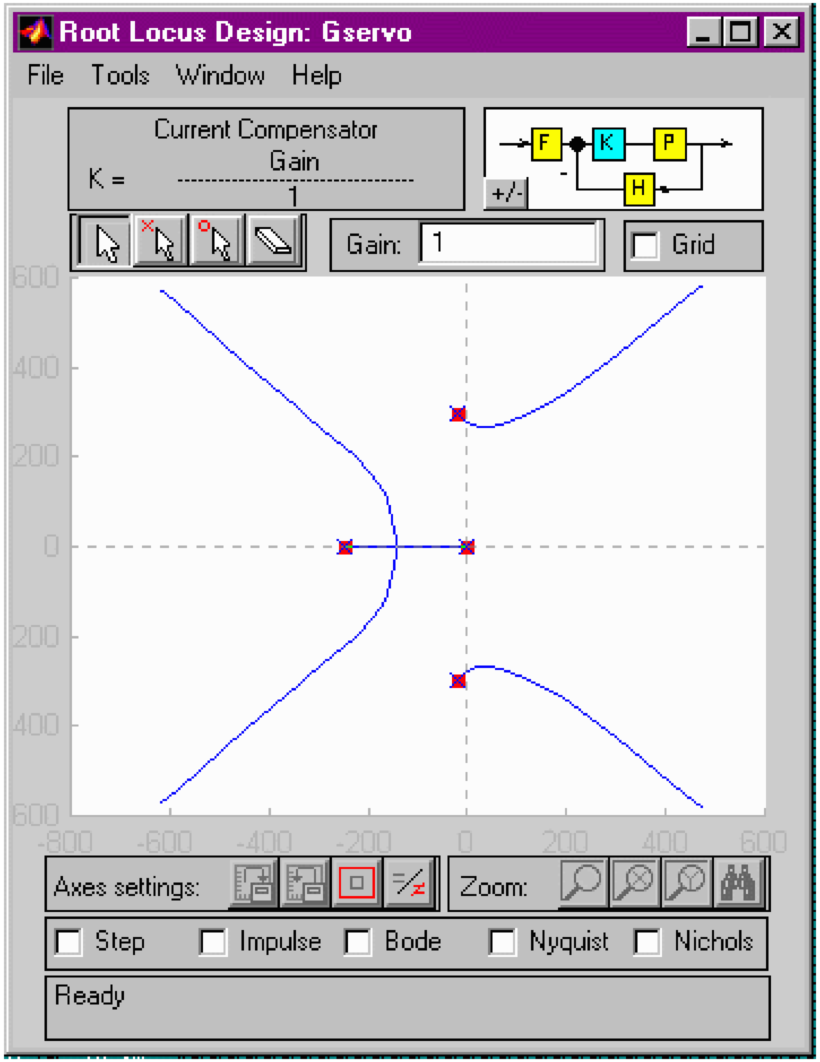
$$\omega^2 + \omega(-3j) + (-2-K) = 0$$

$$\omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2-K)}}{2} = \frac{3j \pm \sqrt{-9+8+4K}}{2} = \frac{3j \pm \sqrt{4K-1}}{2}$$

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.

Root Locus Design GUI (rltool)

The Root Locus Design GUI is an interactive graphical tool to design compensators using the root locus method. This GUI plots the locus of the closed-loop poles as a function of the compensator gains. You can use this GUI to add compensator poles and zeros and analyze how their location affects the root locus and various time and frequency domain responses. Click on the various controls on the GUI to see what they do.



UNIT-IV
FREQUENCY RESPONSE ANALYSIS

Frequency Response Methods and Stability

In previous chapters we examined the use of test signals such as a step and a ramp signal. In this chapter we consider the steady-state response of a system to a sinusoidal input test signal. We will see that the response of a linear constant coefficient system to a sinusoidal input signal is an output sinusoidal signal at the same frequency as the input. However, the magnitude and phase of the output signal differ from those of the input sinusoidal signal, and the amount of difference is a function of the input frequency. Thus we will be investigating the steady-state response of the system to a sinusoidal input as the frequency varies.

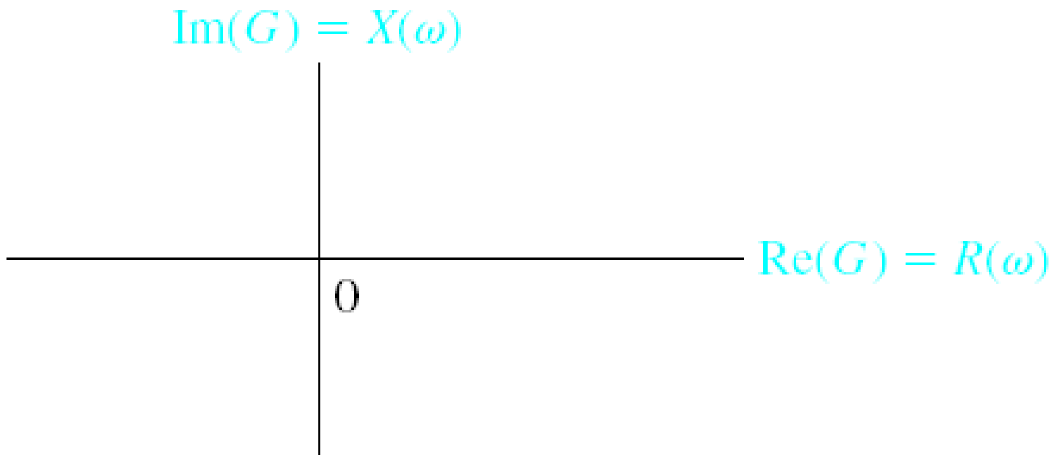
We will examine the transfer function G(s) when s =jw and develop methods for graphically displaying the complex number G(j)as w varies. The Bode plot is one of the most powerful graphical tools for analyzing and designing control systems, and we will cover that subject in this chapter. We will also consider polar plots and log magnitude and phase diagrams. We will develop several time-domain performance measures in terms of the frequency response of the system as well as introduce the concept of system bandwidth.

Introduction

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs form the input waveform only in amplitude and phase.

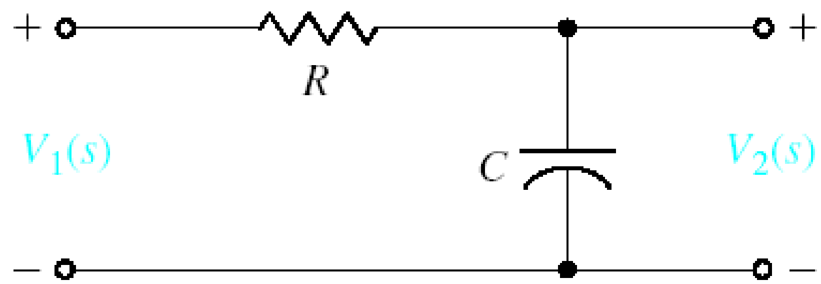
Frequency Response Plots

Polar Plots



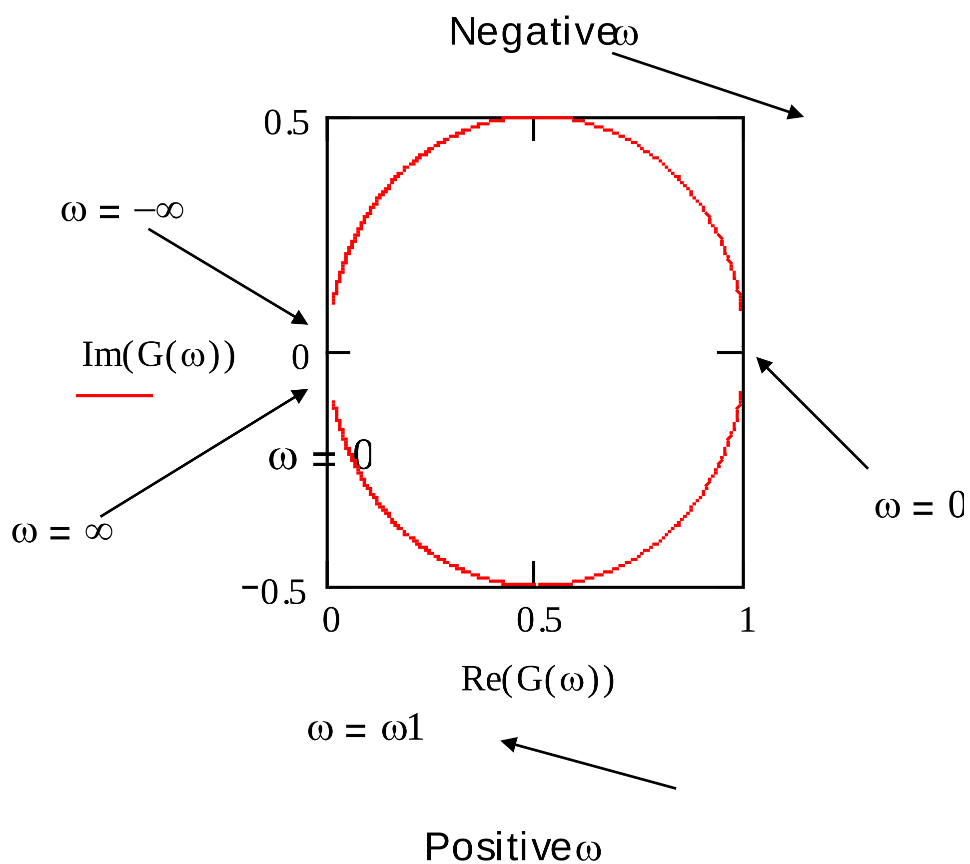
Frequency Response Plots

Polar Plots



$\omega := -1000, -999 .. 1000 \qquad j := \sqrt{-1} \qquad R := 1 \qquad C := 0.01 \qquad \omega 1 := \frac{1}{R \cdot C}$

$$G(\omega) := \frac{1}{\left(j \cdot \frac{\omega}{\omega 1}\right) + 1}$$

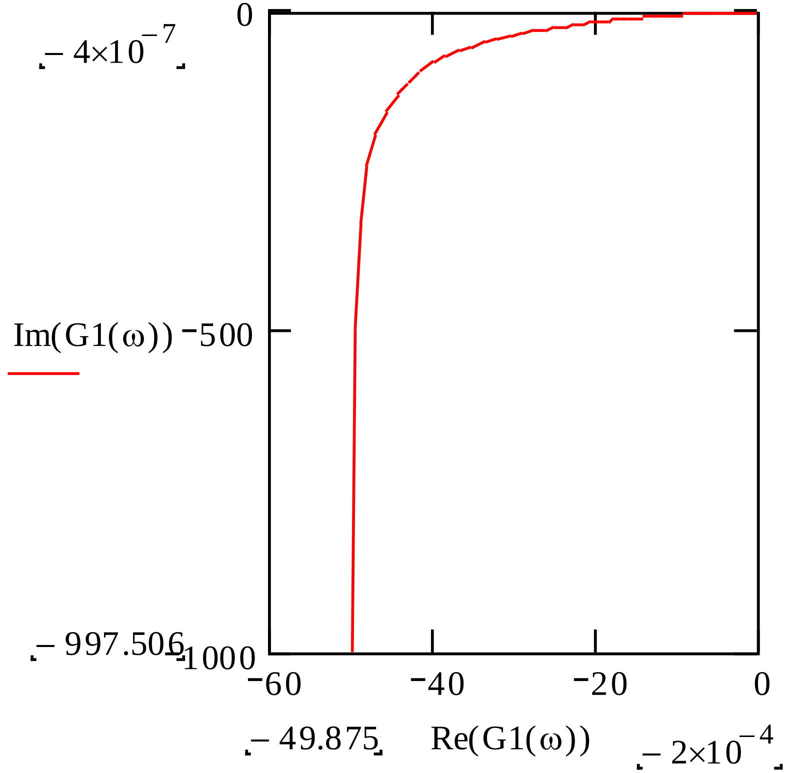


Frequency Response Plots

Polar Plots

$\omega := 0, .1 .. 1000 \qquad \tau := 0.5 \qquad K := 10($

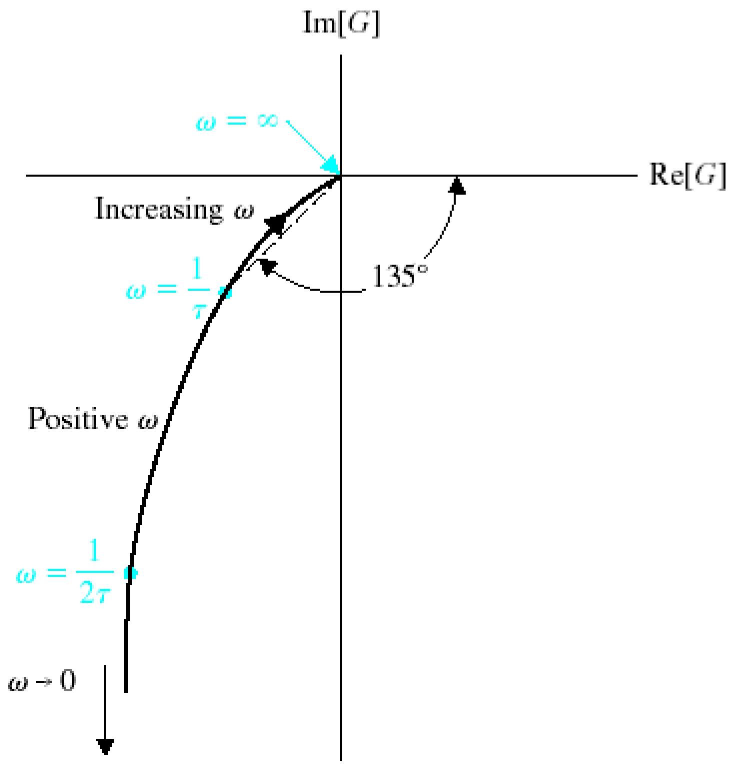
$$G1(\omega) := \frac{\frac{K}{\tau}}{j\cdot\omega\cdot\left(j\cdot\omega + \frac{1}{\tau}\right)}$$



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Frequency Response Plots

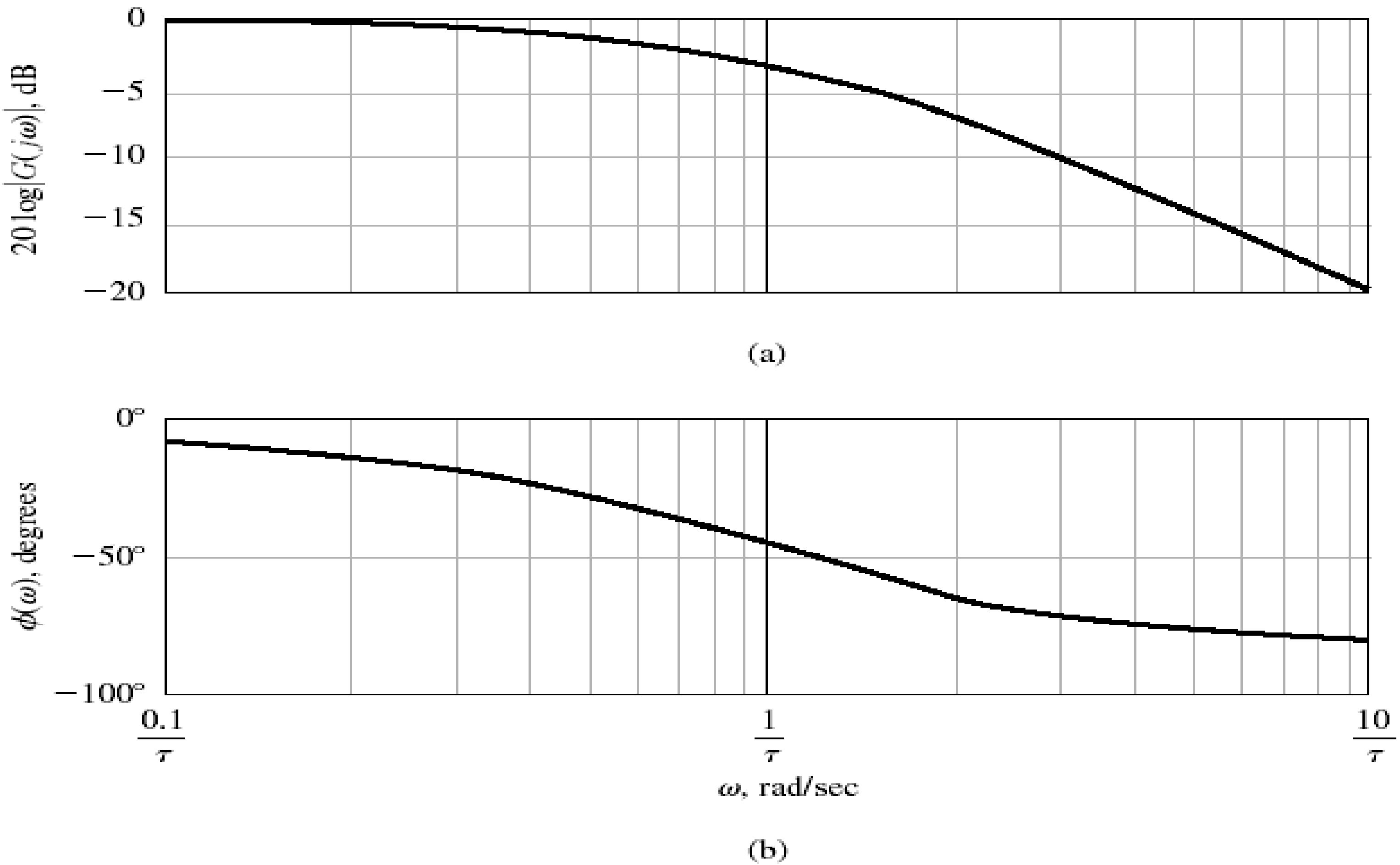
Polar Plots



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Frequency Response Plots

Bode Plots – Real Poles



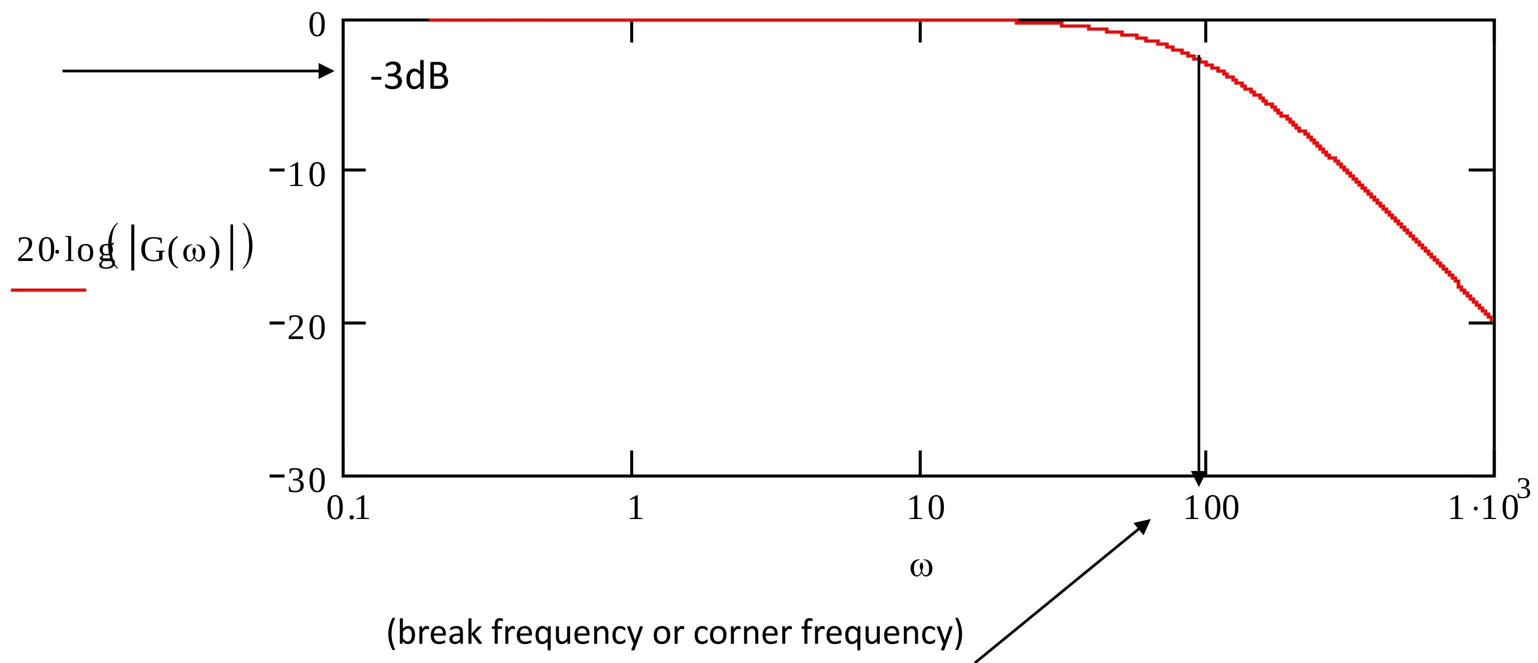
Bode diagram for $G(j\omega) = 1/(j\omega\tau + 1)$: (a) magnitude plot and (b) phase plot.

Frequency Response Plots

Bode Plots – Real Poles

$$\omega := \frac{0.1}{\tau}, \frac{0.11}{\tau} .. 1000 \qquad j := \sqrt{-1} \qquad R := 1 \qquad C := 0.01 \qquad \tau := R \cdot C$$

$$G(\omega) := \frac{1}{j \cdot \omega \cdot \tau + 1} \qquad \omega 1 := \frac{1}{\tau} \qquad \omega 1 = 100$$ (break frequency or corner frequency)

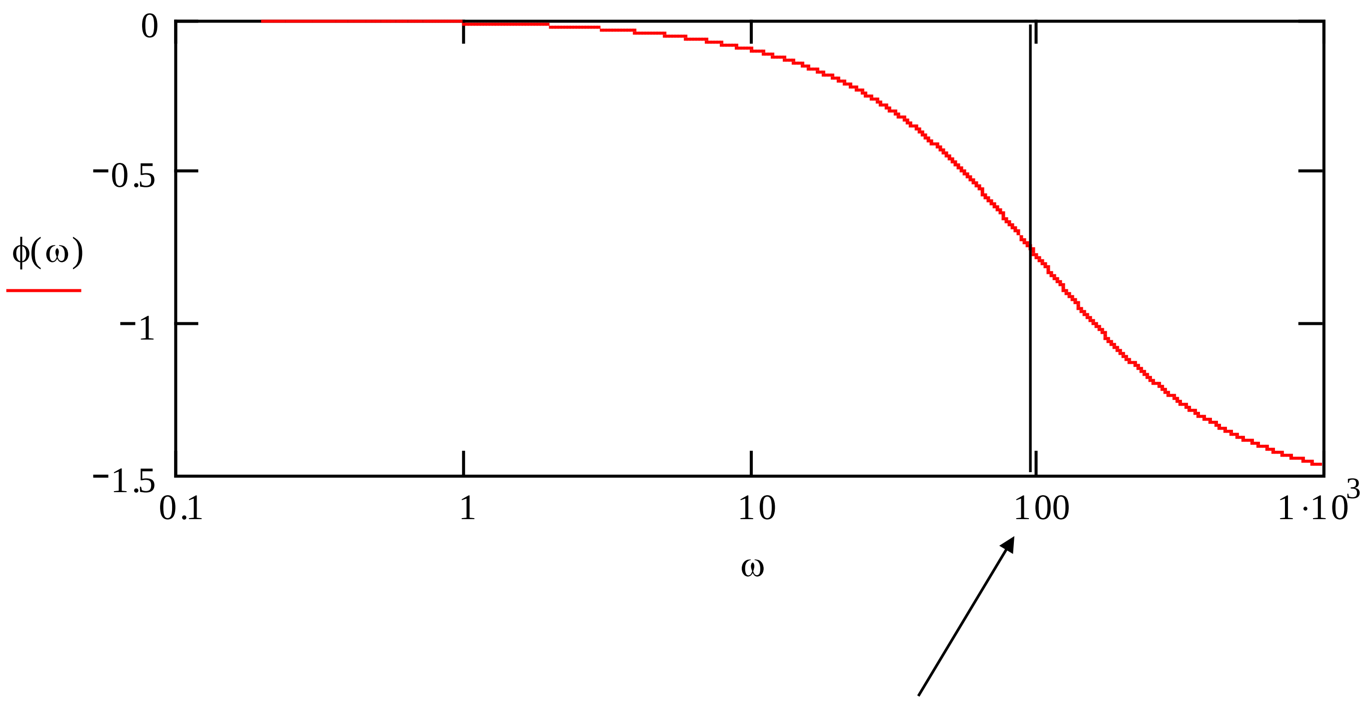


(break frequency or corner frequency)

Frequency Response Plots

Bode Plots – Real Poles

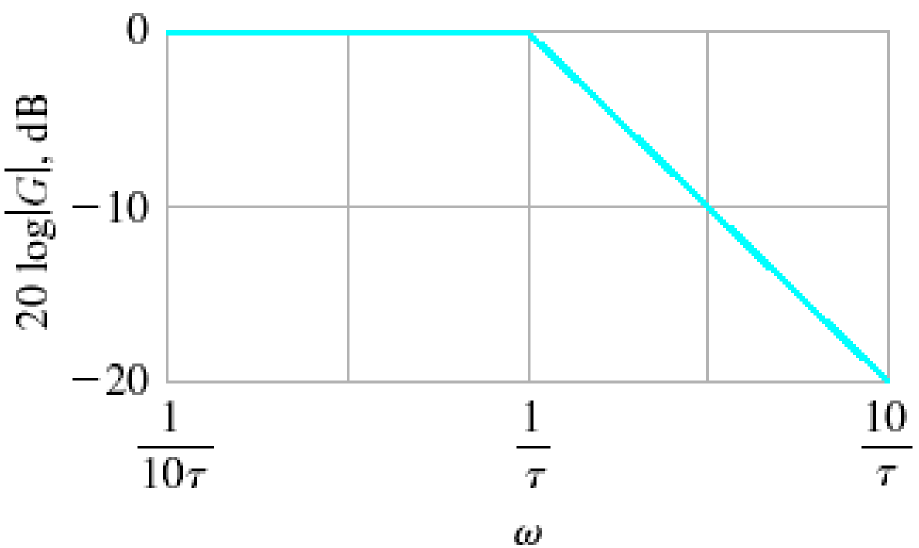
$$\phi(\omega) := -\text{atan}(\omega\cdot\tau)$$



(break frequency or corner frequency)

Frequency Response Plots

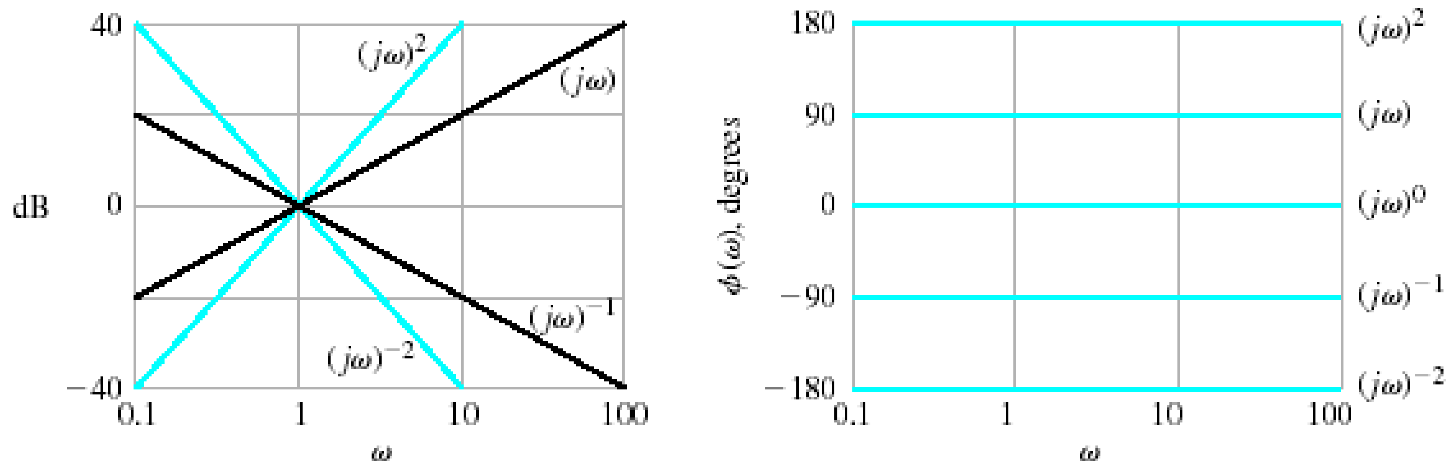
Bode Plots – Real Poles (Graphical Construction)



Asymptotic curve for $(j\omega\tau + 1)^{-1}$.

Frequency Response Plots

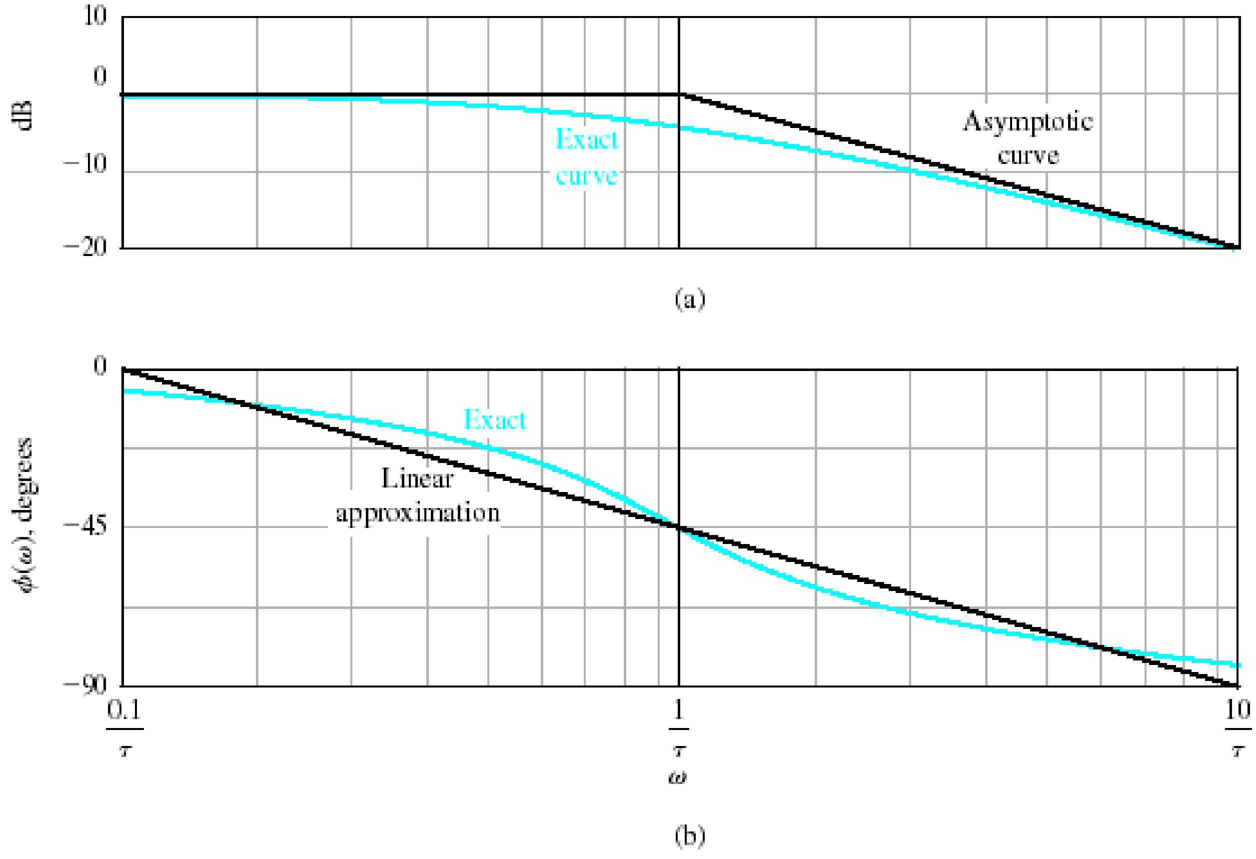
Bode Plots – Real Poles



Bode diagram for $(j\omega)^{\pm N}$.

Frequency Response Plots

Bode Plots – Real Poles



Bode diagram for $(1 + j\omega\tau)^{-1}$.

Frequency Response Plots

Bode Plots – Real Poles

Magnitude:

$$db(G, \omega) := 20 \cdot \log(|G(j \cdot \omega)|)$$

Phase shift:

$$ps(G, \omega) := \frac{180}{\pi} \cdot \arg(G(j \cdot \omega)) - 360 \cdot (\text{if}(\arg(G(j \cdot \omega)) \geq 0, 1, 0))$$

Assume

$$K := 2 \qquad G(s) := \frac{K}{s \cdot (1 + s) \cdot \left(1 + \frac{s}{3}\right)}$$

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): $\omega_{start} := .01$ number of points: $N := 50$

highest frequency (in Hz): $\omega_{end} := 100$

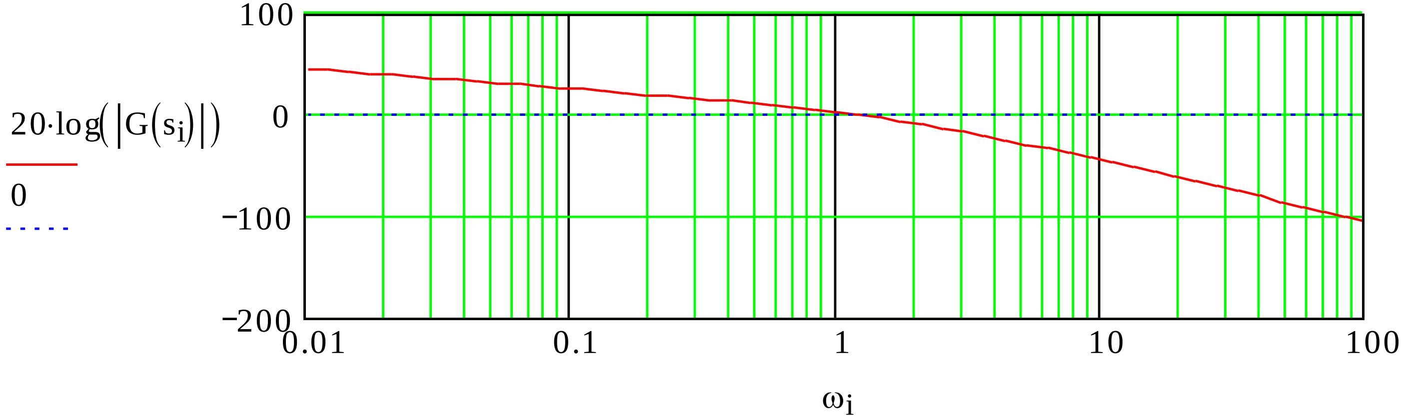
step size: $r := \log\left(\frac{\omega_{start}}{\omega_{end}}\right) \cdot \frac{1}{N}$

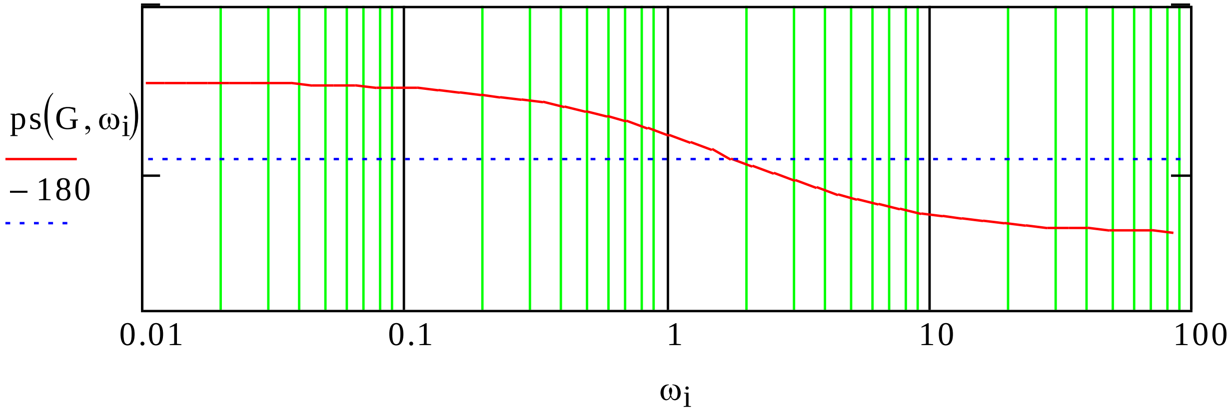
range for plot: $i := 0 .. N$ range variable: $\omega_i := \omega_{end} \cdot 10^{i \cdot r}$ $s_i := j \cdot \omega_i$

Frequency Response Plots

Bode Plots – Real Poles

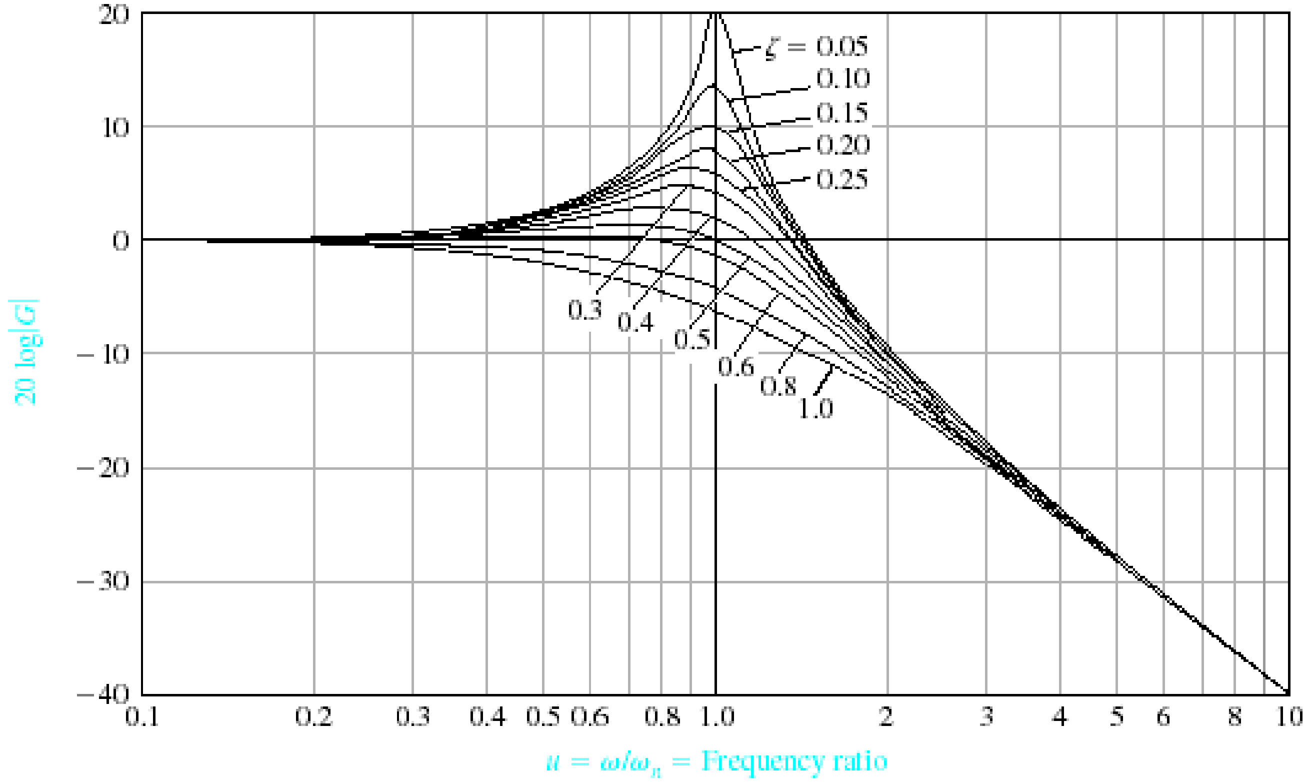
range for plot: $i := 0..N$ range variable: $\omega_i := \omega_{end} \cdot 10^{i \cdot r}$ $s_i := j \cdot \omega_i$





Frequency Response Plots

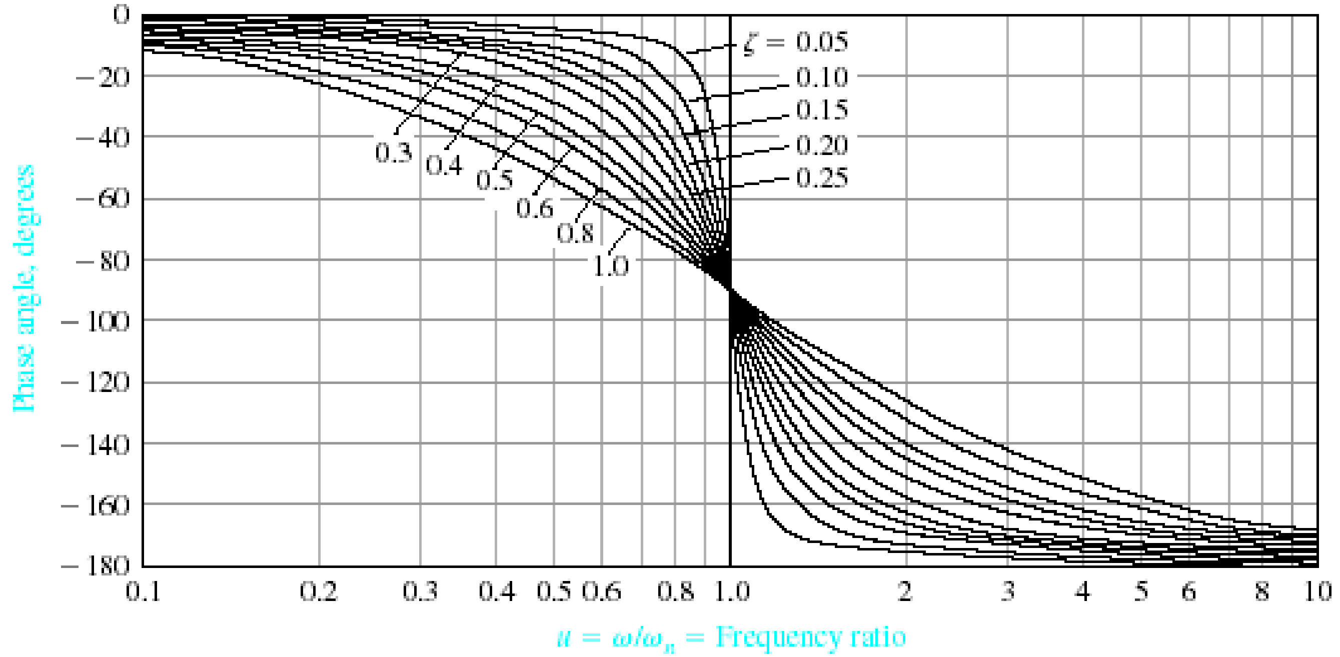
Bode Plots – Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)\,j\omega + (j\omega/\omega_n)^2]^{-1}$.

Frequency Response Plots

Bode Plots – Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2]^{-1}$.

Frequency Response Plots

Bode Plots – Complex Poles

$$\omega_r = \omega_n \cdot \sqrt{1 - 2\cdot\zeta^2} \qquad \zeta < 0.707$$

$$M_{p\omega} = \left|G(\omega_r)\right| = \frac{1}{\left(2\cdot\zeta\cdot\sqrt{1-\zeta^2}\right)} \qquad \zeta < 0.707$$

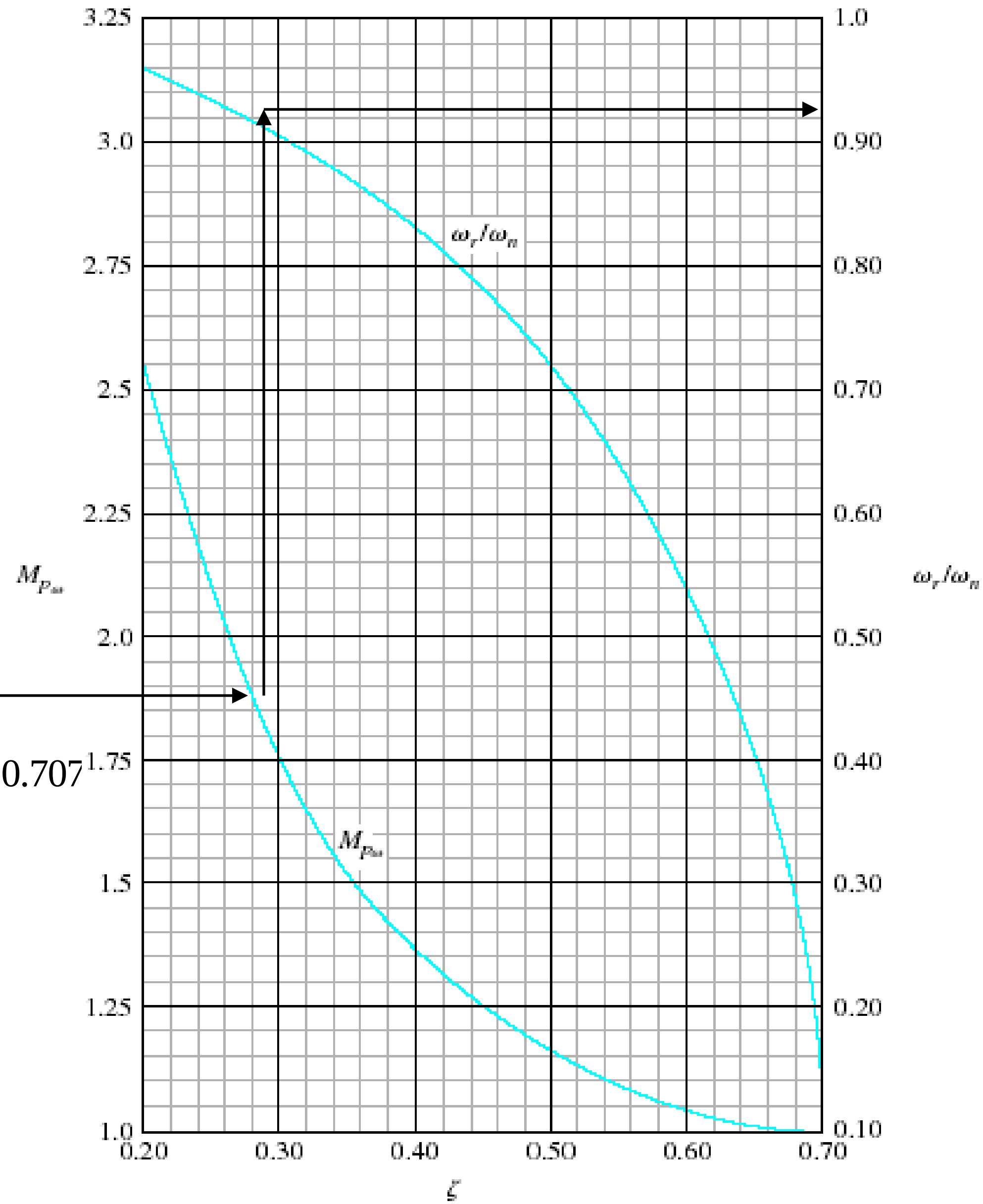
Frequency Response Plots

Bode Plots – Complex Poles

$$\omega_r = \omega_n \cdot \sqrt{1 - 2\cdot\zeta^2} \qquad \zeta < 0.707$$

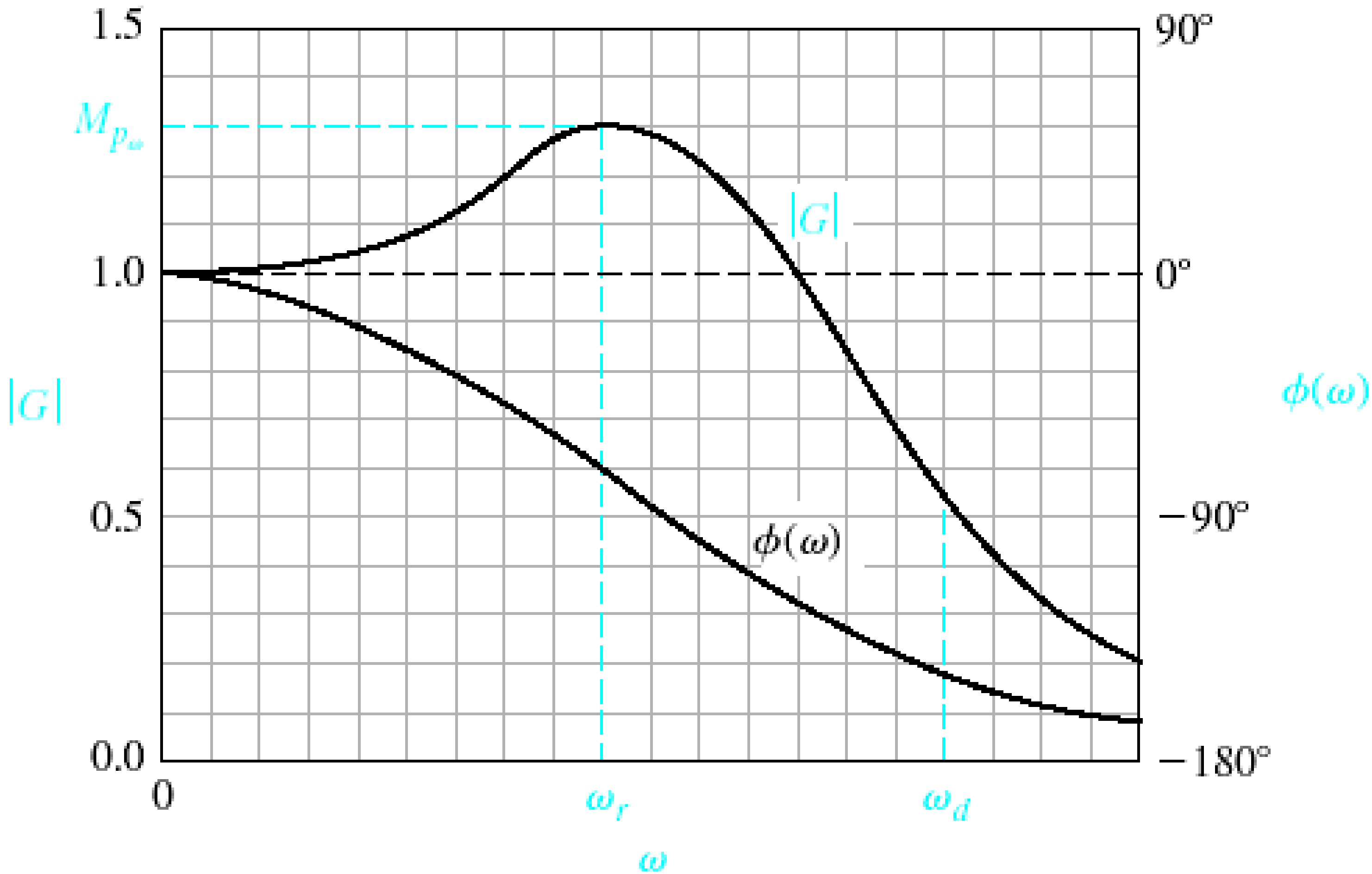
$$M_{p\omega} = \left|G(\omega_r)\right| = \frac{1}{\left(2\cdot\zeta\cdot\sqrt{1-\zeta^2}\right)}$$

$\zeta < 0.707$



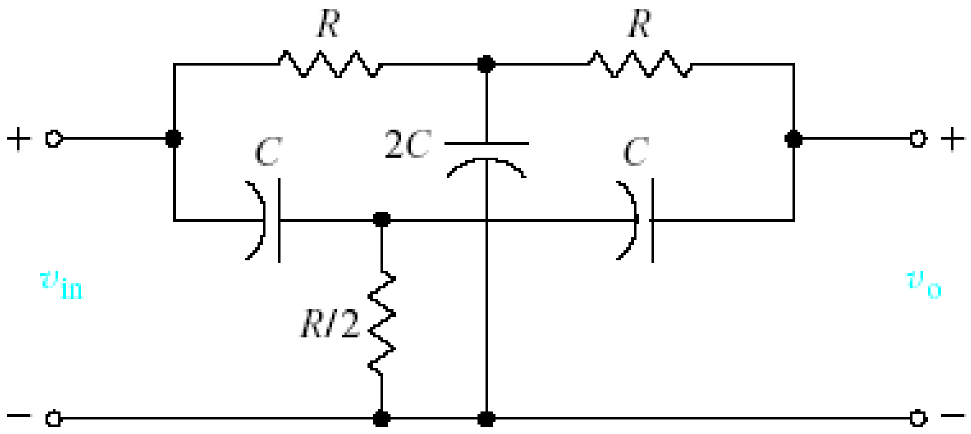
Frequency Response Plots

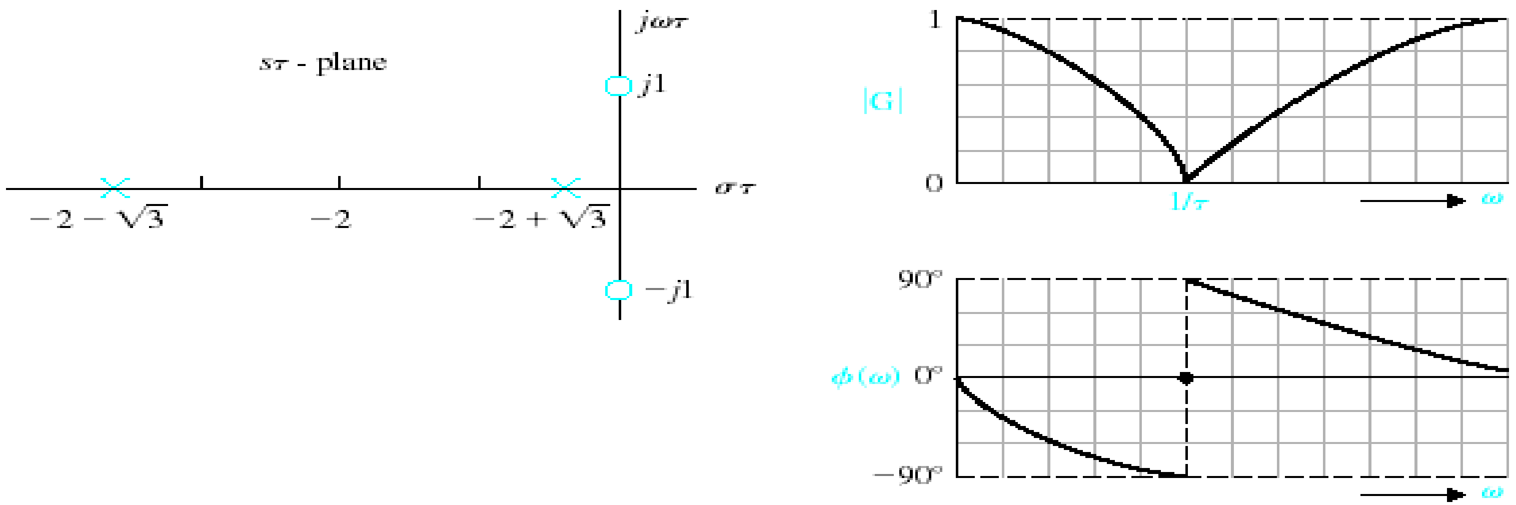
Bode Plots – Complex Poles



Frequency Response Plots

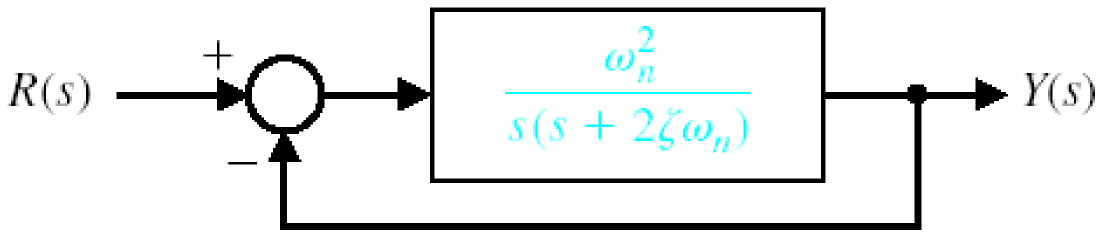
Bode Plots – Complex Poles

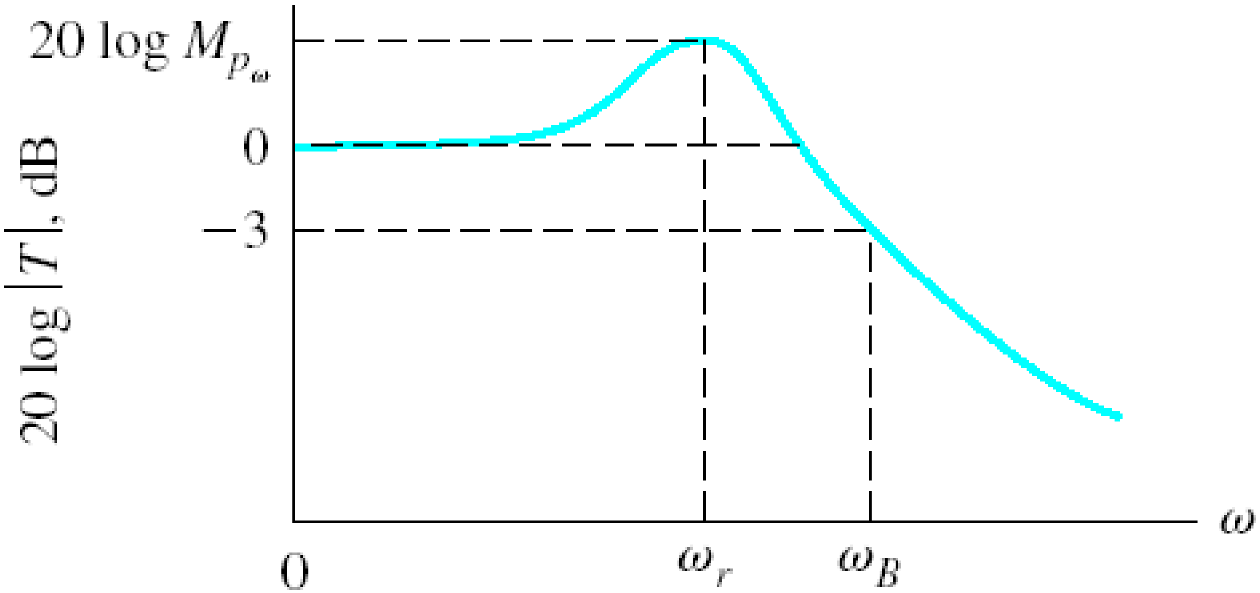




(a) (b)

Performance Specification In the Frequency Domain



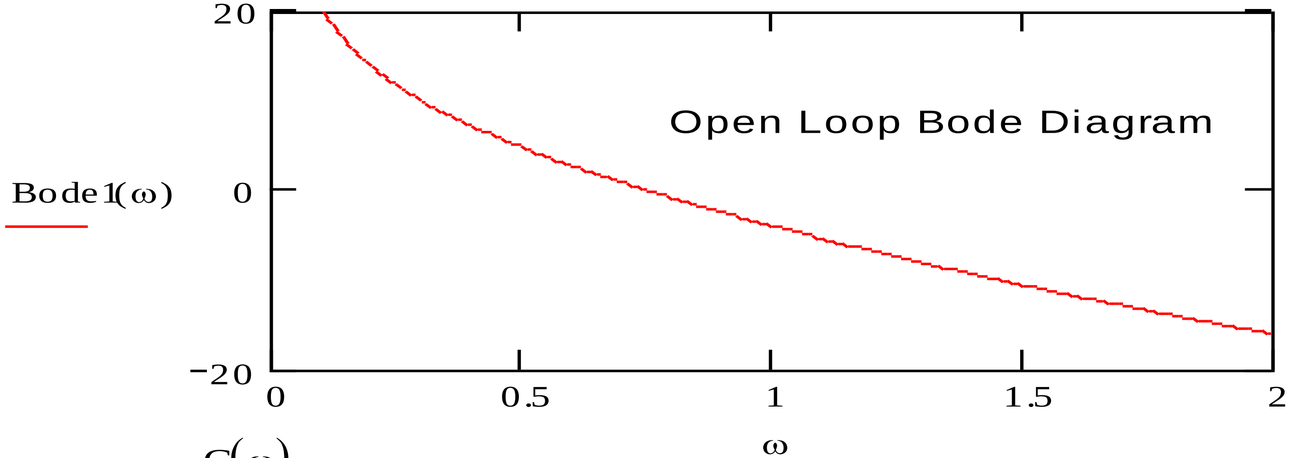


Performance Specification In the Frequency Domain

$\omega := .1, .11 .. 2$ $K := 2$ $j := \sqrt{-1}$

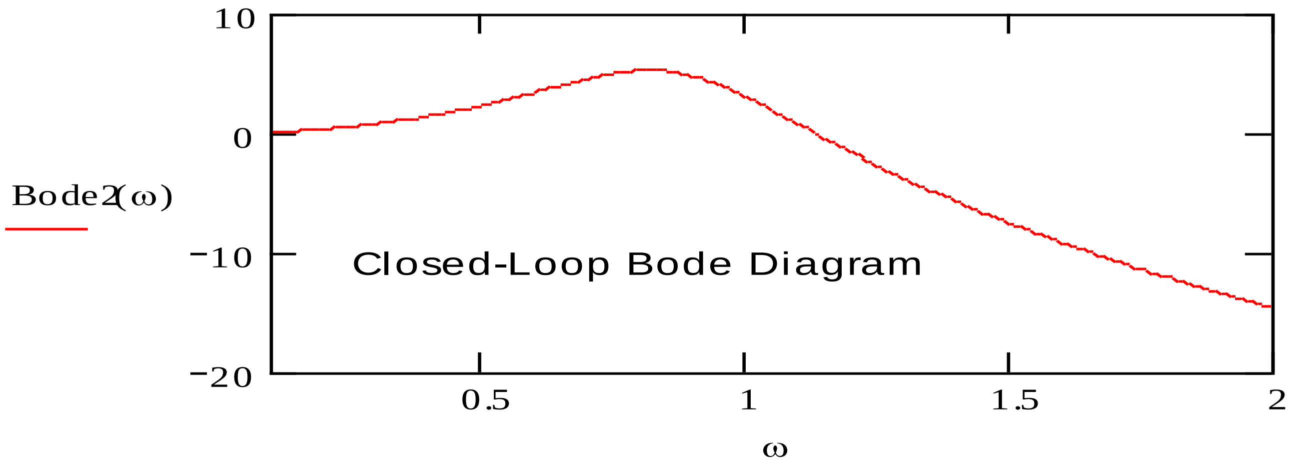
$$G(\omega) := \frac{K}{j \cdot \omega \cdot (j \cdot \omega + 1) \cdot (j \cdot \omega + 2)}$$

$$Bode1(\omega) := 20 \cdot \log(|G(\omega)|)$$



$$T(\omega) := \frac{G(\omega)}{1 + G(\omega)}$$

$$Bode2(\omega) := 20 \cdot \log(|T(\omega)|)$$



Performance Specification In the Frequency Domain

$w := 4$

Finding the Resonance Frequency

Given

$20 \cdot \log(|T(w)|) = 5.282$

$wr := Find(w)$ $wr = 0.813$

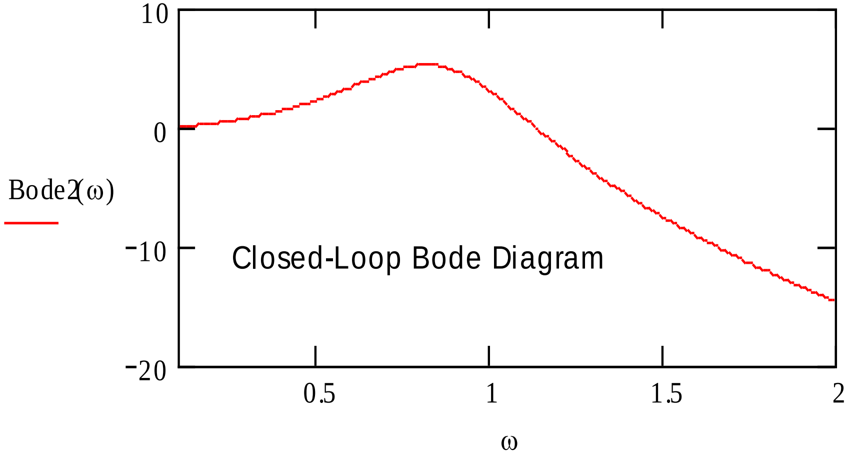
$Mpw := 1$

Given

$20 \cdot \log(Mpw) = 5.282$

Finding Maximum value of the frequency resp

$Mpw := Find(Mpw)$ $Mpw = 1.837$



Performance Specification In the Frequency Domain

Assume that the system has dominant second-order roo

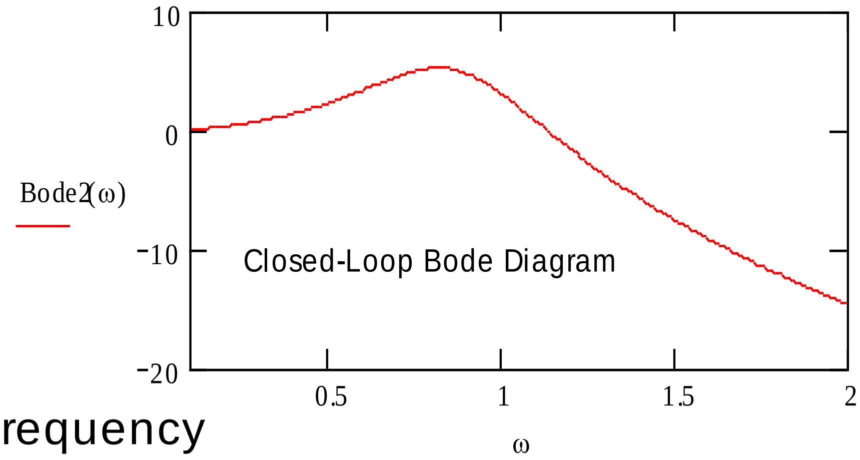
$\zeta := .1$

Finding the damping factor

Given

$$Mpw = \left[2\cdot\zeta\cdot\left(\sqrt{1-\zeta^2}\right)\right]^{-1}$$

$\zeta := Find(\zeta)$ $\qquad \zeta = 0.284$



$wn := .1$

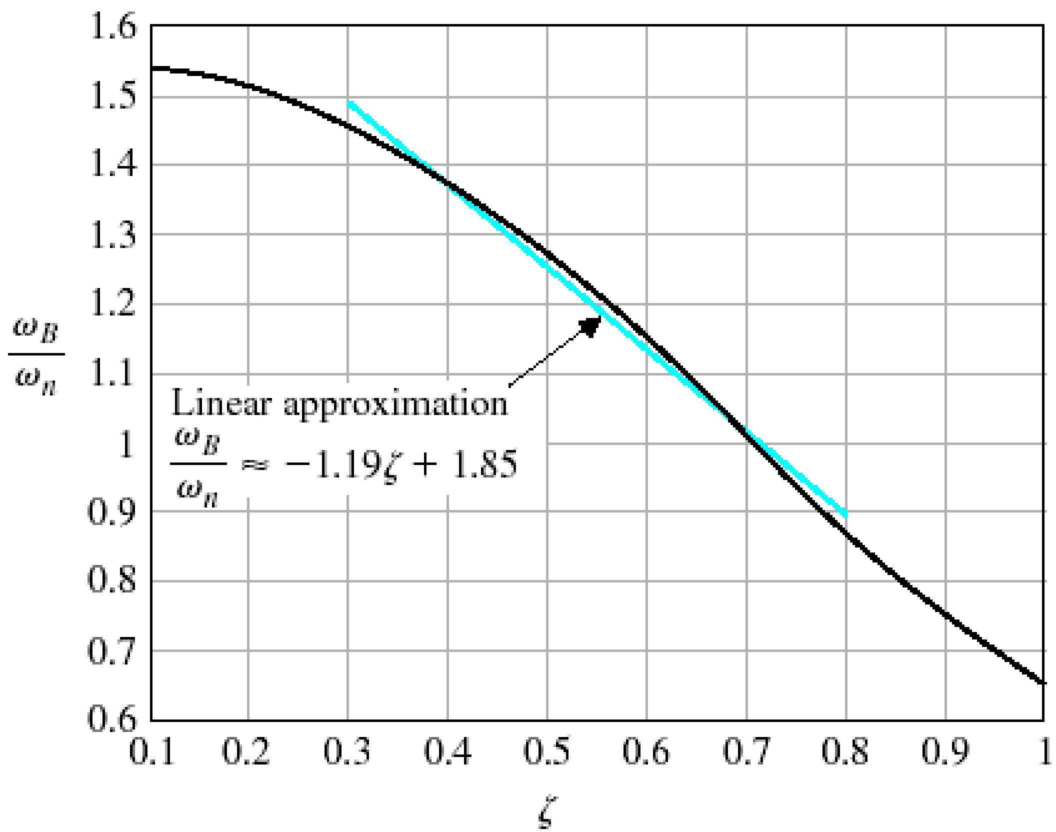
Finding the natural frequency

Given

$$wr = wn\cdot\sqrt{1-2\cdot\zeta^2}$$

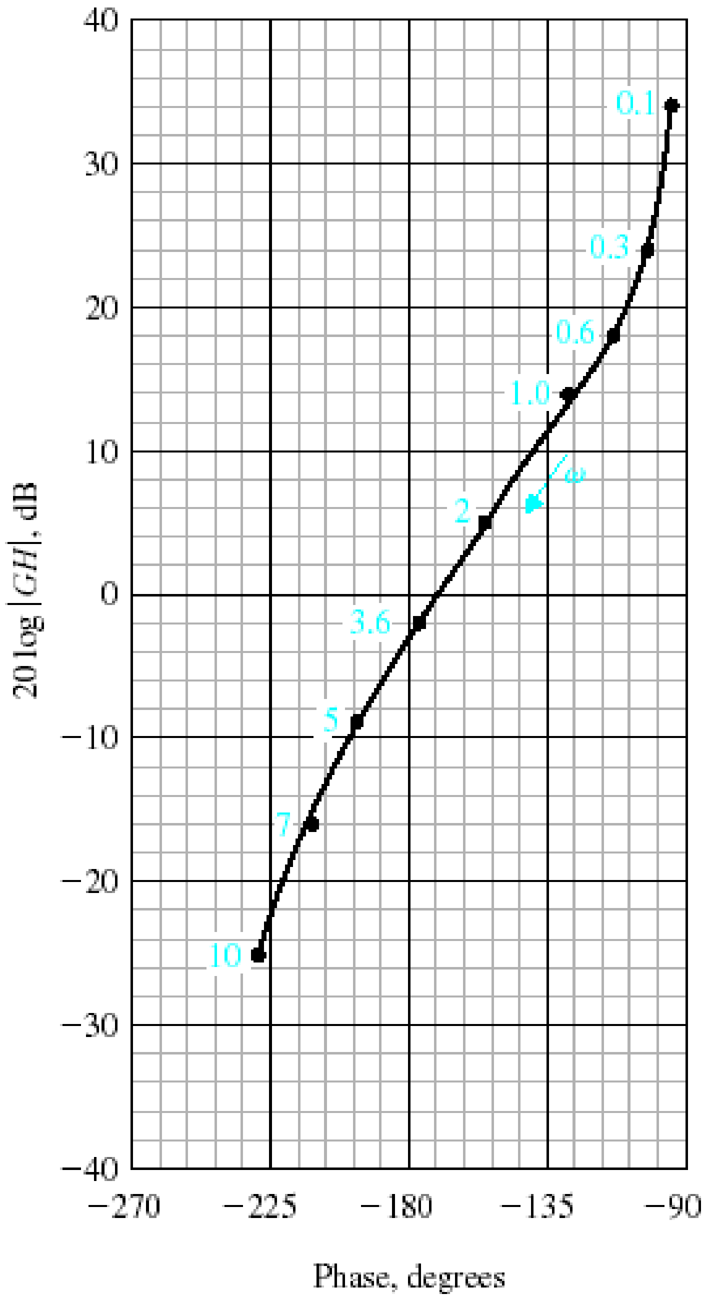
$wn := Find(wn)$ $\qquad wn = 0.888$

Performance Specification In the Frequency Domain

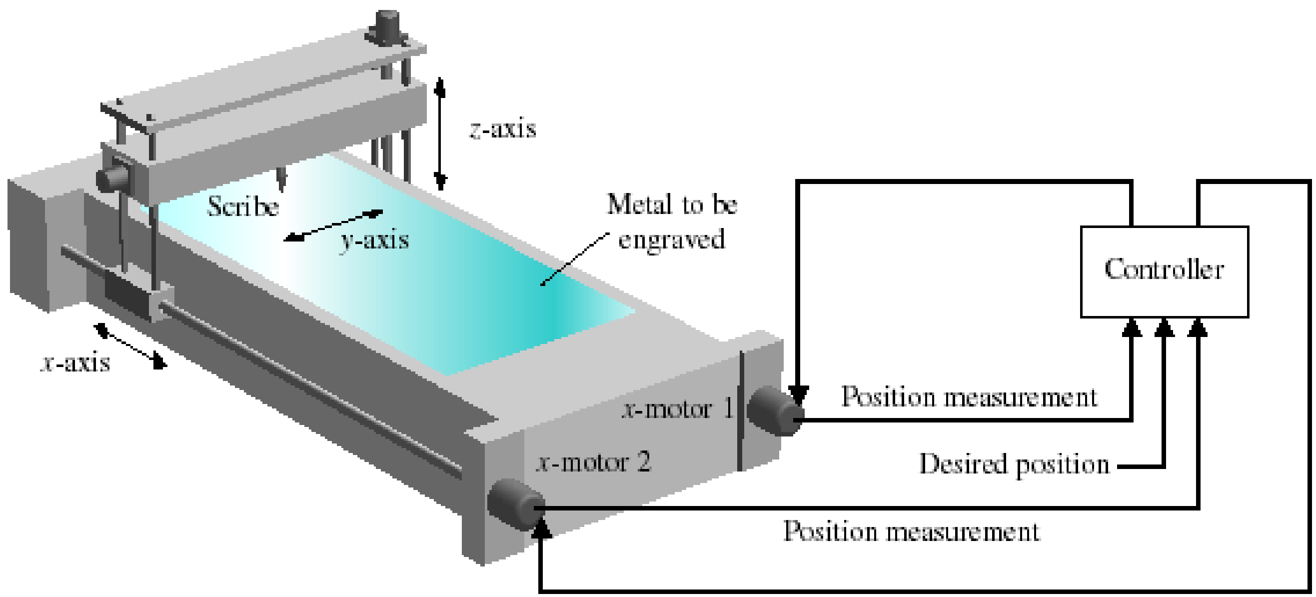


Performance Specification In the Frequency Domain

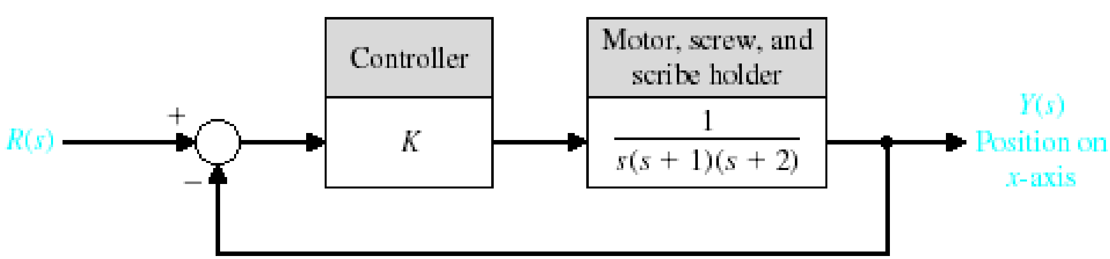
$$GH1(\omega) = \frac{5}{j\cdot\omega\cdot(0.5\,j\cdot\omega + 1)\cdot\left(j\cdot\frac{\omega}{6} + 1\right)}$$



Performance Specification In the Frequency Domain
Example



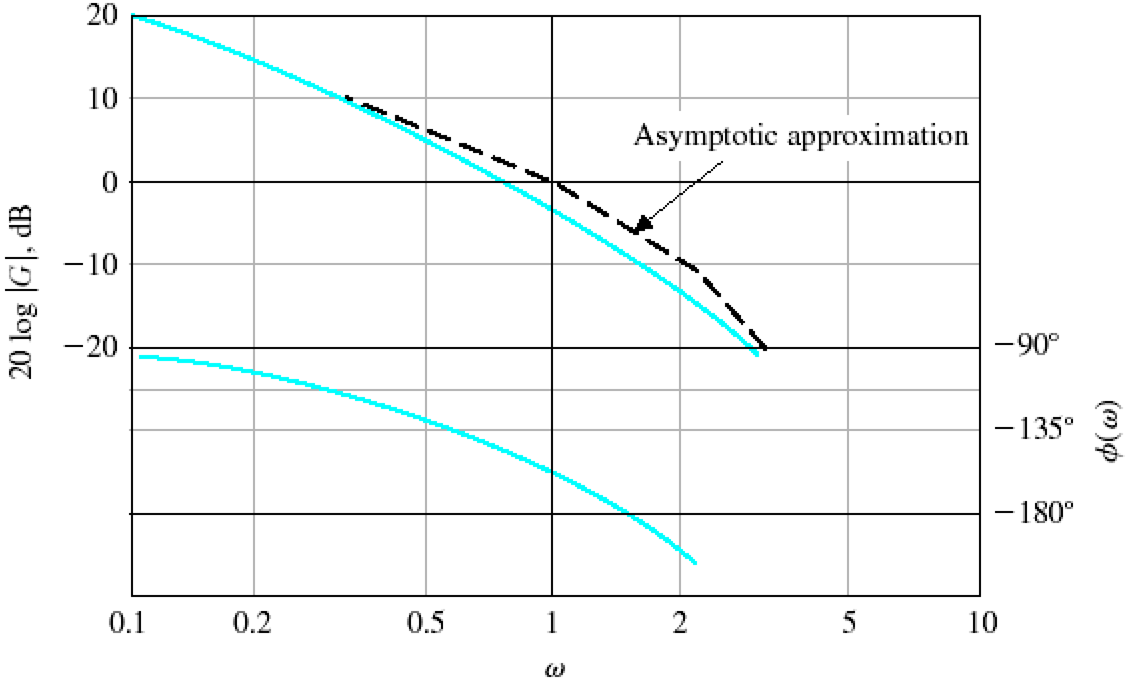
(a)



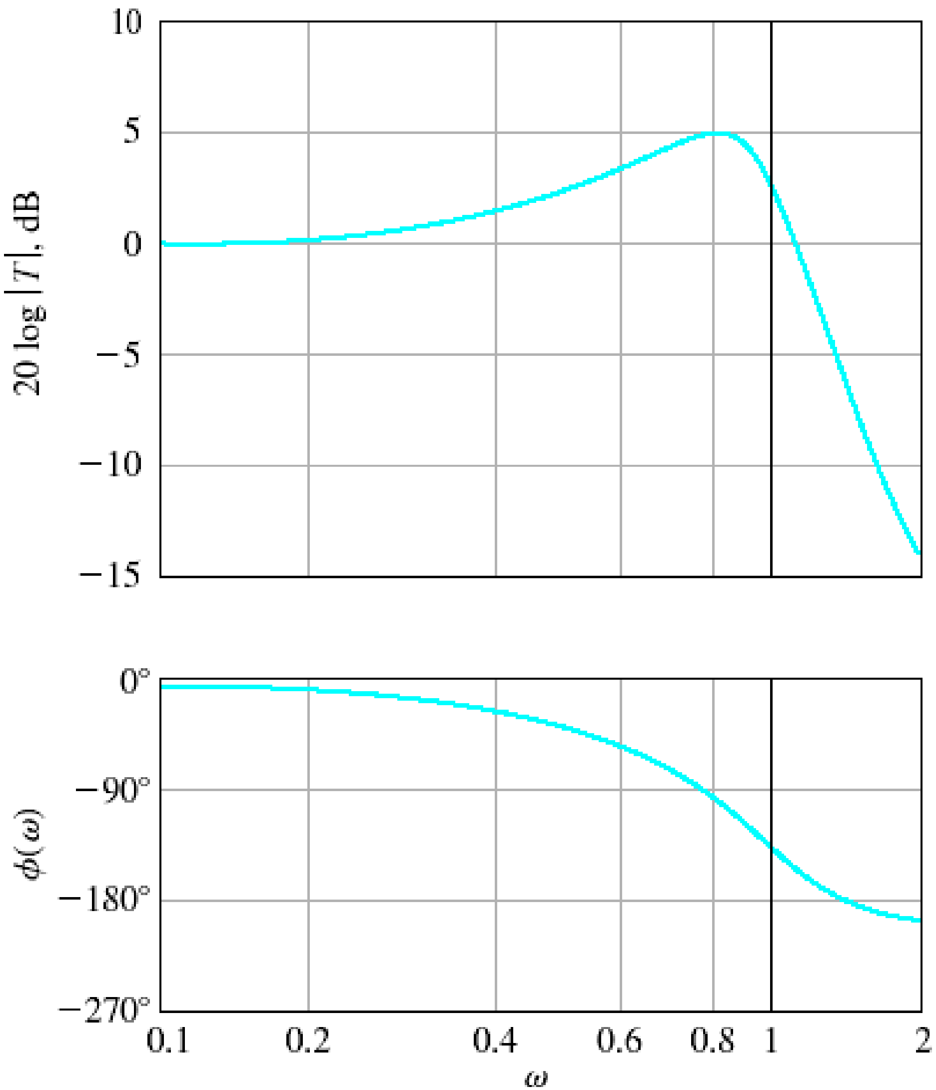
(b)

Performance Specification In the Frequency Domain
Example

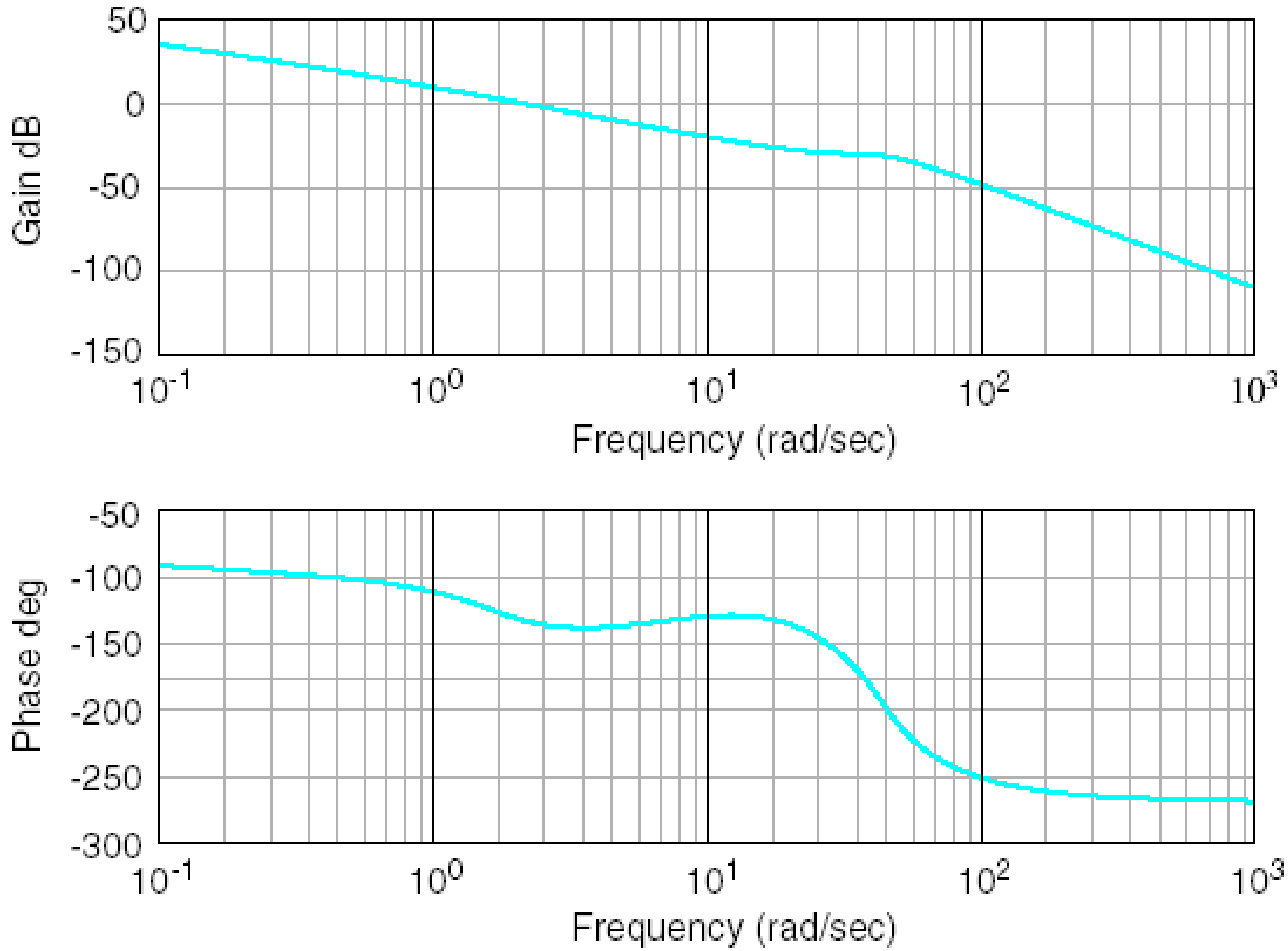
Performance Specification In the Frequency Domain
Example



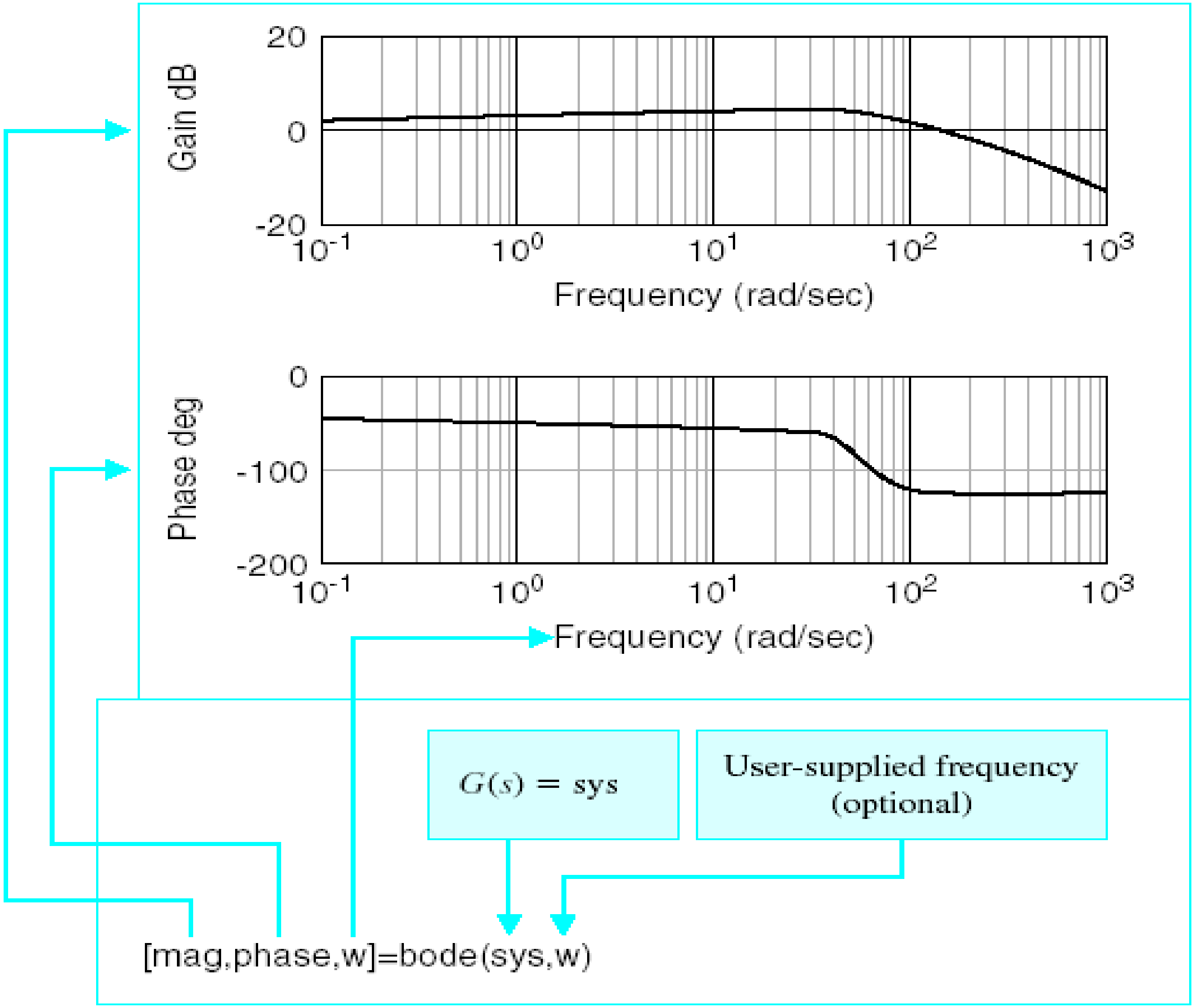
Performance Specification In the Frequency Domain
Example



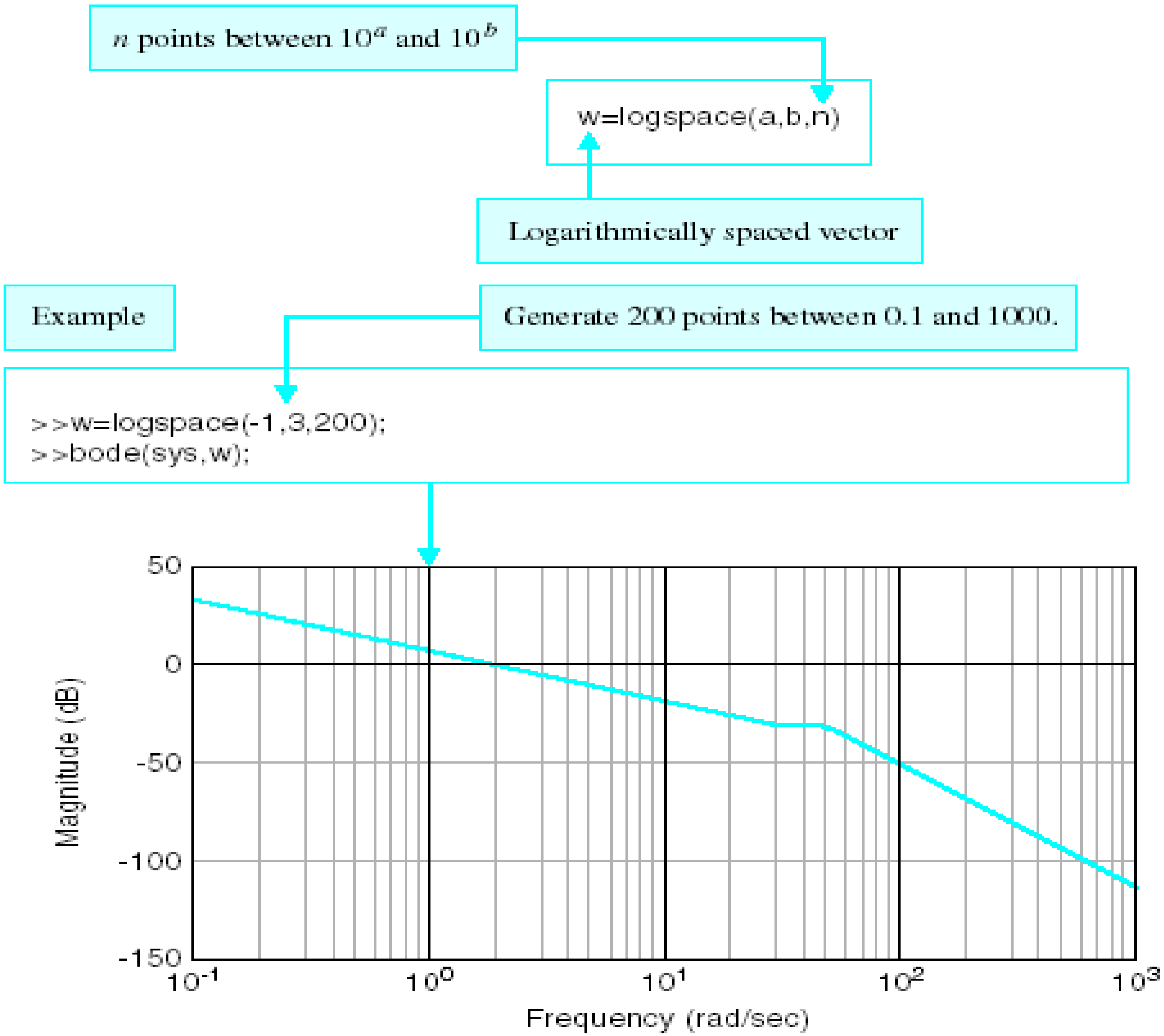
Frequency Response Methods Using MATLAB



Frequency Response Methods Using MATLAB



Frequency Response Methods Using MATLAB



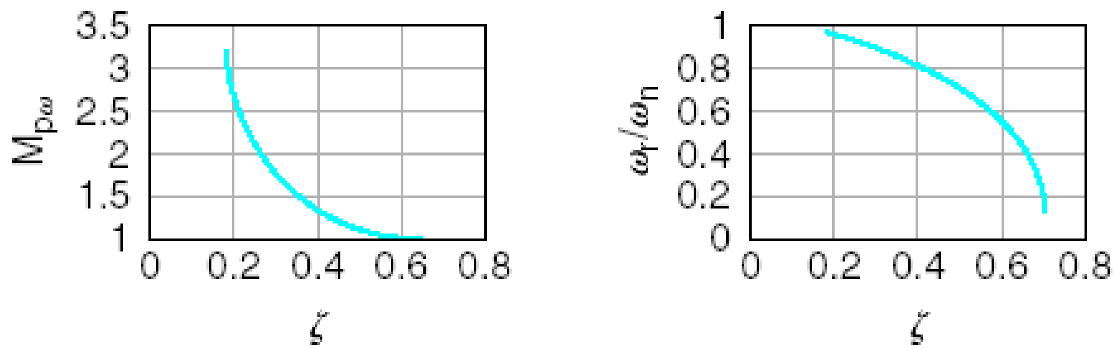
Frequency Response Methods Using MATLAB

```
% Bode plot script for Figure 8.22
%
num=5*[0.1 1];
f1=[1 0]; f2=[0.5 1]; f3=[1/2500 .6/50 1];
den=conv(f1,conv(f2,f3));
%
sys=tf(num,den);
bode(sys)
```

Compute

$$s(1 + 0.5s)(1 + \frac{0.6}{50}s + \frac{1}{50^2}s^2)$$

Frequency Response Methods Using MATLAB



(a)

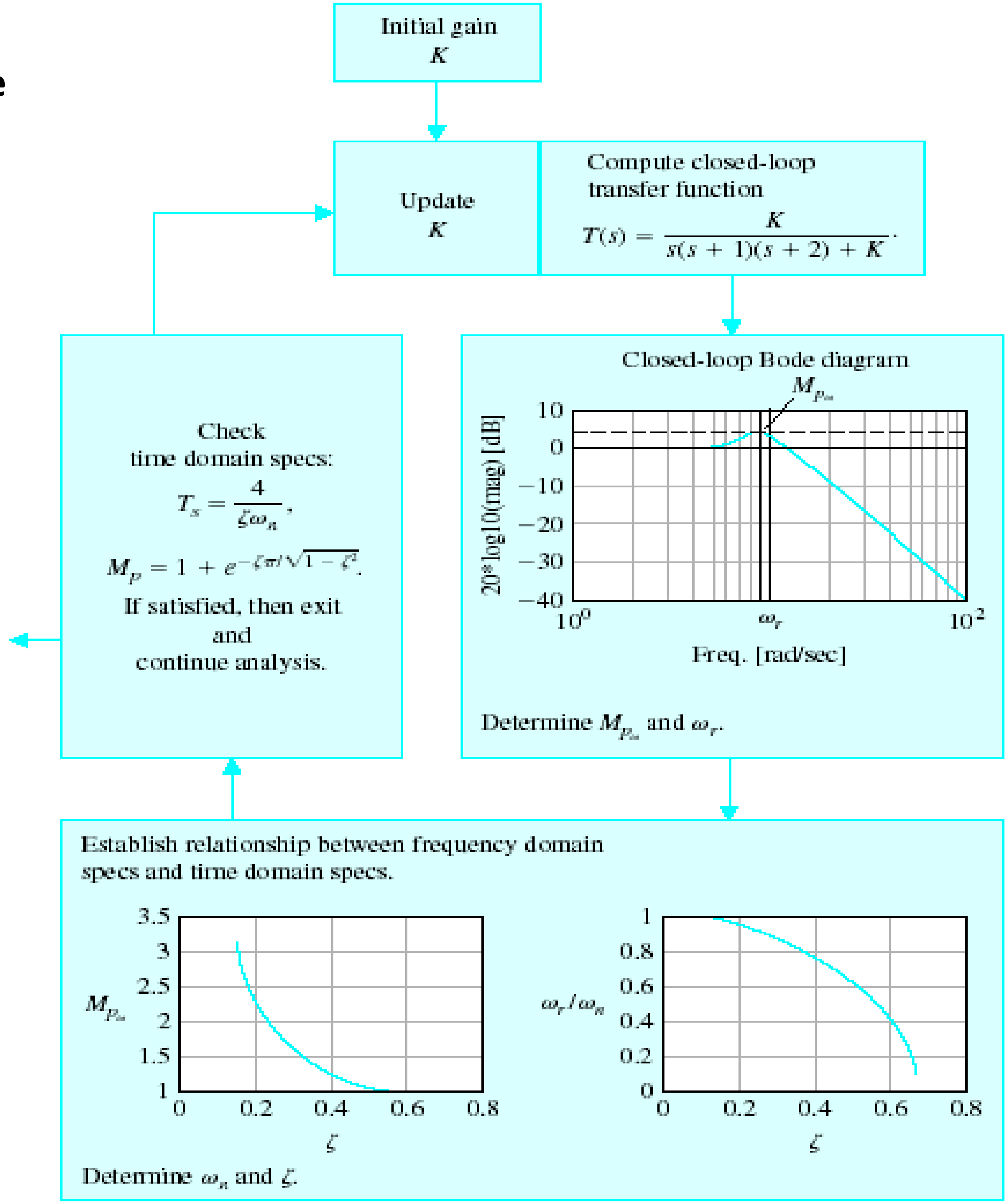
```
zeta=[0.15:0.01:0.7];
wr_over_wn=sqrt(1-2*zeta.^2);
Mp=(2*zeta .* sqrt(1-zeta.^2)).^(-1);
%
subplot(211),plot(zeta,Mp),grid
xlabel('\zeta'), ylabel('M_{p\omega}')
subplot(212),plot(zeta,wr_over_wn),grid
xlabel('\zeta'), ylabel('\omega_r/\omega_n')
```

zeta ranges from 0.15 to 0.70

Generate plots

(a) The relationship between (M_{p_ω}, ω_r) and (ζ, ω_n) for a second-order system. (b) MATLAB script.

Frequency Response Methods Using MATLAB



Frequency Response Methods Using MATLAB

engrave1.m

```
num=[K]; den=[1 3 2 K];
sys=tf(num,den);
w=logspace(-1,1,400);
[mag,phase,w]=bode(sys,w);
[mp,l]=max(mag);wr=w(l);
mp,wr
```

Closed-loop transfer function

Closed-loop Bode diagram

```
>>K=2; engrave1
mp =
     1.8371
wr =
     0.8171
>>
>>
>>
>>zeta=0.29; wn=0.88; engrave2
ts =
     15.6740
po =
     38.5979
```

manual step

Determine ω_n and ζ from Fig. 8.11 using M_{p_ω} and ω_r.

engrave2.m

```
ts=4/zeta/wn
po=100*exp(-zeta*pi/sqrt(1-zeta^2))
```

Check specs and iterate, if necessary.

Bode Plots

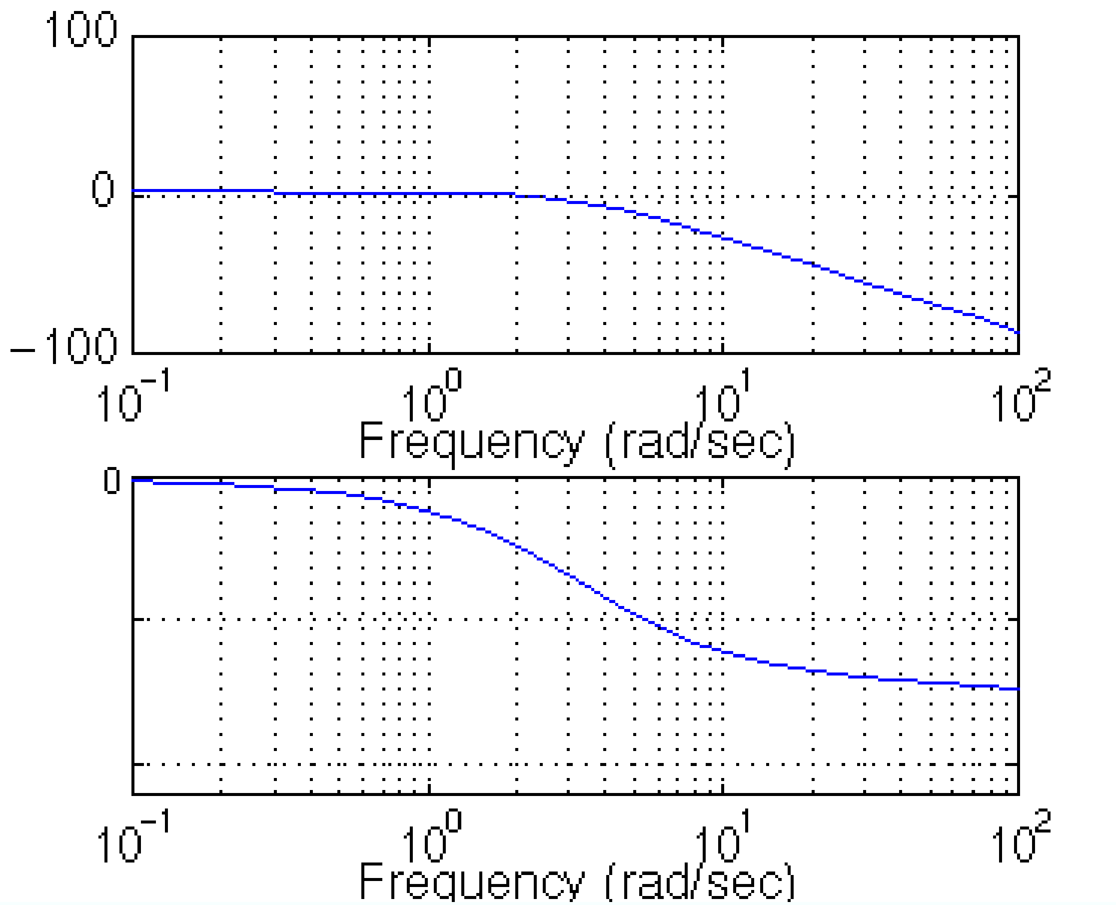
Bode plot is the representation of the magnitude and phase of G(j*w) (where the frequency vector w contains only positive frequencies).
To see the Bode plot of a transfer function, you can use the MATLAB `bode` command.

For example,

bode(50,[1 9 30 40])

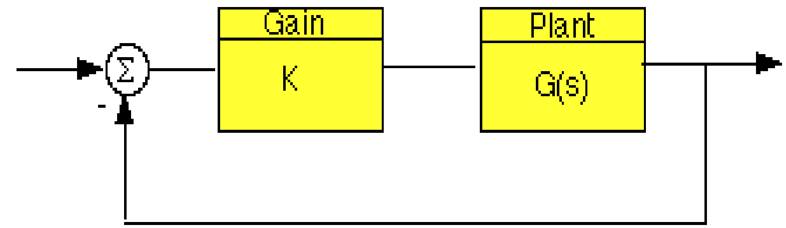
displays the Bode plots for the transfer function:

50 / (s^3 + 9 s^2 + 30 s + 40)



Gain and Phase Margin

Let's say that we have the following system:



where K is a variable (constant) gain and G(s) is the plant under consideration.

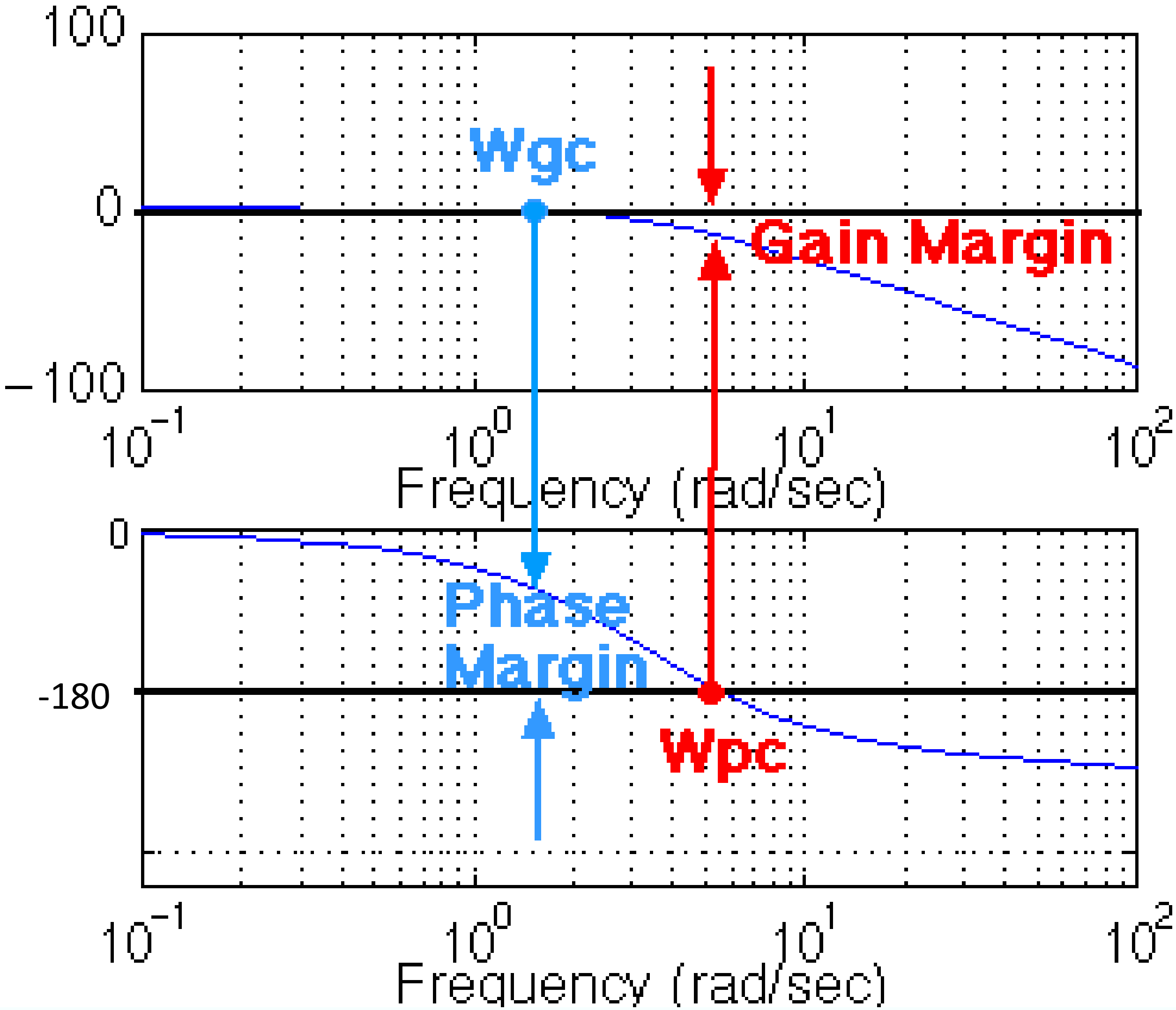
The gain margin is defined as the change in open loop gain required to make the system unstable. Systems with greater gain margins can withstand greater changes in system parameters before becoming unstable in closed loop. Keep in mind that unity gain in magnitude is equal to a gain of zero in dB.

The phase margin is defined as the change in open loop phase shift required to make a closed loop system unstable.

The phase margin is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0dB (the gain cross over frequency, Wgc).

Likewise, the gain margin is the difference between the magnitude curve and 0dB at the point corresponding to the frequency that gives us a phase of -180 deg (the phase cross over frequency, Wpc).

Gain and Phase Margin

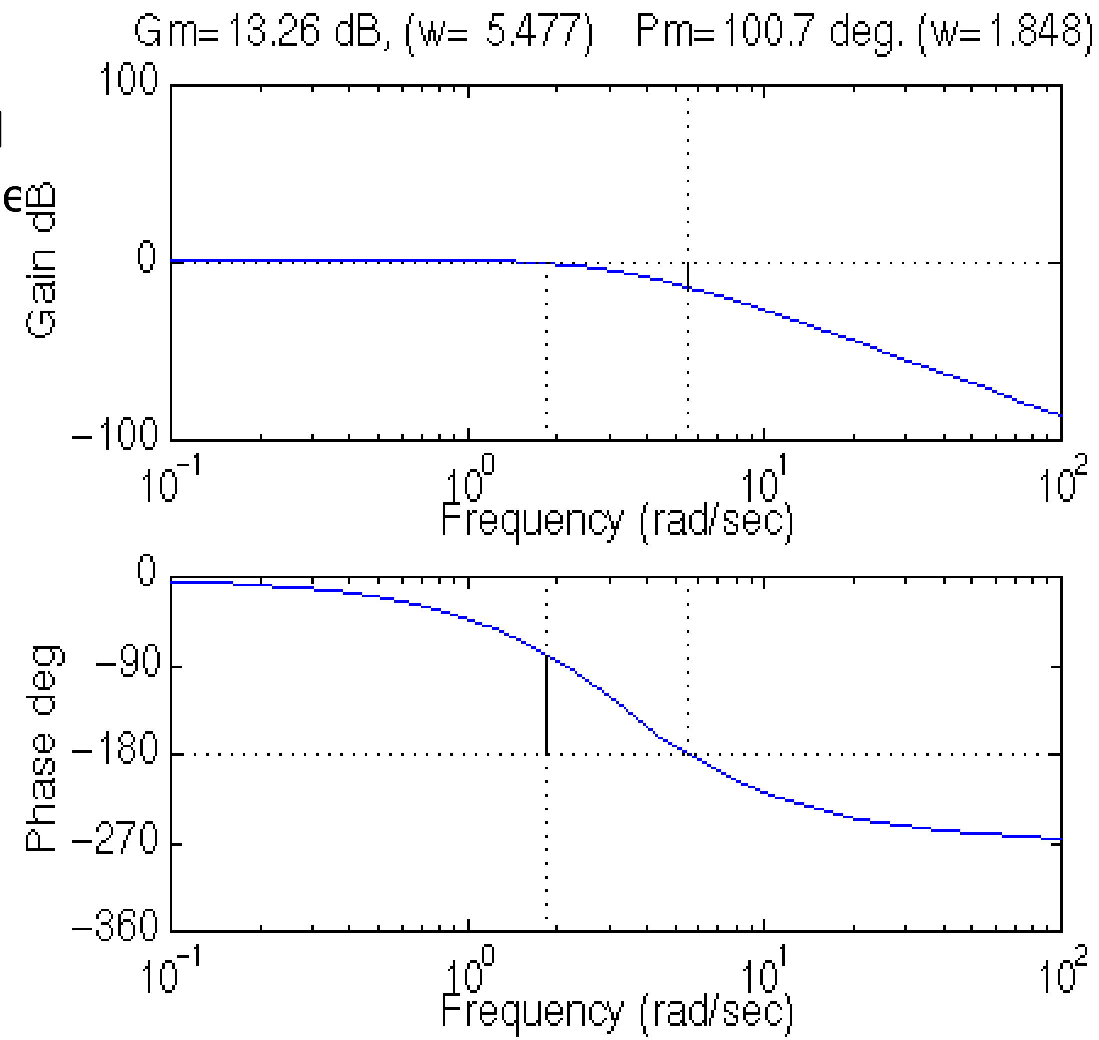


Gain and Phase Margin

We can find the gain and phase margins for a system directly, by using MATLAB. Just enter the `margin` command.

This command returns the gain and phase margins, the gain and phase cross over frequencies, and a graphical representation of these on the Bode plot.

margin(50,[1 9 30 40])



Gain and Phase Margin

Magnitude:

$$db(G,\omega) := 20\cdot\log(|G(j\cdot\omega)|)$$

Phase shift:

$$ps(G,\omega) := \frac{180}{\pi}\cdot\arg(G(j\cdot\omega)) - 360\cdot(\text{if}(\arg(G(j\cdot\omega)) \geq 0,1,0))$$

Assume

$K := 2$ $\qquad G(s) := \dfrac{K}{s\cdot(1+s)\cdot\left(1+\dfrac{s}{3}\right)}$

Next, choose a frequency range for the plots (use powers of 10 for convenient plotting):

lowest frequency (in Hz): $\omega_{start} := .01$ number of points: $N := 50$

highest frequency (in Hz): $\omega_{end} := 100$

step size: $r := \log\left(\dfrac{\omega_{start}}{\omega_{end}}\right)\cdot\dfrac{1}{N}$

range for plot: $i := 0..N$ range variable: $\omega_i := \omega_{end}\cdot 10^{i\cdot r}$ $\qquad s_i := j\cdot\omega_i$

Gain and Phase Margin

Guess for **crossover frequency**: $\omega_c := 1$

Solve for the gain crossover frequency:

$$\omega_c := \mathrm{root}(\mathrm{db}(G, \omega_c), \omega_c) \qquad \omega_c = 1.193$$

Calculate the **phase margin**

$$pm := ps(G, \omega_c) + 180 \qquad pm = 18.265 \text{ degrees}$$

Gain Margin

Now using the phase angle plot, estimate the frequency at which the phase shift crosses 180 degr

$$\omega_{gm} := 1.8$$

Solve for **ω** at the phase shift point of 180 degrees:

$$\omega_{gm} := \mathrm{root}(ps(G, \omega_{gm}) + 180, \omega_{gm})$$

$$\omega_{gm} = 1.732$$

Calculate the **gain margin**

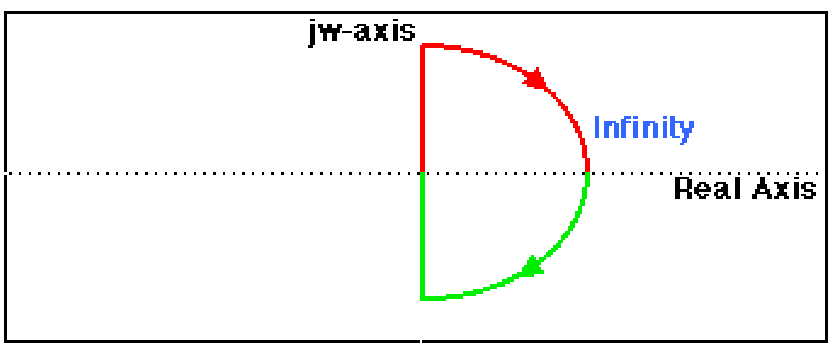
$$gm := -\mathrm{db}(G, \omega_{gm}) \qquad gm = 6.021$$

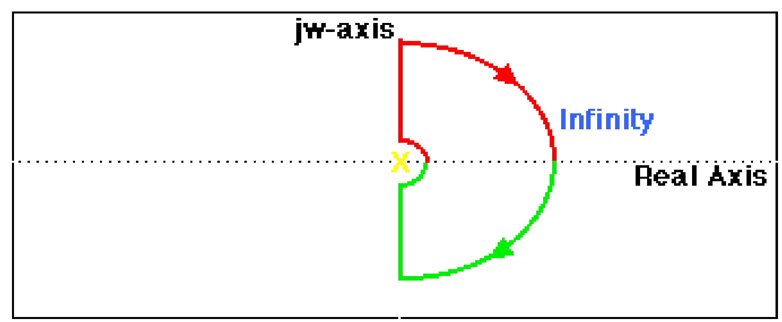
The Nyquist Stability Criterion

The Nyquist plot allows us also to predict the stability and performance of a closed-loop system by observing its open-loop behavior. The Nyquist criterion can be used for design purposes regardless of open-loop stability (Bode design methods assume that the system is stable in open loop). Therefore, we use this criterion to determine closed-loop stability when the Bode plots display confusing information.

The Nyquist diagram is basically a plot of G(j* w) where G(s) is the open-loop transfer function and w is a vector of frequencies which encloses the entire right-half plane. In drawing the Nyquist diagram, both positive and negative frequencies (from zero to infinity) are taken into account. In the illustration below we represent positive frequencies in red and negative frequencies in green. The frequency vector used in plotting the Nyquist diagram usually looks like this (if you can imagine the plot stretching out to infinity):

However, if we have open-loop poles or zeros on the jw axis, G(s) will not be defined at those points, and we must loop around them when we are plotting the contour. Such a contour would look as follows:





The Cauchy criterion

The Cauchy criterion (from complex analysis) states that when taking a closed contour in the complex plane, and mapping it through a complex function G(s), the number of times that the plot of G(s) encircles the origin is equal to the number of zeros of G(s) enclosed by the frequency contour minus the number of poles of G(s) enclosed by the frequency contour. Encirclements of the origin are counted as positive if they are in the same direction as the original closed contour or negative if they are in the opposite direction.

When studying feedback controls, we are not as interested in G(s) as in the closed-loop transfer function:

```
   G(s)
---------
1 + G(s)
```

If 1+ G(s) encircles the origin, then G(s) will enclose the point -1.
Since we are interested in the closed-loop stability, we want to know if there are any closed-loop poles (zeros of 1 + G(s)) in the right-half plane.

Therefore, the behavior of the Nyquist diagram around the -1 point in the real axis is very important; however, the axis on the standard `nyquist` diagram might make it hard to see what's happening around this point

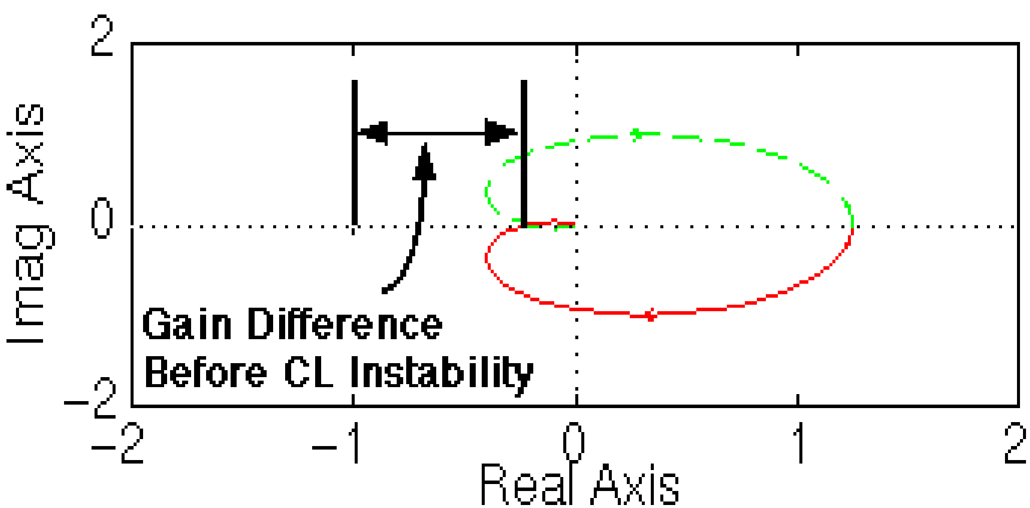
Gain and Phase Margin

Gain Margin is defined as the change in open-loop gain expressed in decibels (dB), required at 180 degrees of phase shift to make the system unstable. First of all, let's say that we have a system that is stable if there are no Nyquist encirclements of -1, such as :

$$\frac{50}{s^3 + 9 s^2 + 30 s + 40}$$

Looking at the roots, we find that we have no open loop poles in the right half plane and therefore no closed-loop poles in the right half plane if there are no Nyquist encirclements of -1. Now, how much can we vary the gain before this system becomes unstable in closed loop?

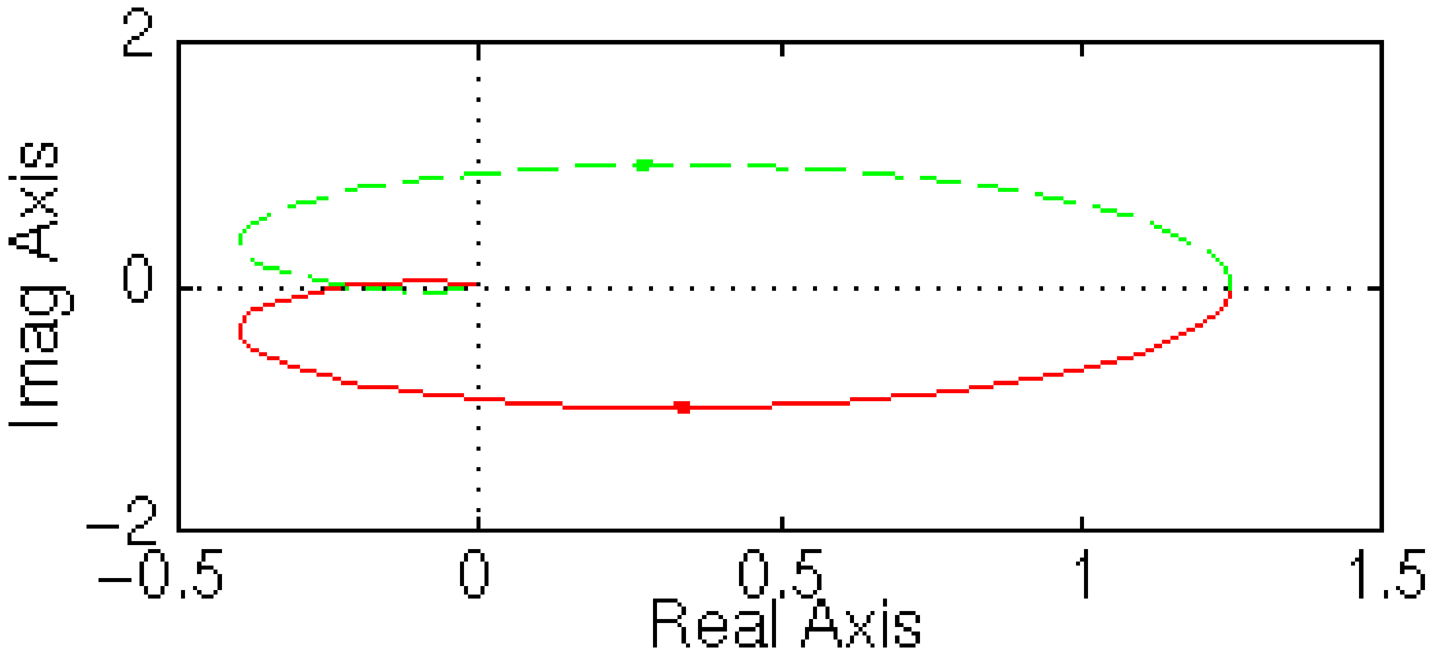
The open-loop system represented by this plot will become unstable in closed loop if the gain is increased past a certain boundary.



The Nyquist Stability Criterion

and that the Nyquist diagram can be viewed by typing:

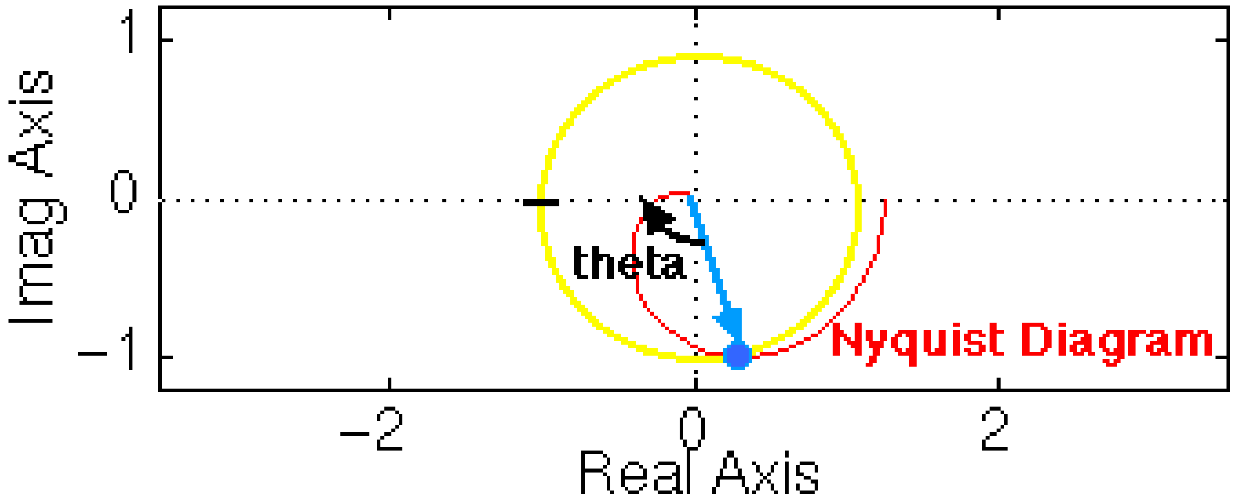
nyquist (50, [1 9 30 40])



Gain and Phase Margin

Phase margin as the change in open-loop phase shift required at unity gain to make a closed-loop system unstable.

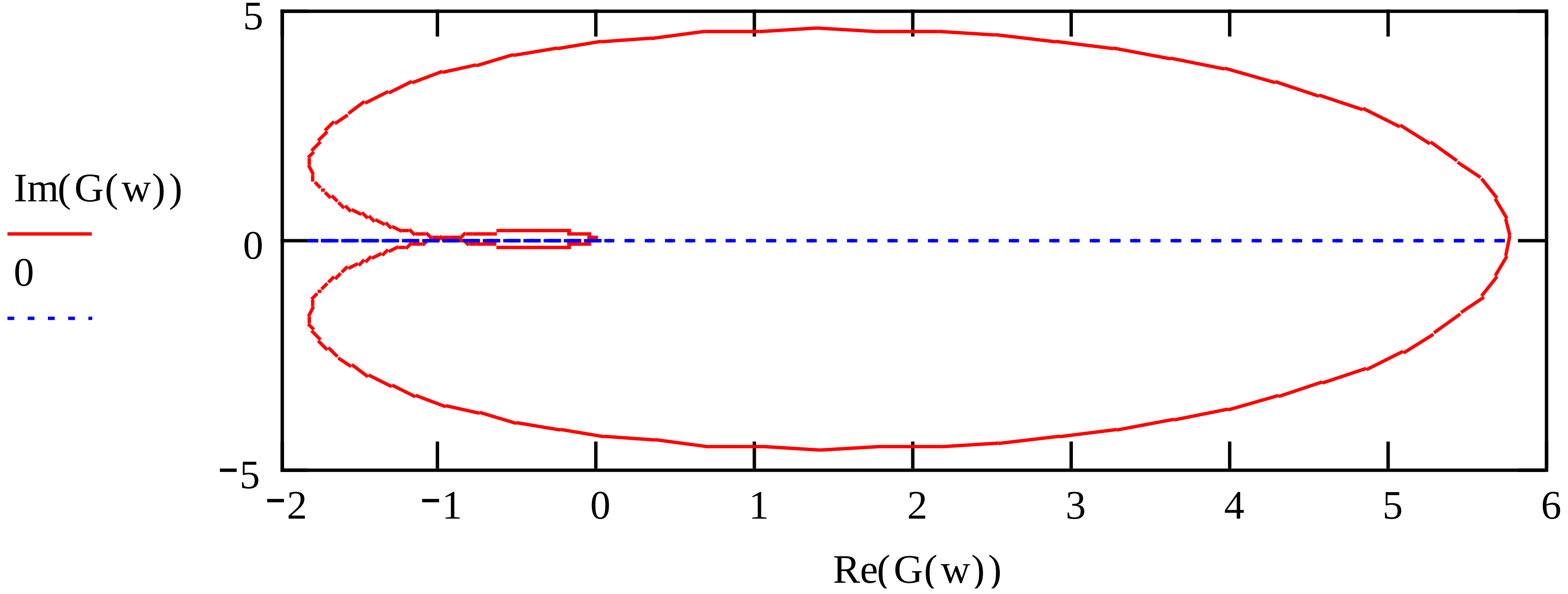
From our previous example we know that this particular system will be unstable in closed loop if the Nyquist diagram encircles the -1 point. However, we must also realize that if the diagram is shifted by theta degrees, it will then touch the -1 point at the negative real axis, making the system marginally stable in closed loop. Therefore, the angle required to make this system marginally stable in closed loop is called the phase margin (measured in degrees). In order to find the point we measure this angle from, we draw a circle with radius of 1, find the point in the Nyquist diagram with a magnitude of 1 (gain of zero dB), and measure the phase shift needed for this point to be at an angle of 180 deg.



The Nyquist Stability Criterion

$w := -100, -99.9 .. 100 \qquad j := \sqrt{-1} \qquad s(w) := j \cdot w \qquad f(w) := -1$

$$G(w) := \frac{50 \cdot 4.6}{s(w)^3 + 9 \cdot s(w)^2 + 30 \cdot s(w) + 40}$$



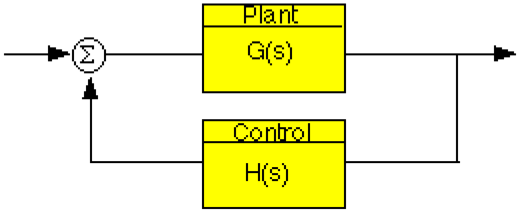
Consider the Negative Feedback System

Remember from the Cauchy criterion that the number N of times that the plot of G(s)H(s) encircles -1 is equal to the number Z of zeros of 1 + G(s)H(s) enclosed by the frequency contour minus the number P of poles of 1 + G(s)H(s) enclosed by the frequency contour (N = Z - P).

Keeping careful track of open- and closed-loop transfer functions, as well as numerators and denominators, you should convince yourself that:

- the zeros of 1 + G(s)H(s) are the poles of the closed-loop transfer function
- the poles of 1 + G(s)H(s) are the poles of the open-loop transfer function.

The Nyquist criterion then states that:



- P = the number of open-loop (unstable) poles of G(s)H(s)
- N = the number of times the Nyquist diagram encircles -1
- clockwise encirclements of -1 count as positive encirclements
- counter-clockwise (or anti-clockwise) encirclements of -1 count as negative encirclements
- Z = the number of right half-plane (positive, real) poles of the closed-loop system

The important equation which relates these three quantities is:

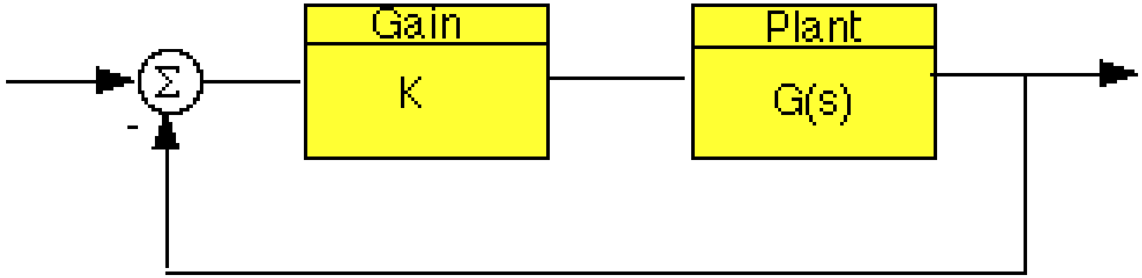
Z = P + N

The Nyquist Stability Criterion - Application

Knowing the number of right-half plane (unstable) poles in open loop (P), and the number of encirclements of -1 made by the Nyquist diagram (N), we can determine the closed-loop stability of the system.

If Z = P + N is a positive, nonzero number, the closed-loop system is unstable.

We can also use the Nyquist diagram to find the range of gains for a closed-loop unity feedback system to be stable. The system we will test looks like this:



where G(s) is :

$$\frac{s^2 + 10s + 24}{s^2 - 8s + 15}$$

The Nyquist Stability Criterion

This system has a gain K which can be varied in order to modify the response of the closed-loop system. However, we will see that we can only vary this gain within certain limits, since we have to make sure that our closed-loop system will be stable. This is what we will be looking for: the range of gains that will make this system stable in the closed loop.

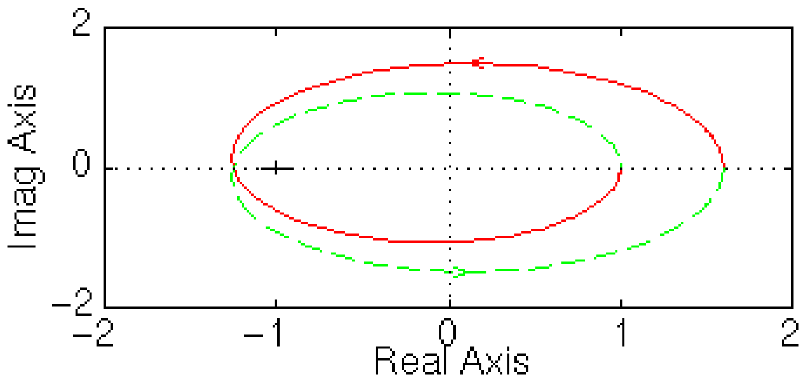
The first thing we need to do is find the number of positive real poles in our open-loop transfer function:

```
roots([1 -8 15])
        ans =
        5
        3
```

The poles of the open-loop transfer function are both positive. Therefore, we need two anti-clockwise (N = -2) encirclements of the Nyquist diagram in order to have a stable closed-loop system (Z = P + N). If the number of encirclements is less than two or the encirclements are not anti-clockwise, our system will be unstable.

Let's look at our Nyquist diagram for a gain of 1:

nyquist([1 10 24], [1 -8 15])



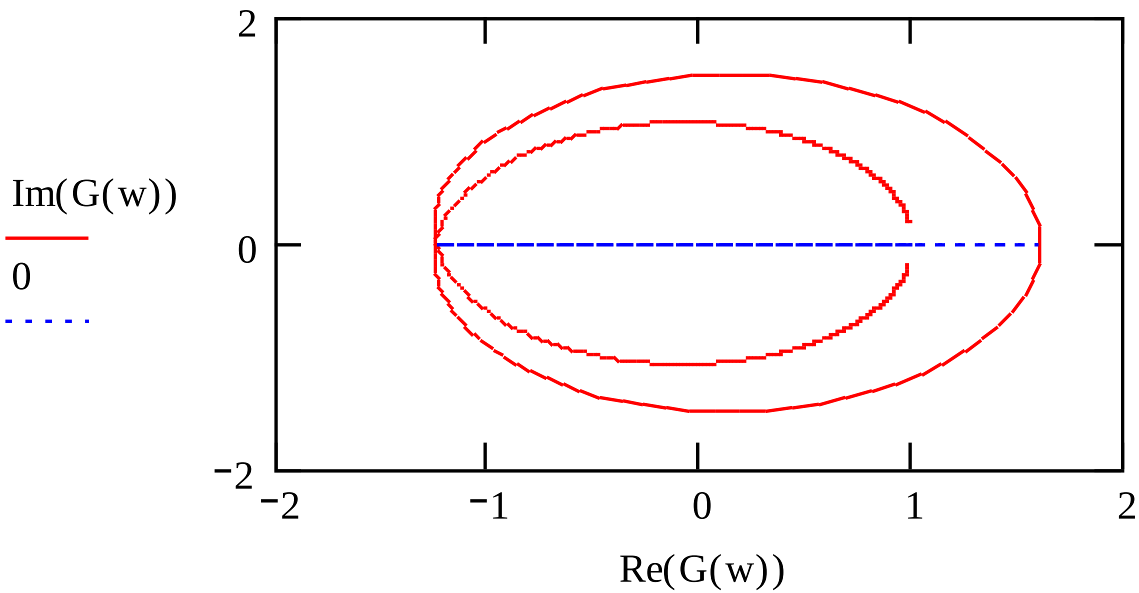
There are two anti-clockwise encirclements of -1. Therefore, the system is stable for a gain of 1.

The Nyquist Stability Criterion

MathCAD Implementation

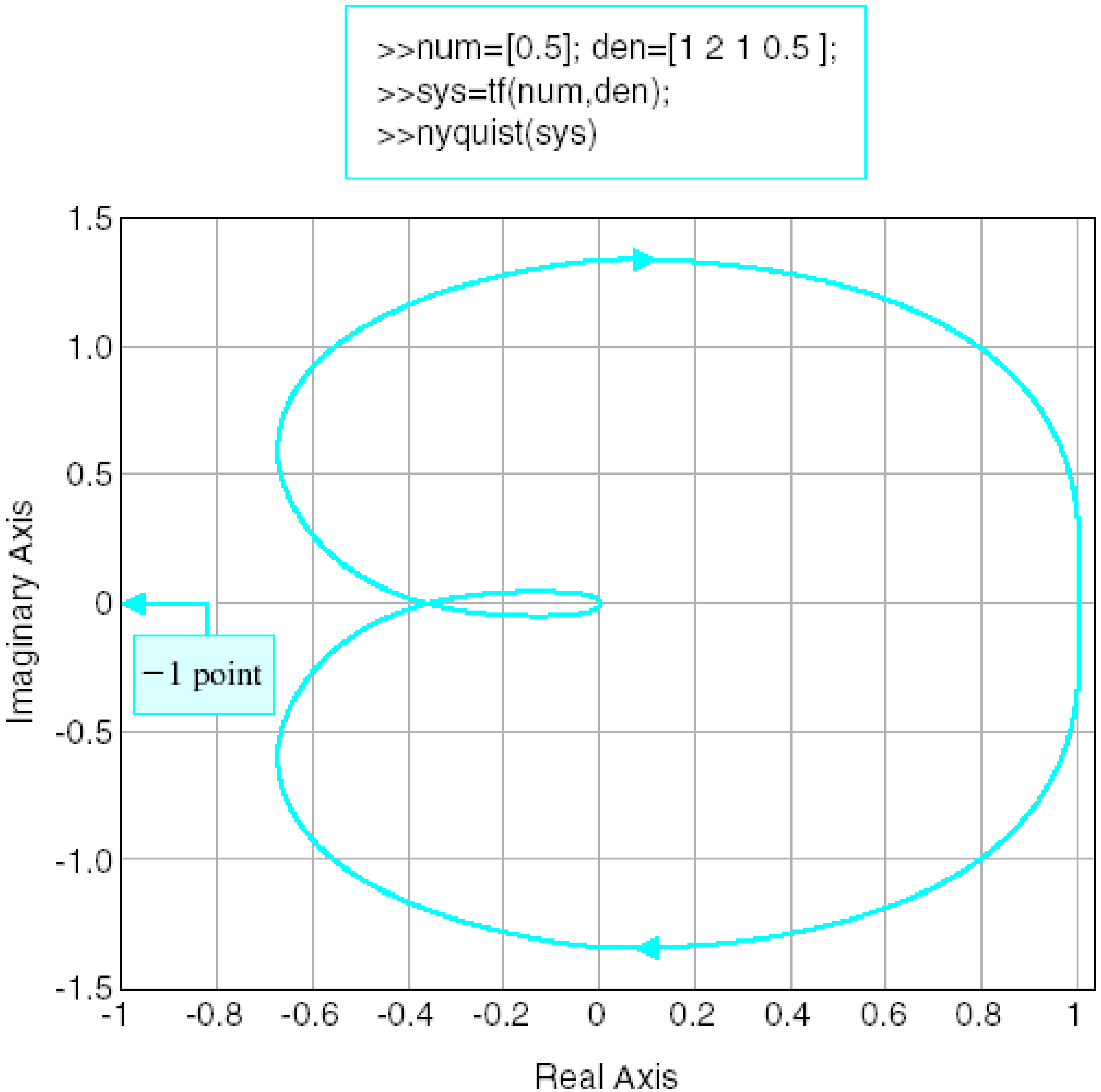
$w := -100, -99.9 .. 100$ $\qquad j := \sqrt{-1}$ $\qquad s(w) := j \cdot w$

$$G(w) := \frac{s(w)^2 + 10 \cdot s(w) + 24}{s(w)^2 - 8 \cdot s(w) + 15}$$

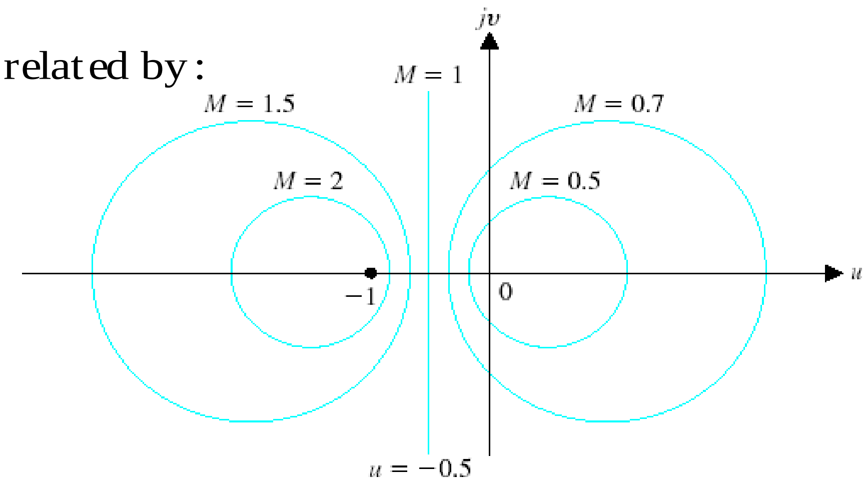


There are two anti-clockwise encirclements of -1.
Therefore, the system is stable for a gain of 1.

The Nyquist Stability Criterion



Time-Domain Performance Criteria Specified In The Frequency Domain



Open and closed-loop frequency responses are related by:

$$T(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)}$$

$$M_{pw} = \frac{1}{2 \cdot \zeta \cdot \sqrt{1 - \zeta^2}} \qquad \zeta < 0.707$$

$$G(\omega) = u + j \cdot v \qquad M = M(\omega)$$

$$M(\omega) = \left| \frac{G(j\omega)}{1 + G(j\omega)} \right| = \left| \frac{u + jv}{1 + u + jv} \right| = \frac{\sqrt{u^2 + v^2}}{\sqrt{(1 + u)^2 + v^2}}$$

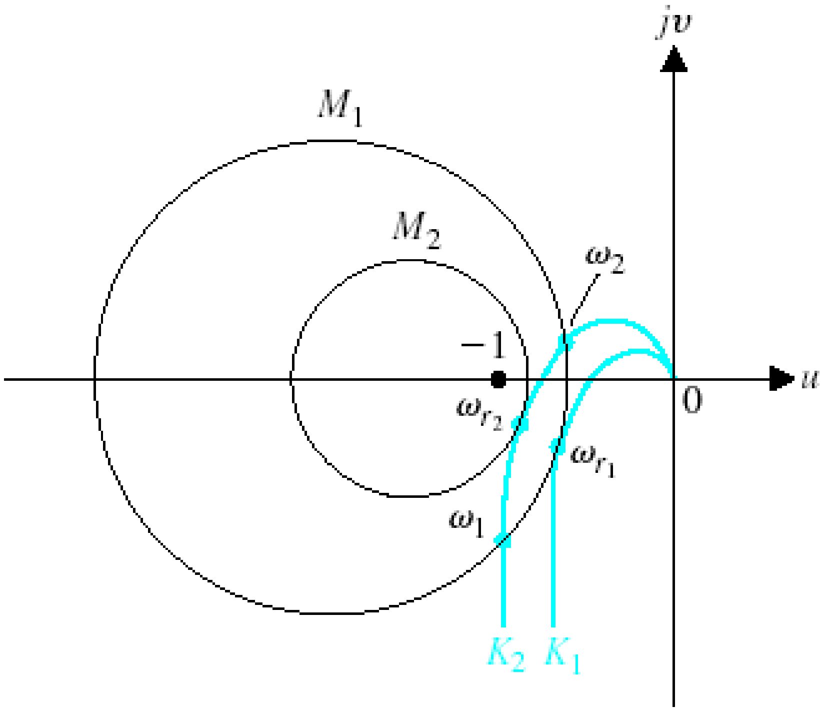
Squaring and rearrenging

$$\left(u - \frac{M^2}{1 - M^2} \right)^2 + v^2 = \left(\frac{M}{1 - M^2} \right)^2$$

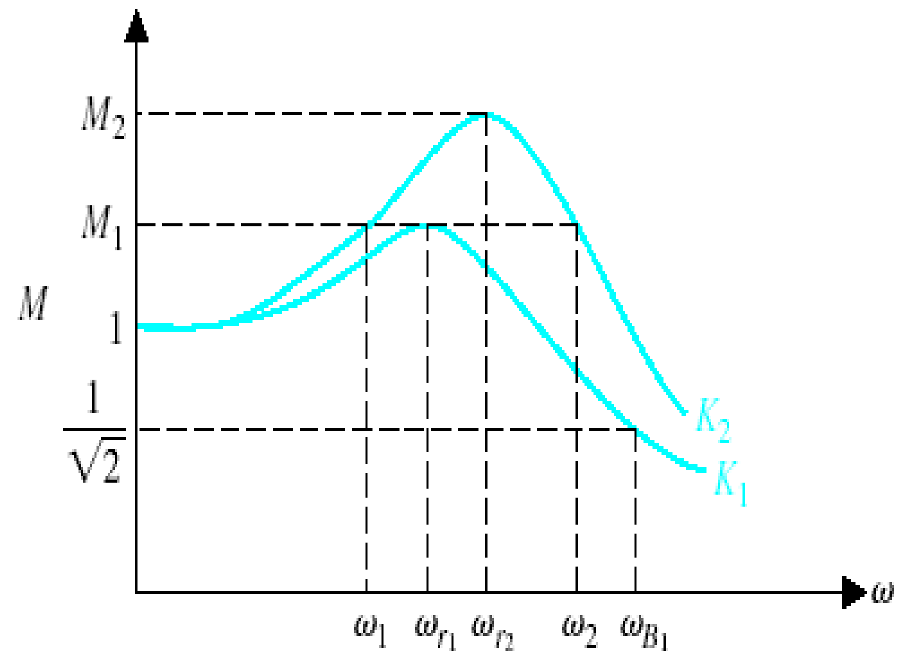
which is the equation of a circle on u-v planwe with a center at

$$u = \frac{M^2}{1 - M^2} \qquad v = 0$$

Time-Domain Performance Criteria Specified In The Frequency Domain



Polar plot of $G(j\omega)$ for two values of a gain ($K_2 > K_1$).



Closed-loop frequency response of $T(j\omega) = G(j\omega)/1 + G(j\omega)$. Note that $K_2 > K_1$.

The Nichols Stability Method

Polar Stability Plot - Nichols Mathcad Implementation

This example makes a polar plot of a transfer function and draws one contour of constant closed-loop magnitude. To draw the plot, enter a definition for the transfer function **G(s)**:

$$G(s) := \frac{45000}{s\cdot(s+2)\cdot(s+30)}$$

The frequency range defined by the next two equations provides a logarithmic frequency scale running from 1 to 100. You can change this range by editing the definitions for **m** and $\boldsymbol{\omega_m}$:

$$m := 0..100 \qquad \omega_m := 10^{.02\cdot m}$$

Now enter a value for **M** to define the closed-loop magnitude contour that will be plotted.

$$M := 1.1$$

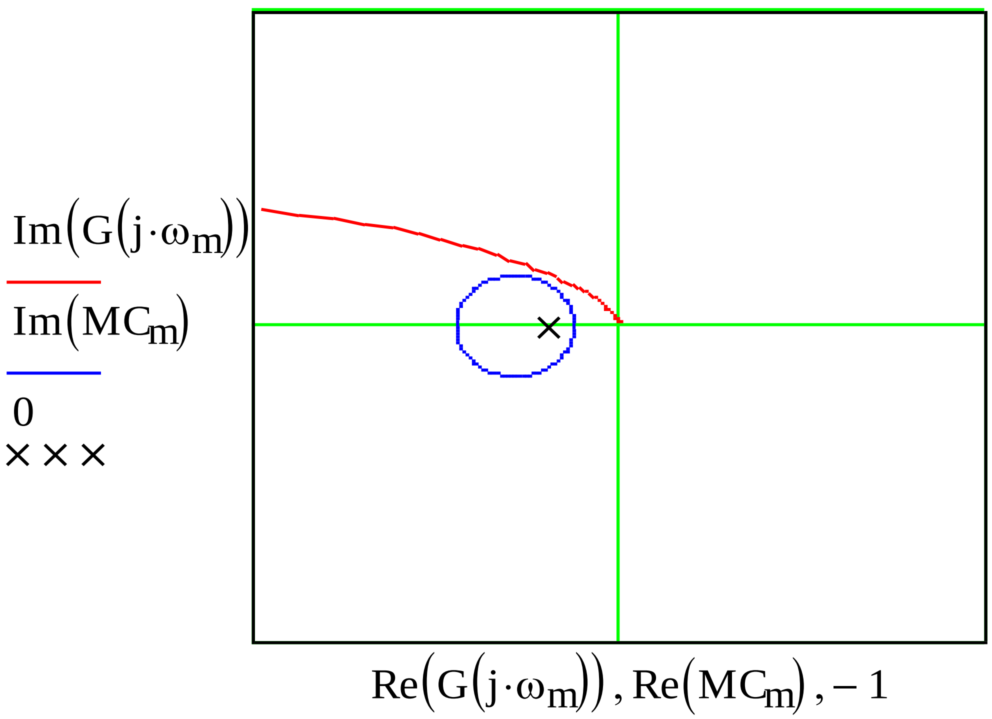
Calculate the points on the M-circle:

$$MC_m := \left(\frac{-M^2}{M^2-1} + \left|\frac{M}{M^2-1}\right|\cdot\exp(2\cdot\pi\cdot j\cdot .01\cdot m)\right)$$

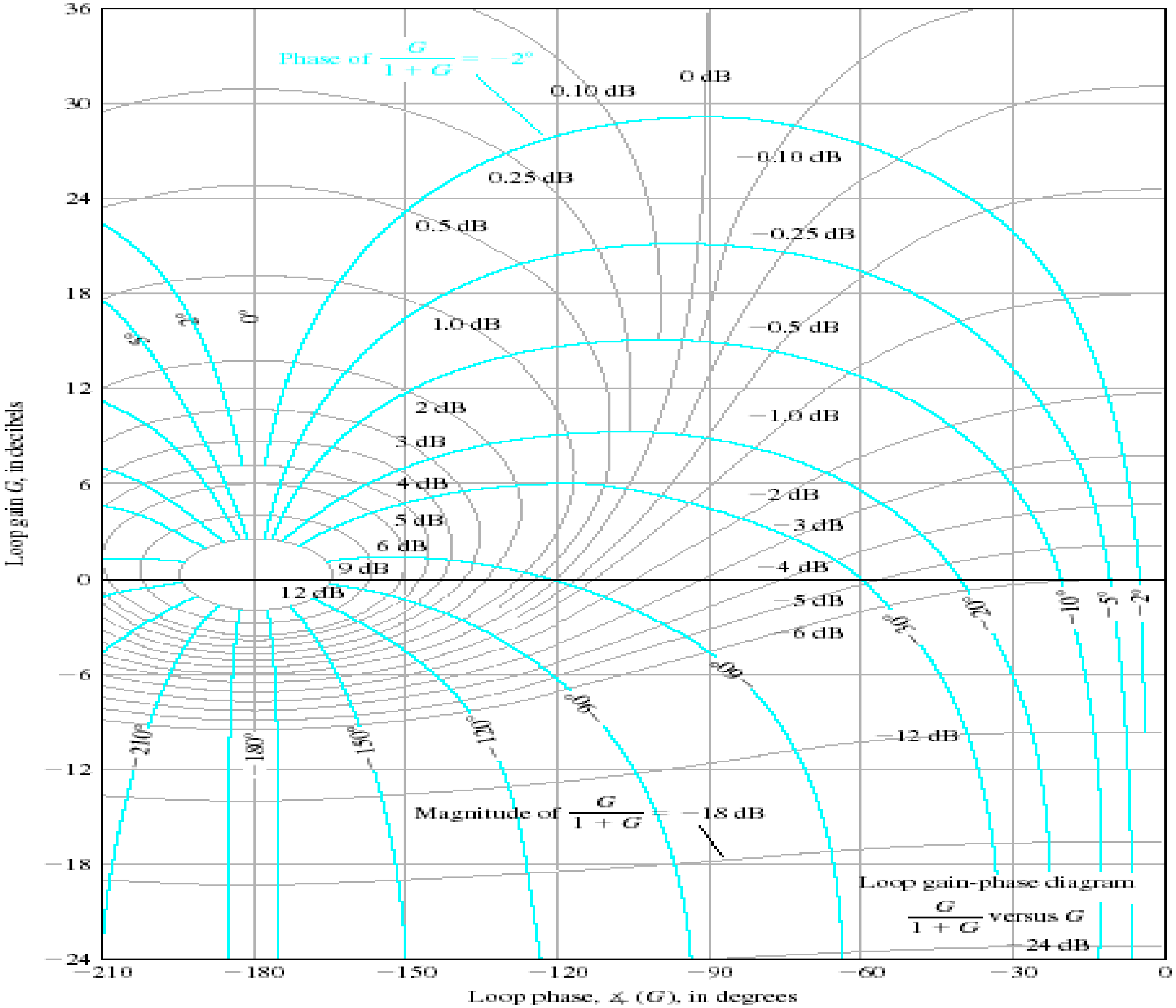
The first plot shows **G**, the contour of constant closed-loop magnitude, **M**

The Nichols Stability Method

The first plot shows **G**, the contour of constant closed-loop magnitude **M**, and the Nyquist of the open loop system



The Nichols Stability Method



Nichols chart. The phase curves for the closed-loop system are shown in color.

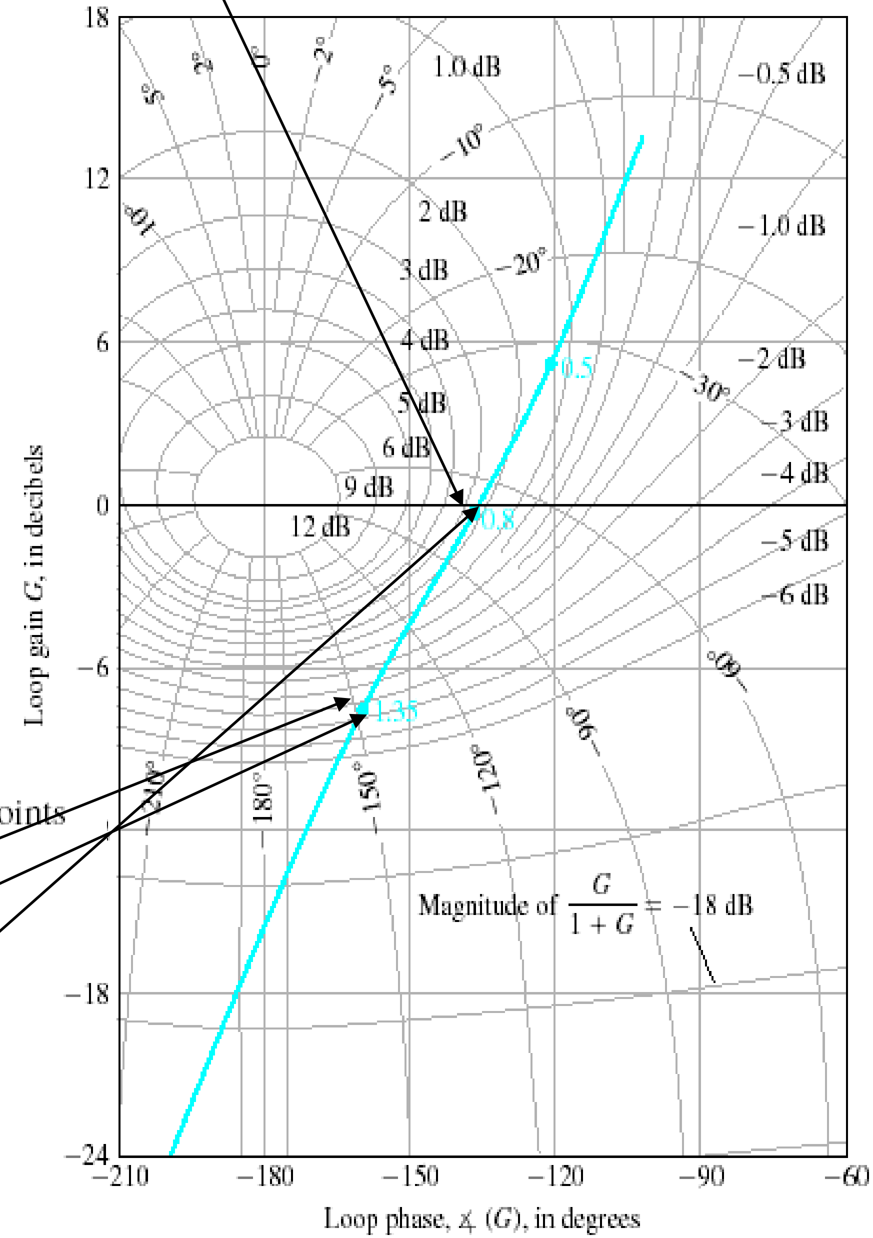
The Nichols Stability Method

$$G(\omega) := \frac{1}{j\cdot\omega\cdot(j\cdot\omega + 1)\cdot(0.2\cdot j\cdot\omega + 1)}$$

$M_{pw} := 2.5$ dB $\qquad \omega_r := 0.8$

The closed-loop phase angle at ωr is equal to -72 degrees and ωb = 1.33
The closed-loop phase angle at ωb is equal to -142 degrees

Mpw



Nichols diagram for $G(j\omega) = 1/j\omega(j\omega + 1)(0.2j\omega + 1)$. Three points on curve are shown for ω = 0.5, 0.8, and 1.35, respectively.

-3dB

-72 deg wr=0.8

-142 deg

The Nichols Stability Method

$$G(\omega) := \frac{0.64}{j\cdot\omega\cdot\left[(j\cdot\omega)^2 + j\cdot\omega + 1\right]}$$

Phase Margin = 30 degrees

On the basis of the phase we estimate $\zeta := 0.30$

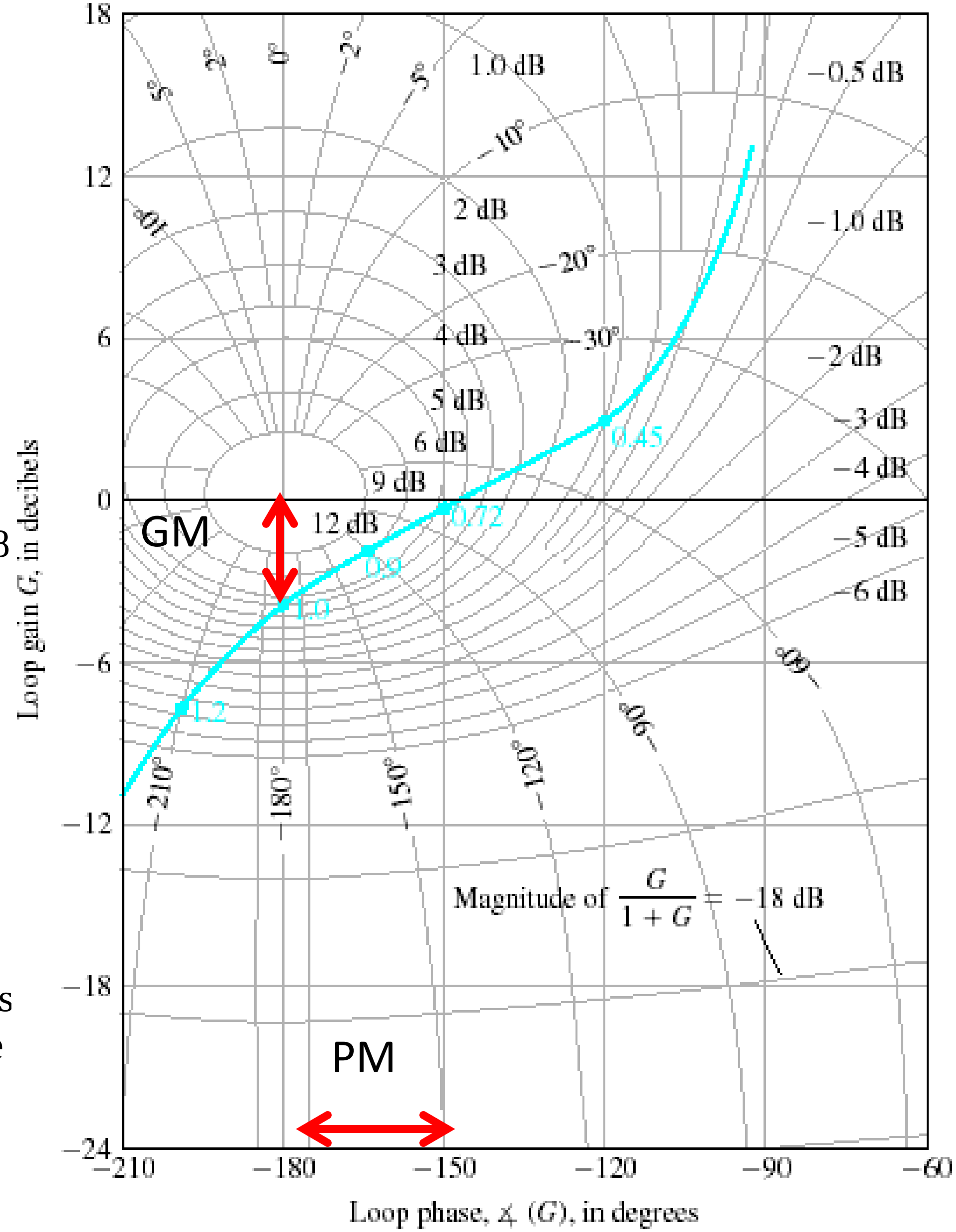
$M_{pw} := 9$ dB $\quad M_{pw} := 2.8 \quad \omega_r := 0.88$

From equation

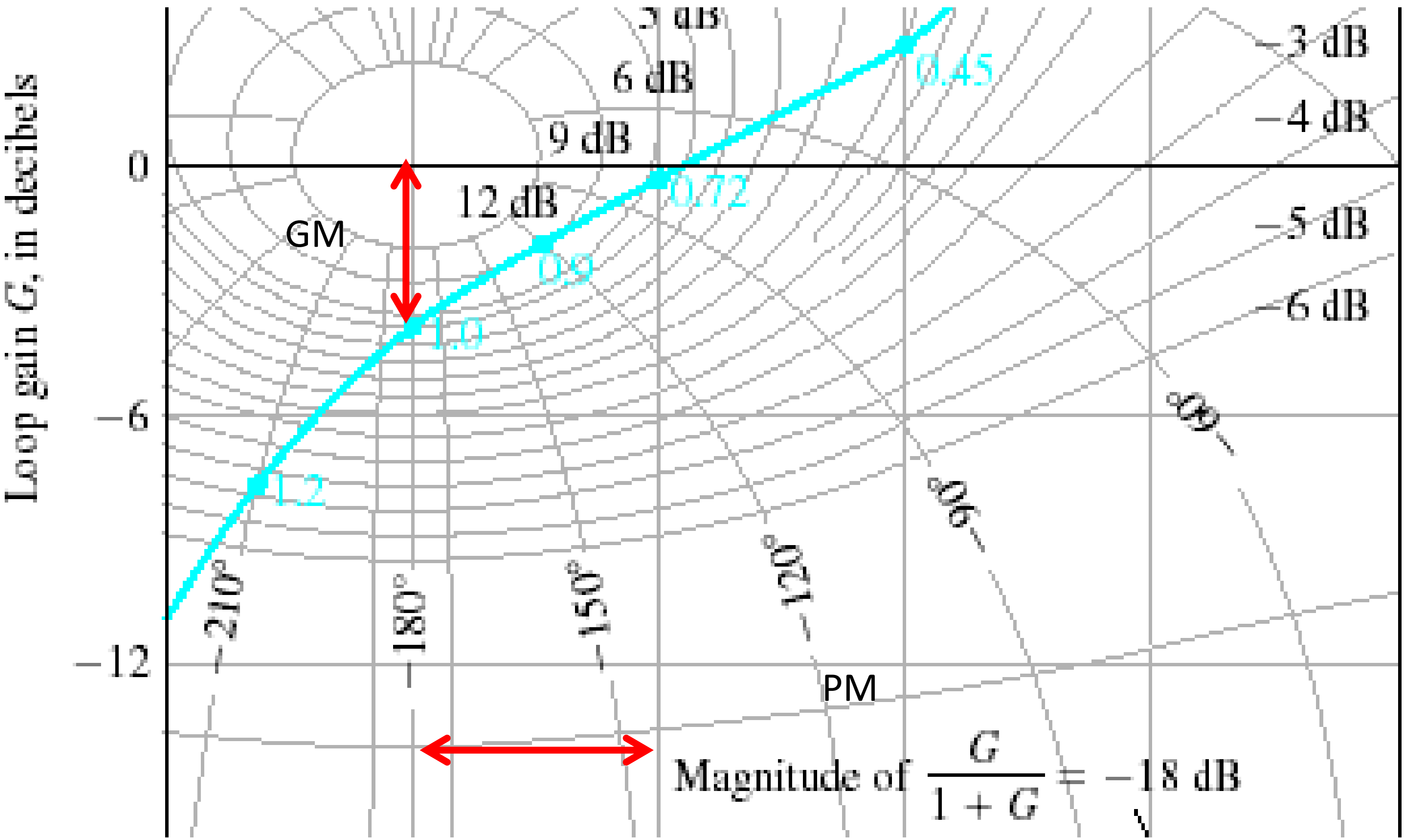
$$M_{pw} = \frac{1}{2\cdot\zeta\cdot\sqrt{1-\zeta^2}} \qquad \zeta := 0.18$$

We are confronted with comflecting ζs

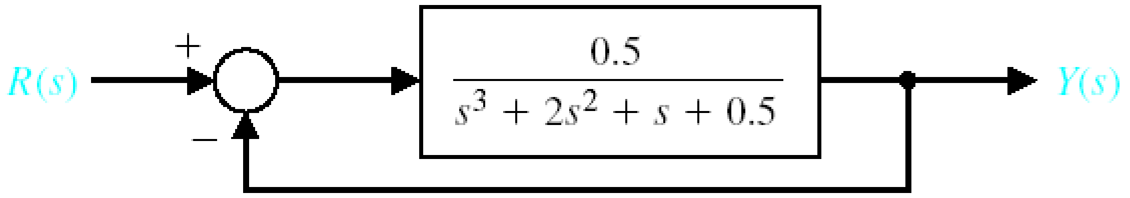
The apparent conflict is caused by the nature of G(jω) which slopes rapidally toward 180 degrees line from the 0-dB axis. The designer must use the frequency-domain-time-domain correlation with caution



The Nichols Stability Method

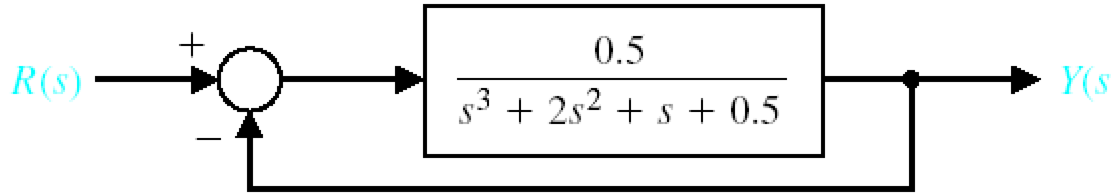


Examples – Bode and Nyquist



A closed-loop control system example for Nyquist and Bode with relative stability.

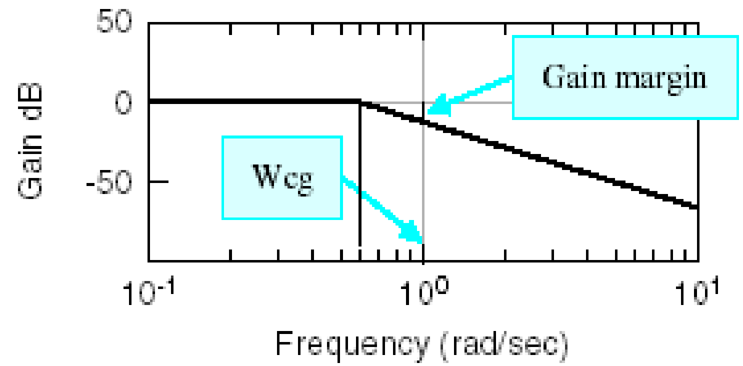
Examples - Bode




```
[mag,phase,w]=bode(sys);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
```

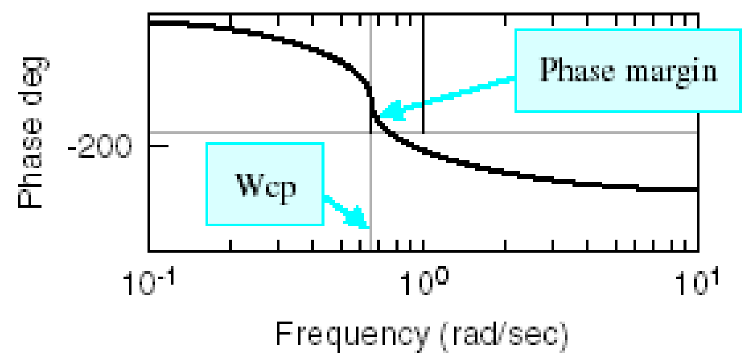
or

```
[Gm,Pm,Wcg,Wcp]=margin(sys);
```



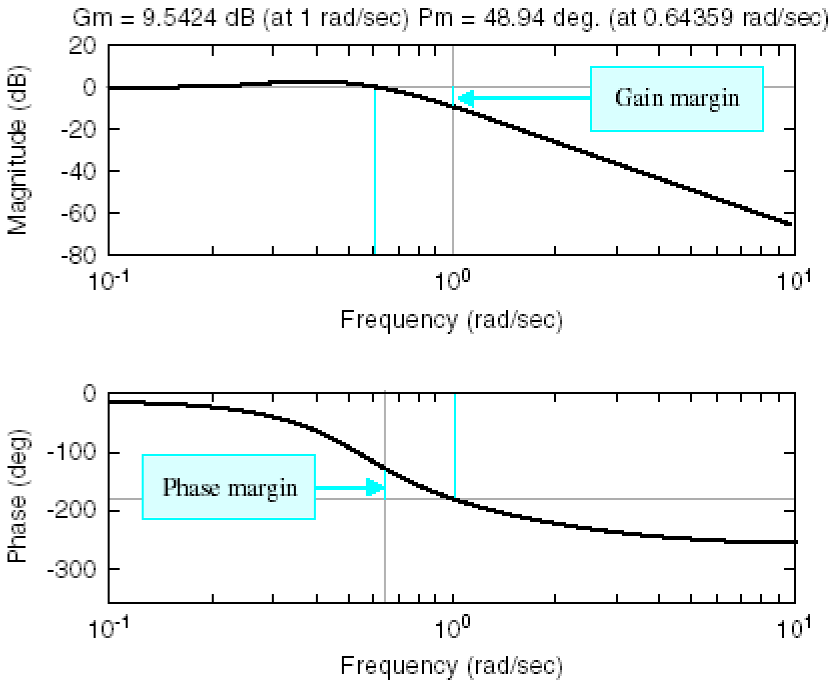
Example

```
num=[0.5]; den=[1 2 1 0.5];
sys=tf(num,den);
margin(sys);
```



Gm = gain margin (dB)
Pm = phase margin (deg)
Wcg = freq. for phase = −180
Wcp = freq. for gain = 0 dB

Examples - Bode

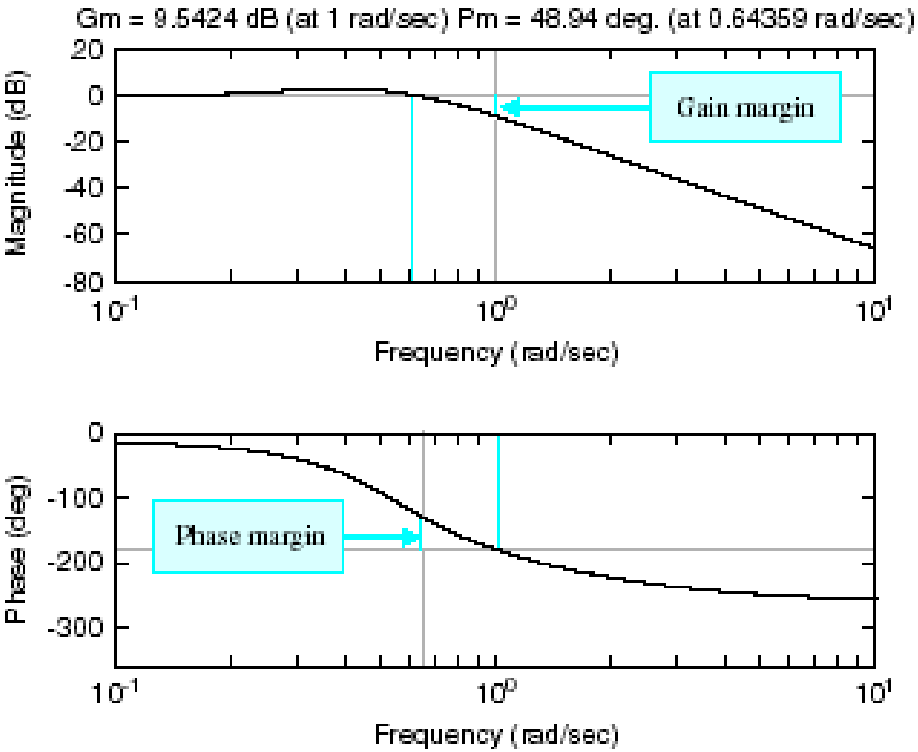


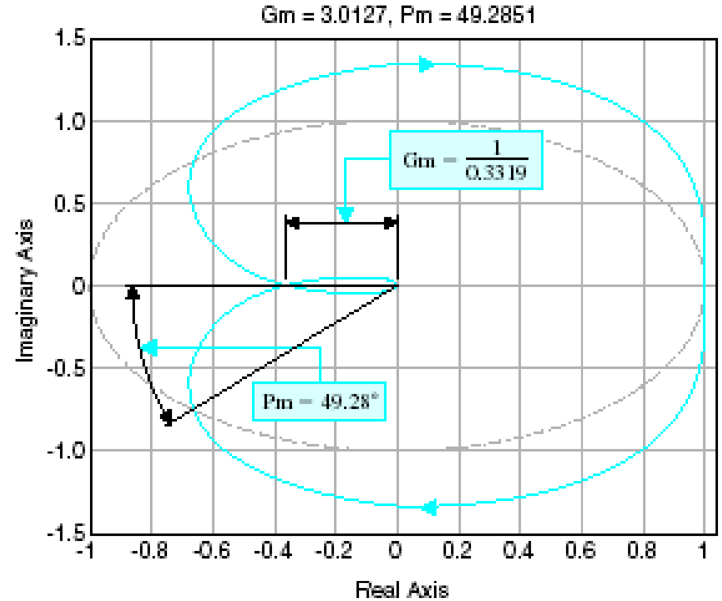

```
num=[0.5];
den=[1 2 1 0.5 ];
sys=tf(num,den);
%
w=logspace(-1,1,200);
%
[mag,phase,w]=bode(sys,w);
%
margin(mag,phase,w);
```

Open-loop system

Specify frequency range

Examples – Bode and Nyquist

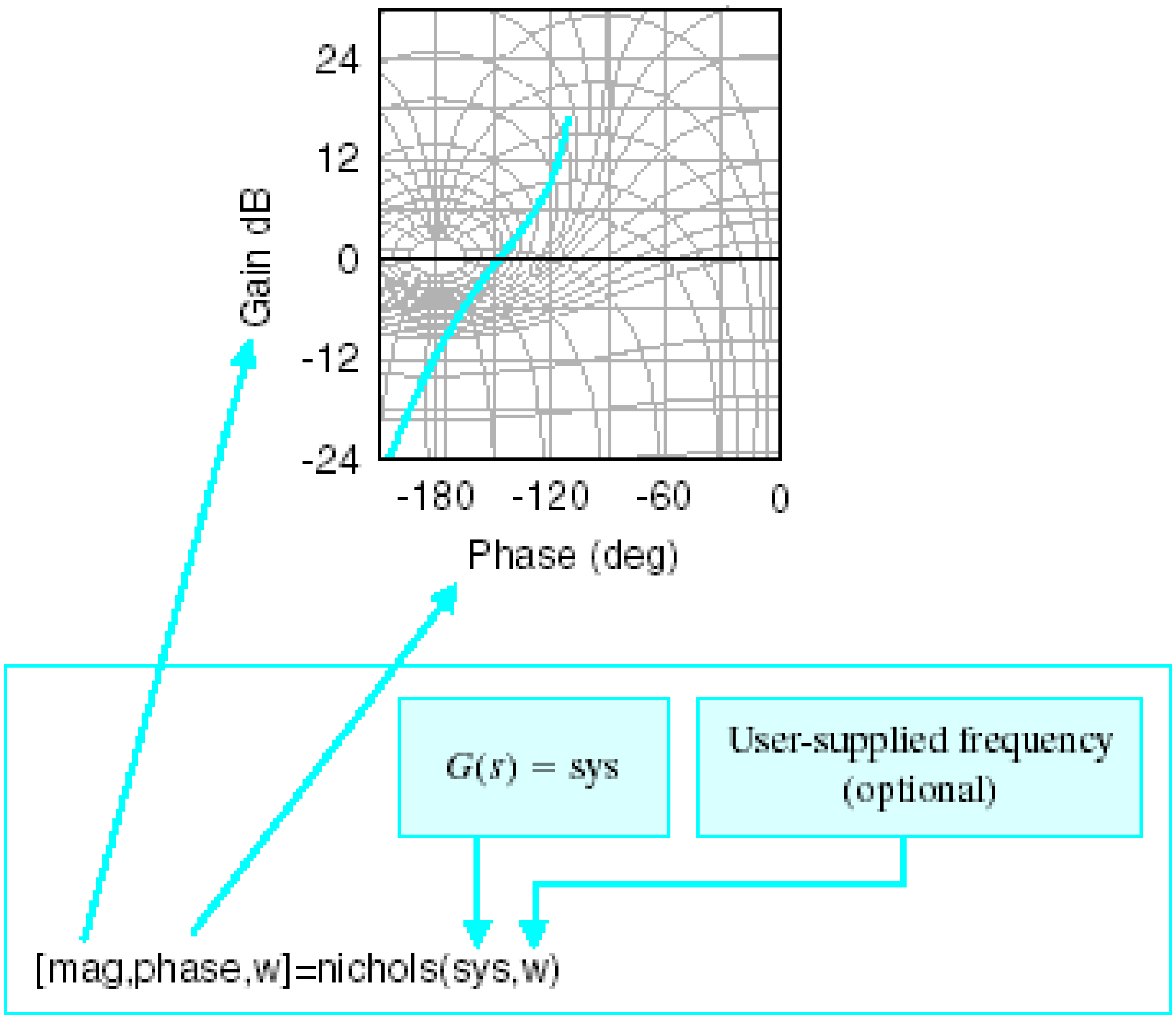




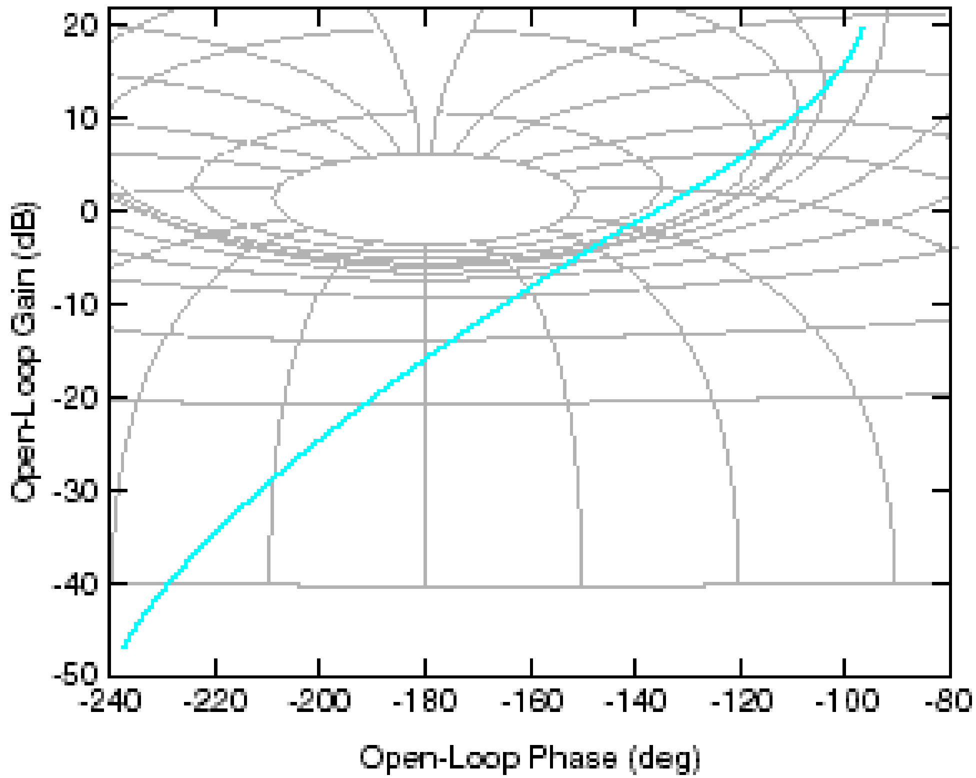

```
num=[0.5];          <- Open-loop system
den=[1 2 1 0.5 ];
sys=tf(num,den);
%                   <- Specify frequency range
w=logspace(-1,1,200);
%
[mag,phase,w]=bode(sys,w);
%
margin(mag,phase,w);
```

```
%   The Nyquist plot of
%
%                    0.5
%   G(s) = -----------------------------
%          s^3 + 2 s^2 + s + 0.5
%
%   with gain and phase margin calculation.
%
num=[0.5]; den=[1 2 1 0.5]; sys=tf(num,den);
%
[mag,phase,w]=bode(sys);          <- Compute gain and phase margins.
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
%
nyquist(sys);                     <- Nyquist plot
title(['Gm = ',num2str(gm),' Pm = ',num2str(Pm)])   <- Label gain and phase margins on plot.
```

Examples - Nichols



Examples - Nichols




```
num=[1]; den=[0.2 1.2 1 0 ];
sys=tf(num,den);
w=logspace(-1,1,400);
nichols(sys,w);
ngrid
```

Set up to generate Fig. 9.27

Plot Nichols chart and add grid lines.

The Design of Feedback Control Systems

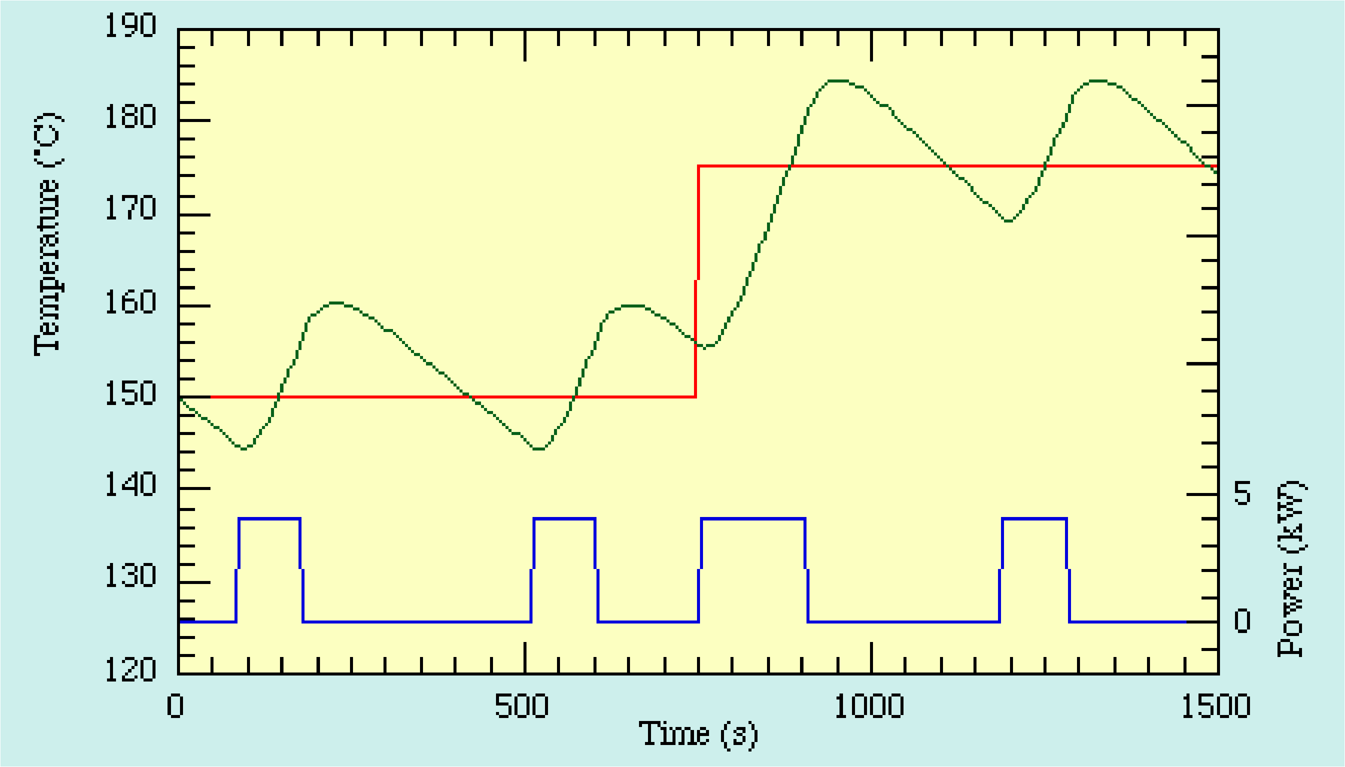
PID

Compensation Networks

Different Types of Feedback Control

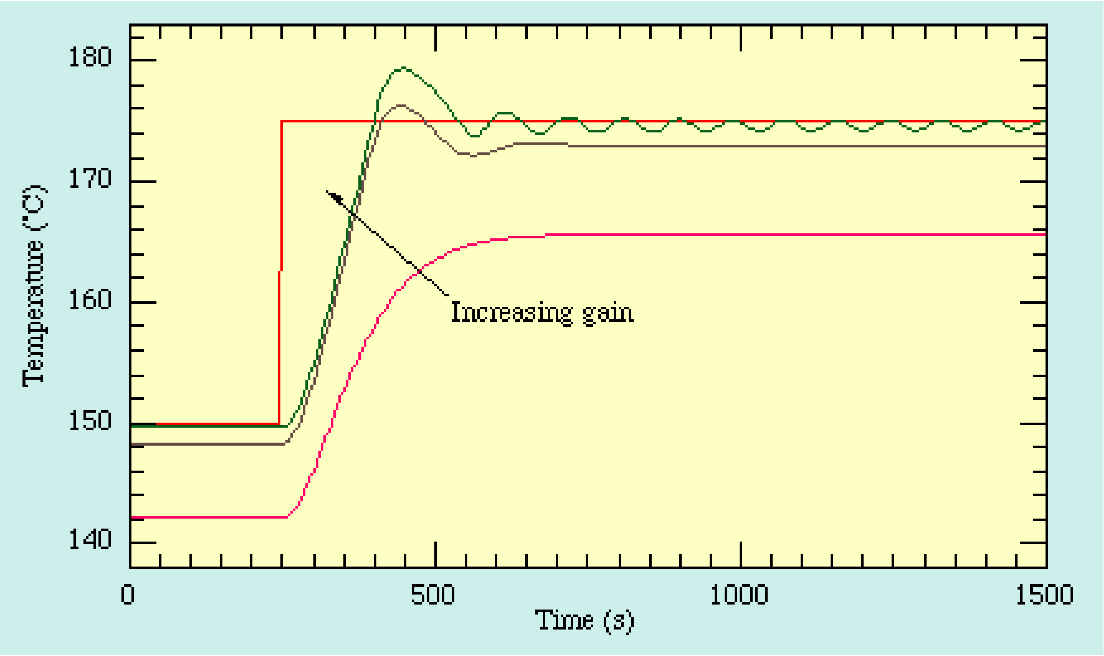
On-Off Control

This is the simplest form of control.



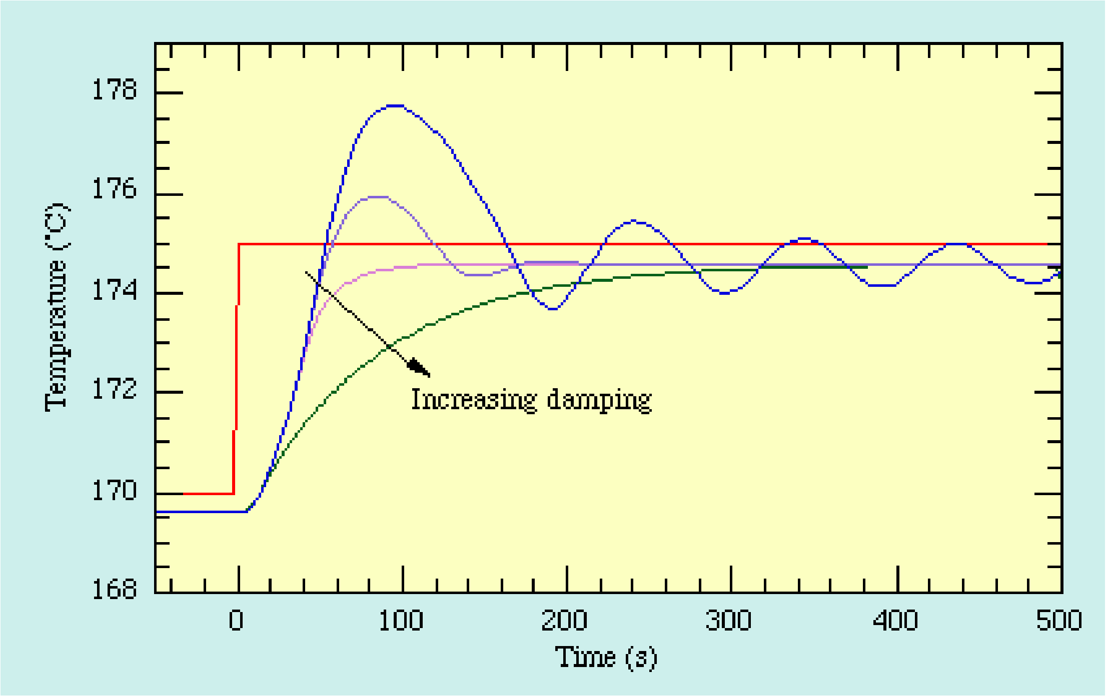
Proportional Control

A proportional controller attempts to perform better than the On-off type by applying power in proportion to the difference in temperature between the measured and the set-point. As the gain is increased the system responds faster to changes in set-point but becomes progressively underdamped and eventually unstable. The final temperature lies below the set-point for this system because some difference is required to keep the heater supplying power.



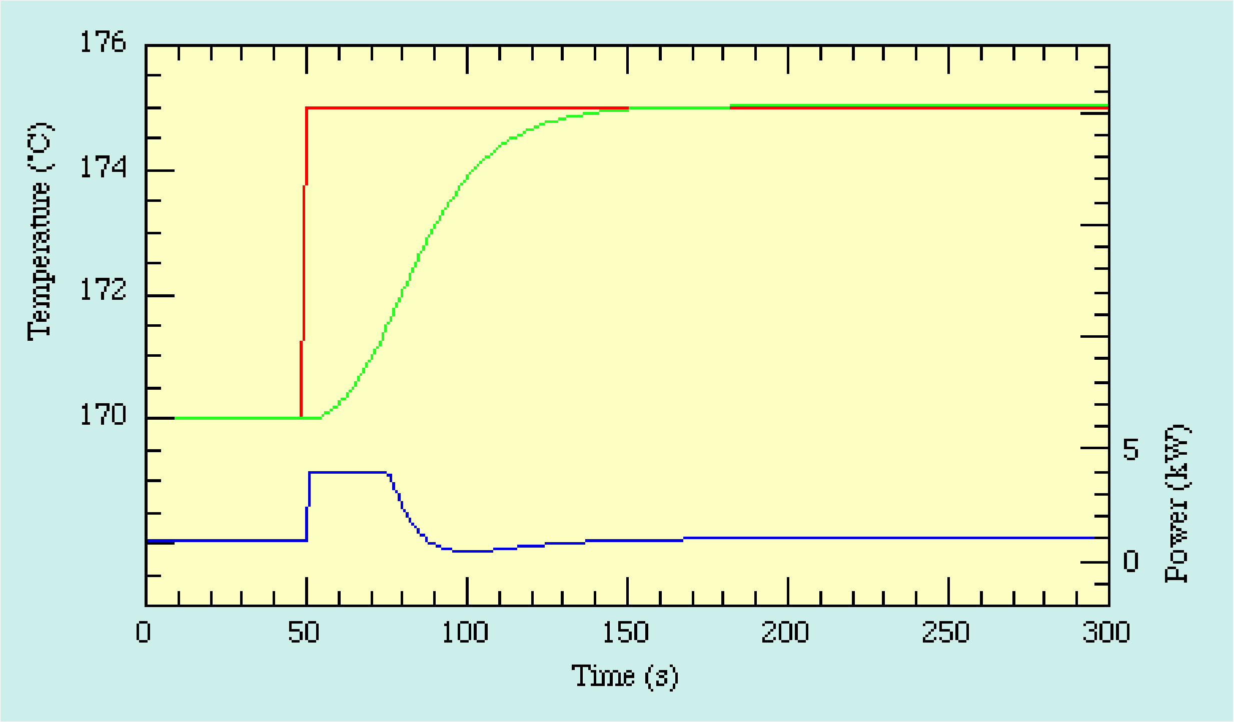
Proportional, Derivative Control

The stability and overshoot problems that arise when a proportional controller is used at high gain can be mitigated by adding a term proportional to the time-derivative of the error signal. The value of the damping can be adjusted to achieve a critically damped response.



Proportional+Integral+Derivative Control

Although PD control deals neatly with the overshoot and ringing problems associated with proportional control it does not cure the problem with the steady-state error. Fortunately it is possible to eliminate this while using relatively low gain by adding an integral term to the control function which becomes



The Characteristics of P, I, and D controllers

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.

An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Proportional Control

By only employing proportional control, a steady state error occurs.

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control

All design specifications can be reached.

The Characteristics of P, I, and D controllers

CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
Kp ↑	Decrease	Increase	Small Change	Decrease
Ki	Decrease	Increase	Increase	Eliminate
Kd	Small Change	Decrease	Decrease	Small Change

Tips for Designing a PID Controller

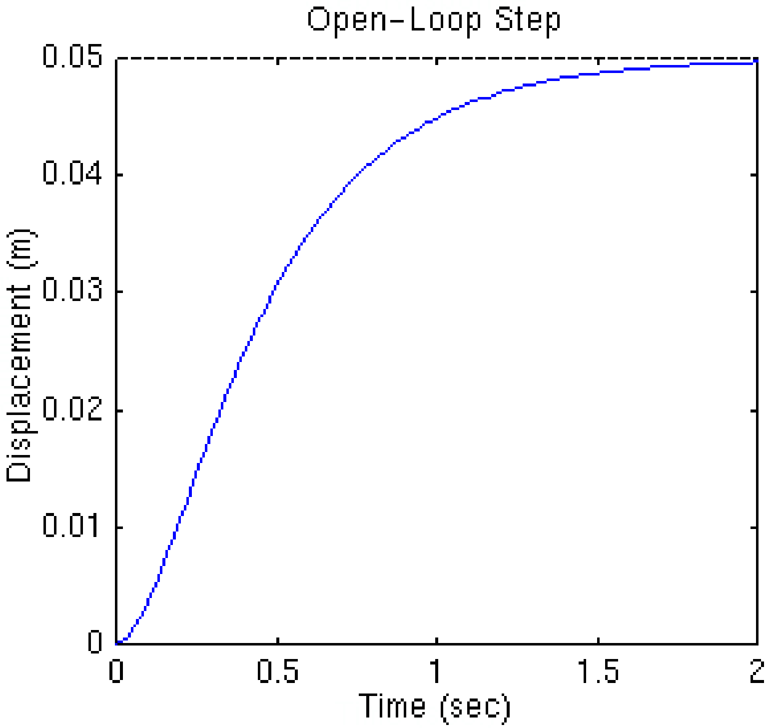
1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

Open-Loop Control - Example

$$G(s) = \frac{1}{s^2 + 10s + 20}$$

```
num=1;
den=[1 10 20];
step(num,den)
```



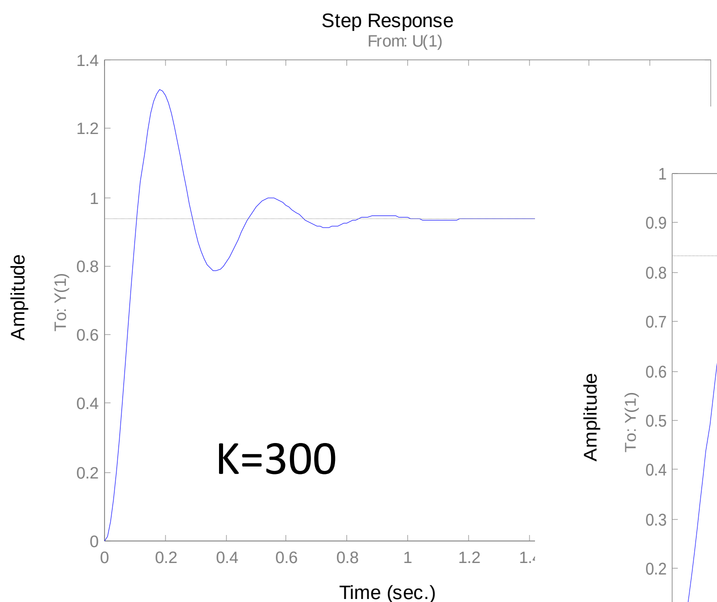
Proportional Control - Example

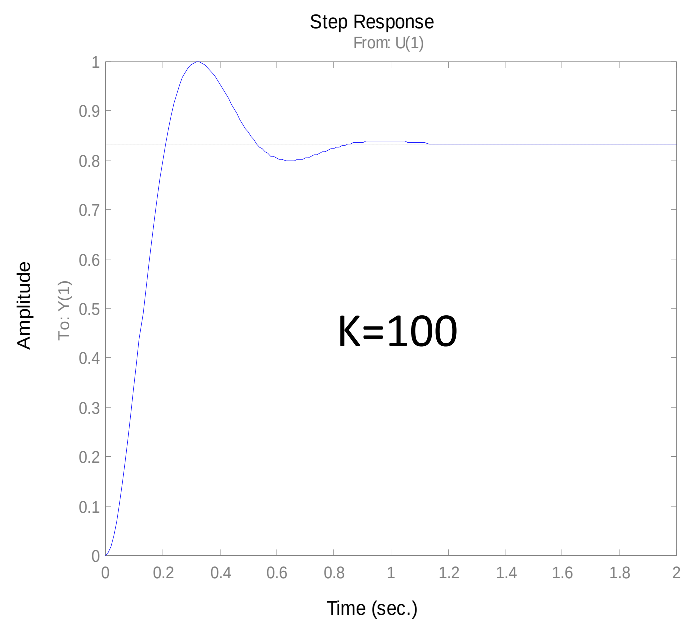
The proportional controller (Kp) reduces the rise time, increases the overshoot, and reduces the steady-state error.

MATLAB Example

$$T(s) = \frac{Kp}{s^2 + 10 \cdot s + (20 + Kp)}$$

```
Kp=300;

num=[Kp];

den=[1 10 20+Kp];

t=0:0.01:2;

step(num,den,t)
```





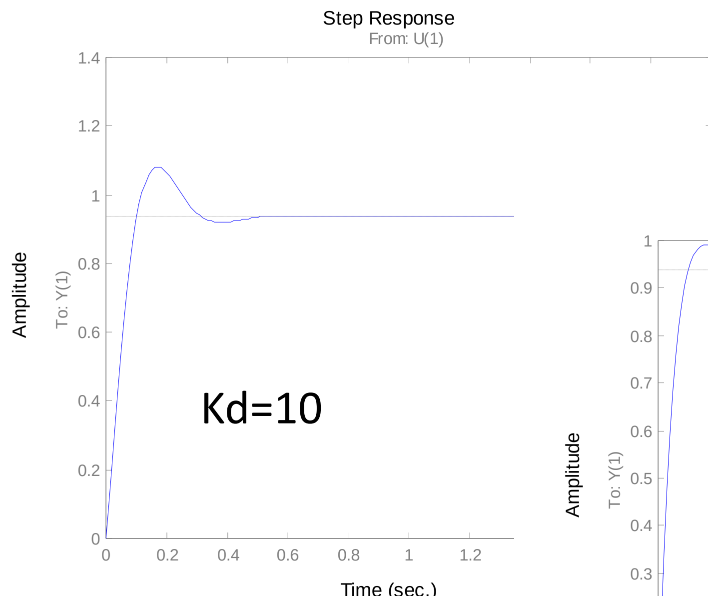
Proportional - Derivative - Example

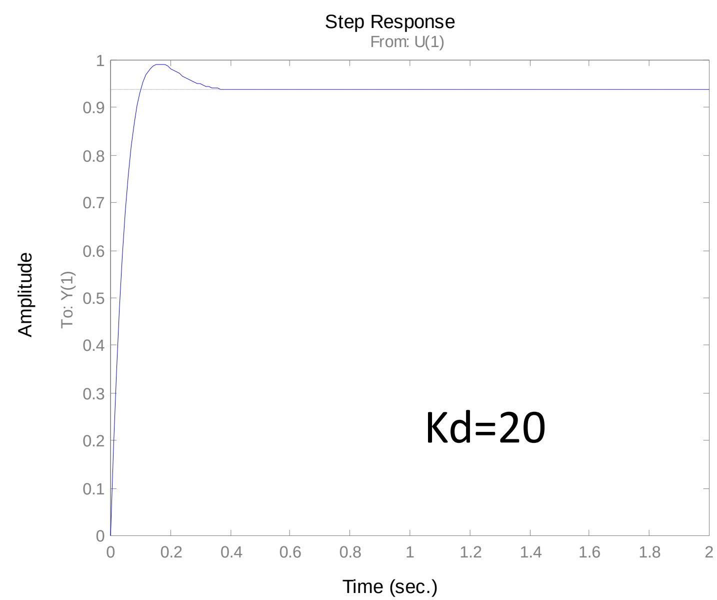
The derivative controller (Kd) reduces both the overshoot and the settling time.

MATLAB Example

$$T(s) = \frac{Kd \cdot s + Kp}{s^2 + (10 + Kd) \cdot s + (20 + Kp)}$$

```
Kp=300;
Kd=10;
num=[Kd Kp];
den=[1 10+Kd 20+Kp];
t=0:0.01:2;
step(num,den,t)
```





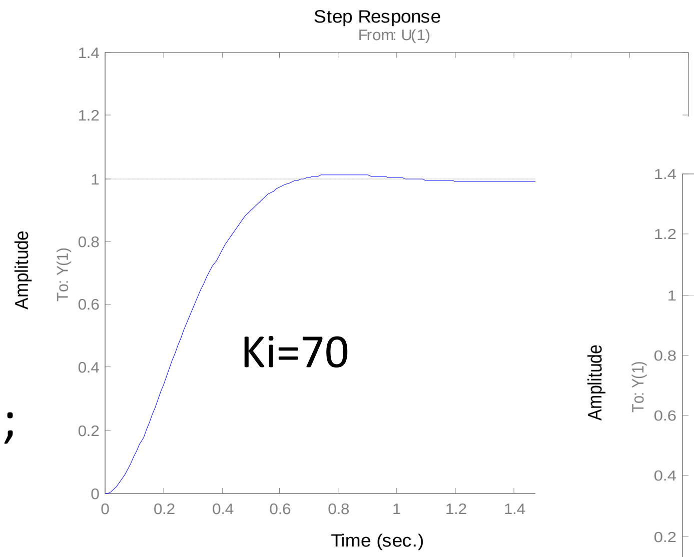
Proportional - Integral - Example

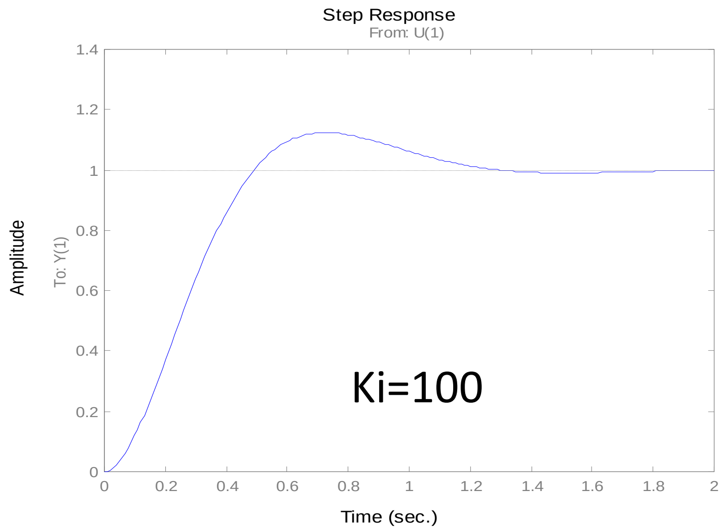
The integral controller (Ki) decreases the rise time, increases both the overshoot and the settling time, and eliminates the steady-state error

MATLAB Example

$$T(s) = \frac{Kp \cdot s + Ki}{s^3 + 10 \cdot s^2 + (20 + Kp) \cdot s + Ki}$$

```
Kp=30;
Ki=70;
num=[Kp Ki];
den=[1 10 20+Kp Ki];
t=0:0.01:2;
step(num,den,t)
```





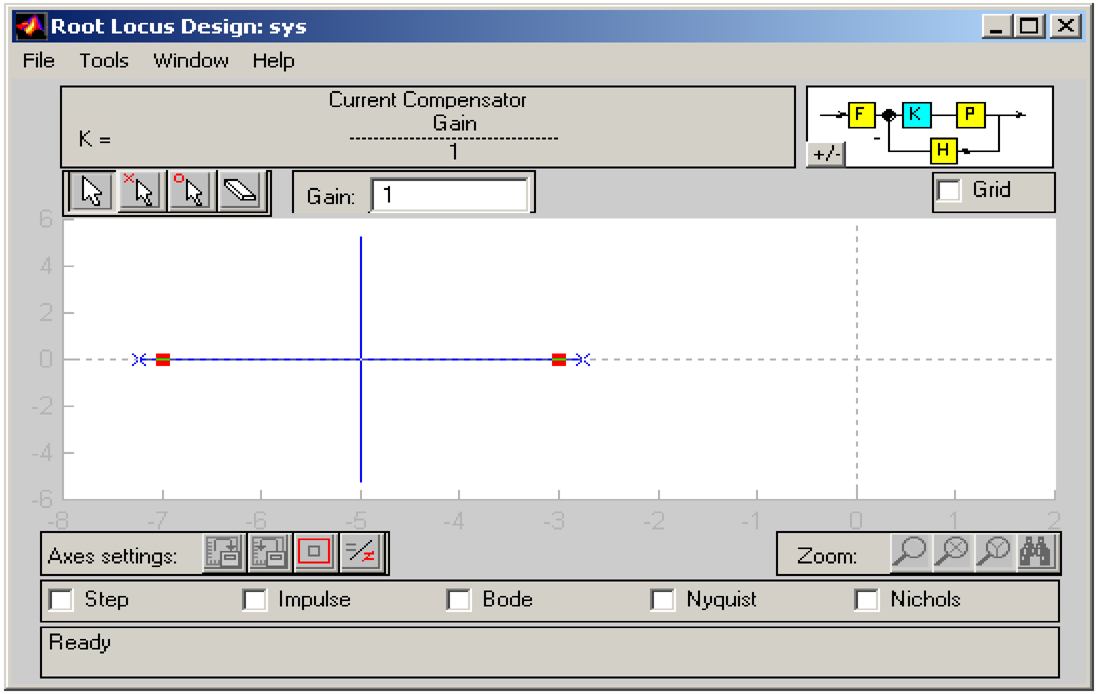
RLTOOL

Syntax

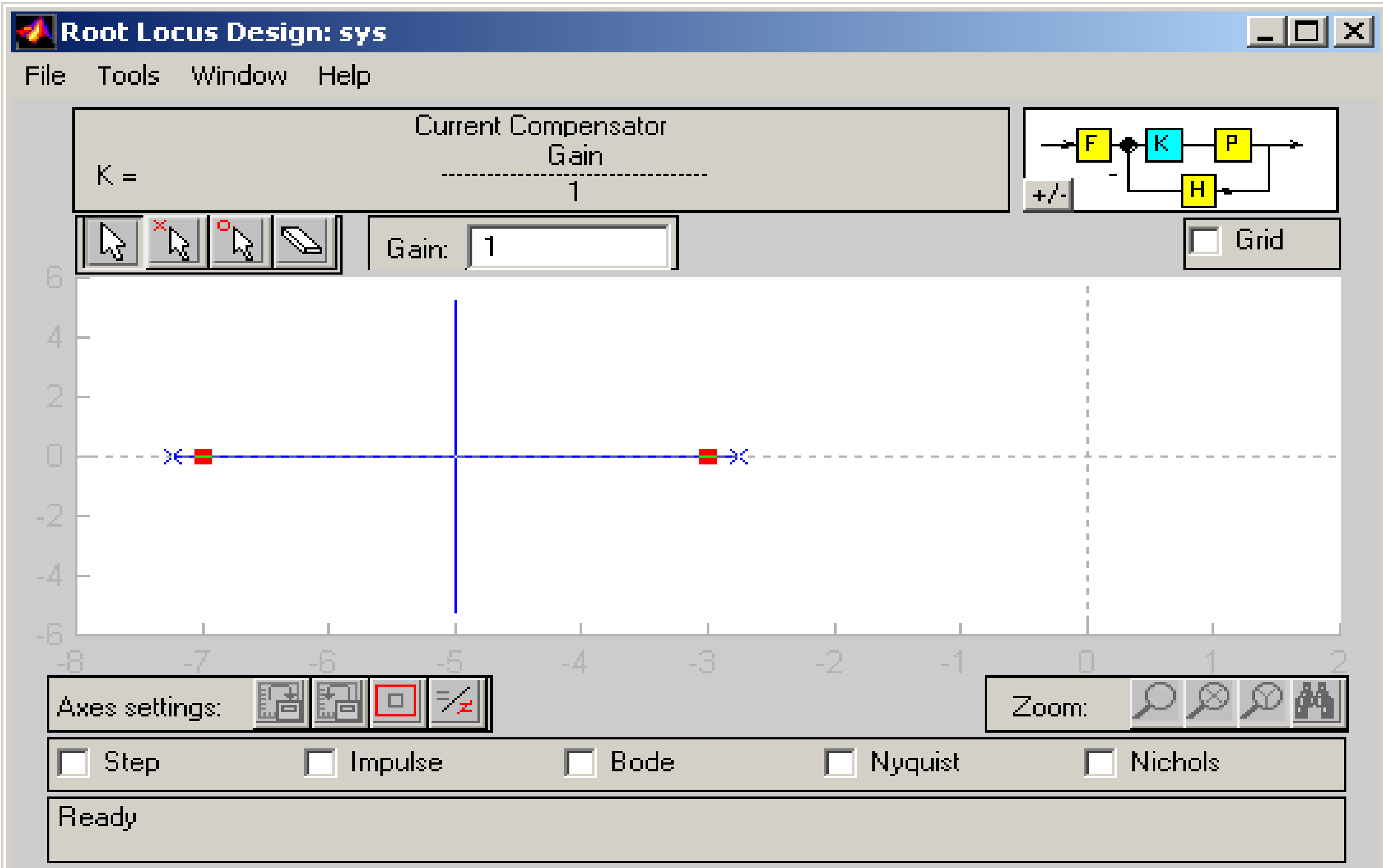
rltool

rltool(sys)

rltool(sys,comp)

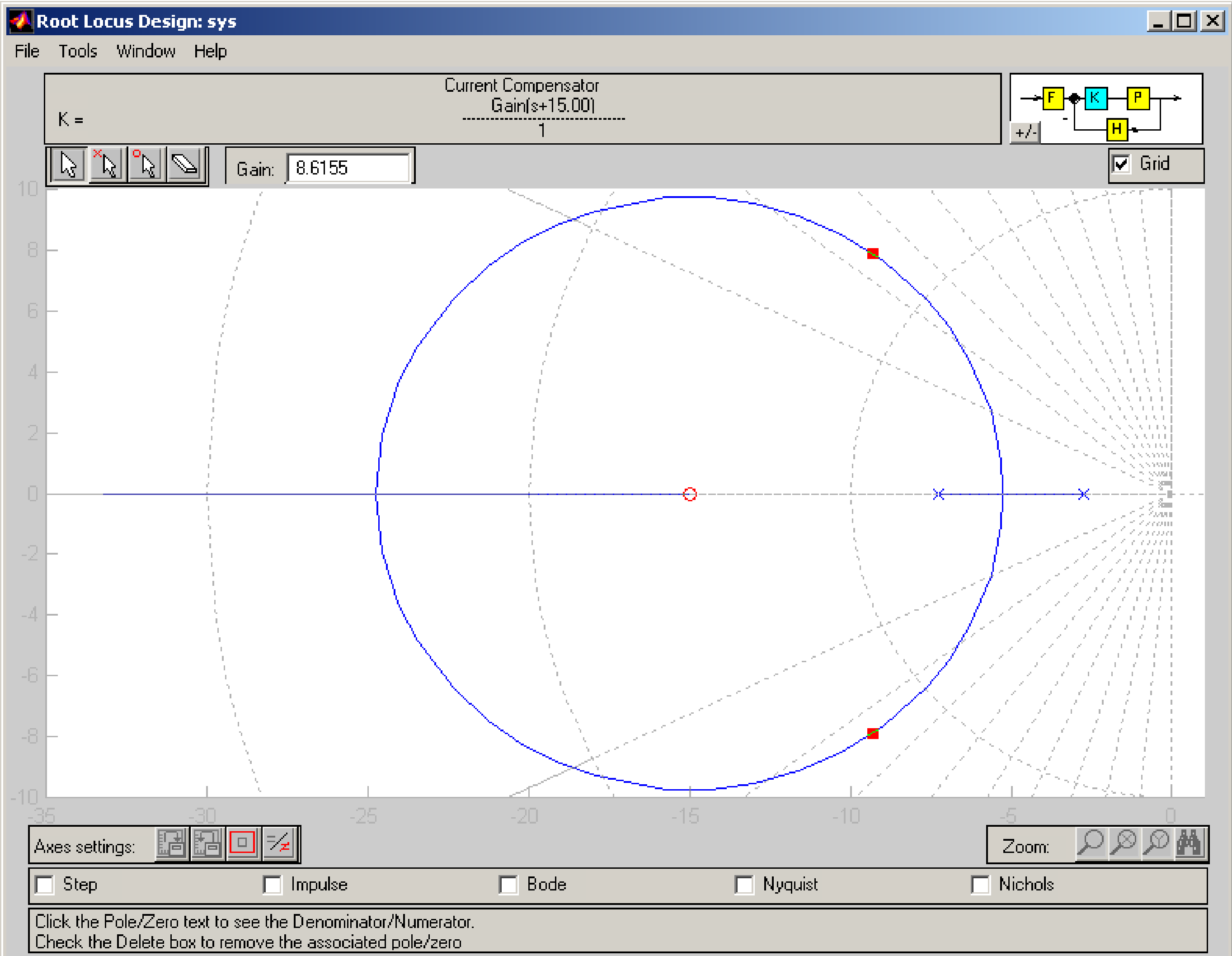


RLTOOL



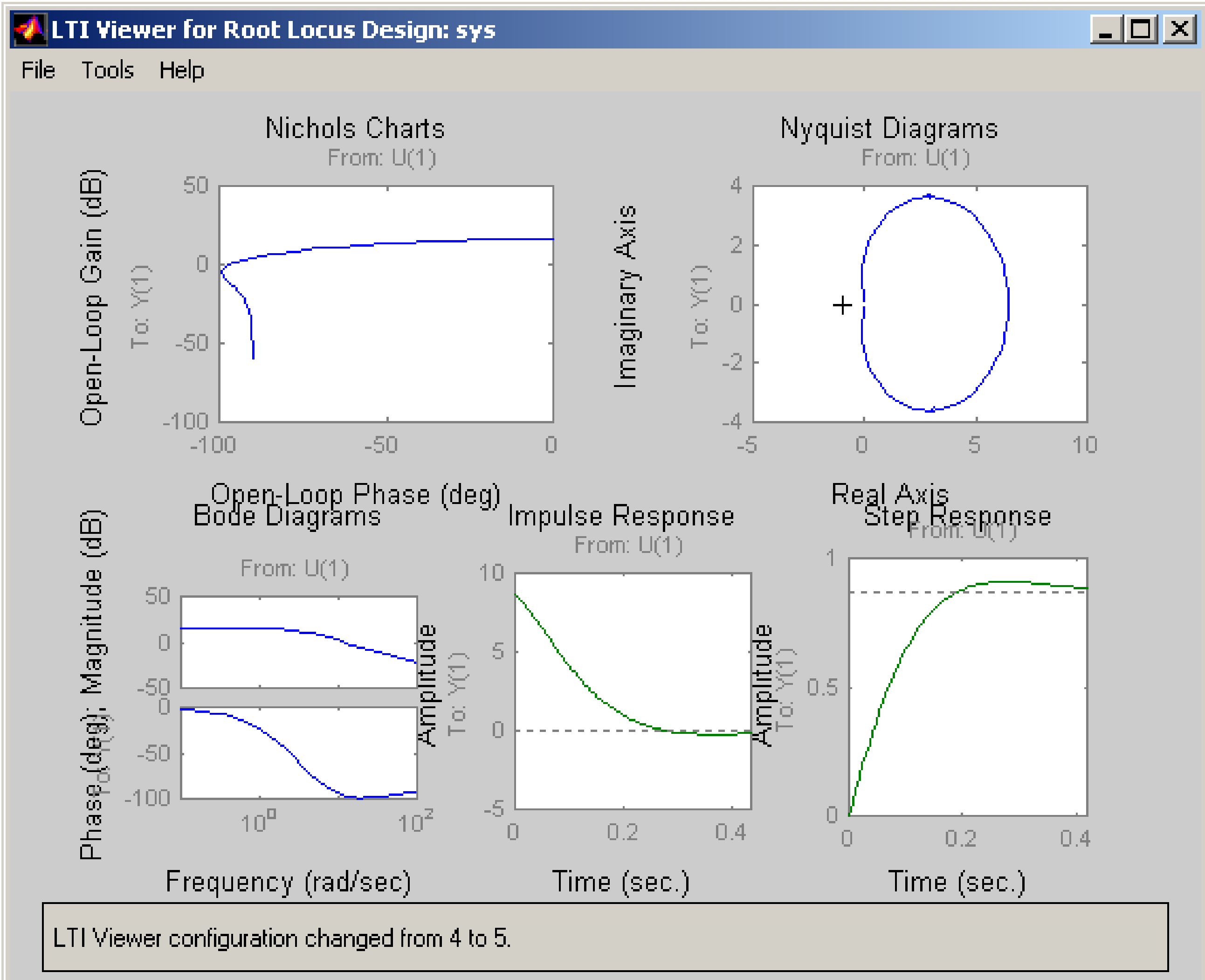
Root Locus Design: sys
File
Tools
Window
Help
Current Compensator
K =
Gain
1
F
K
P
H
+/-
Gain:
1
Grid
6
4
2
0
-2
-4
-6
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
Axes settings:
Zoom:
Step
Impulse
Bode
Nyquist
Nichols
Ready

RLTOOL

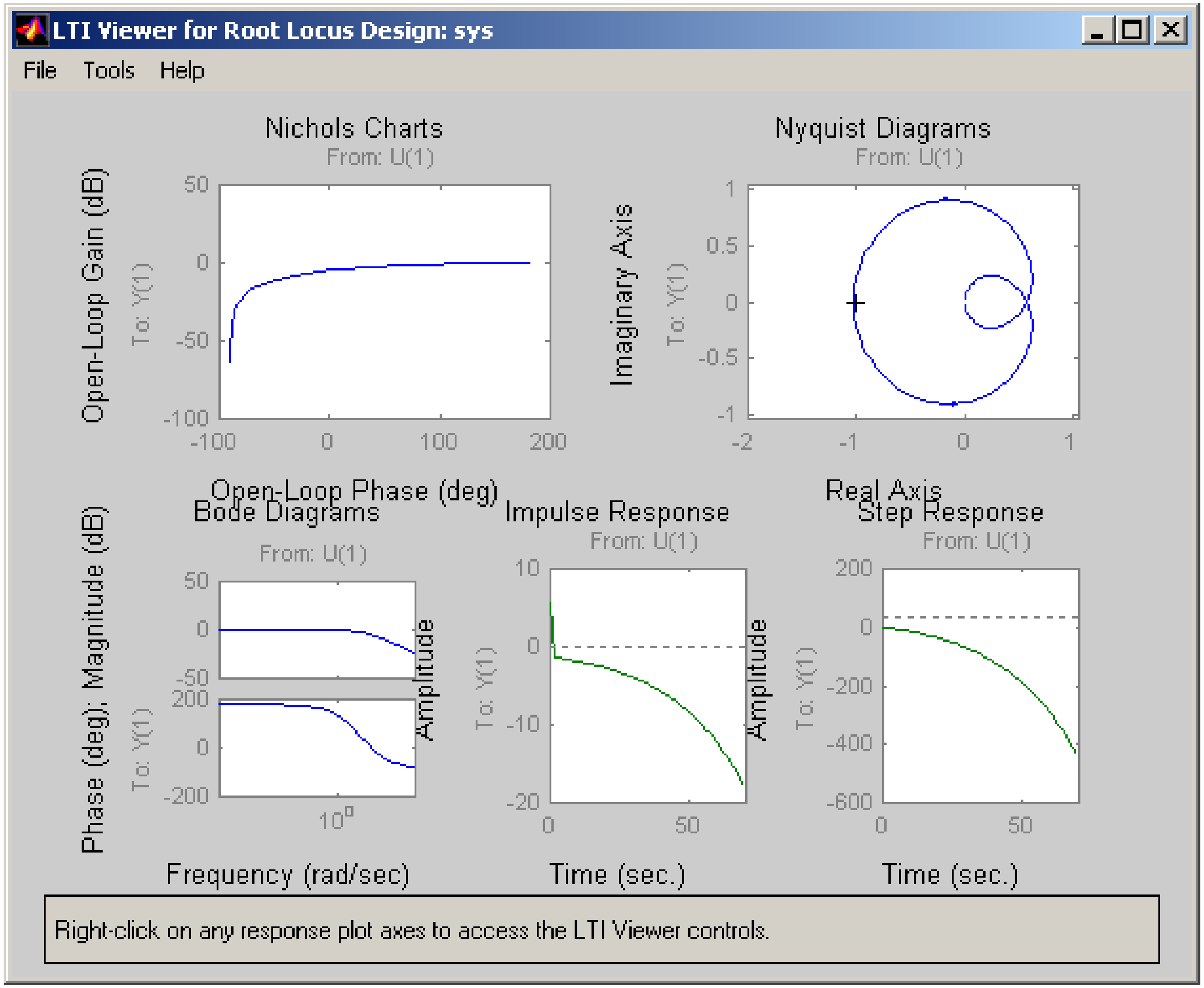


Root Locus Design: sys
File Tools Window Help
Current Compensator
K =
Gain(s+15.00)
1
F
K
P
H
+/-
Gain: 8.6155
Grid
10
8
6
4
2
0
-2
-4
-6
-8
-10
-35
-30
-25
-20
-15
-10
-5
0
Axes settings:
Zoom:
Step
Impulse
Bode
Nyquist
Nichols
Click the Pole/Zero text to see the Denominator/Numerator.
Check the Delete box to remove the associated pole/zero

RLTOOL


LTI Viewer for Root Locus Design: sys
File
Tools
Help
Nichols Charts
From: U(1)
Open-Loop Gain (dB)
To: Y(1)
Open-Loop Phase (deg)
Nyquist Diagrams
From: U(1)
Imaginary Axis
To: Y(1)
Real Axis
Bode Diagrams
From: U(1)
Phase (deg); Magnitude (dB)
Frequency (rad/sec)
Impulse Response
From: U(1)
Amplitude
To: Y(1)
Time (sec.)
Step Response
From: U(1)
Amplitude
To: Y(1)
Time (sec.)
LTI Viewer configuration changed from 4 to 5.

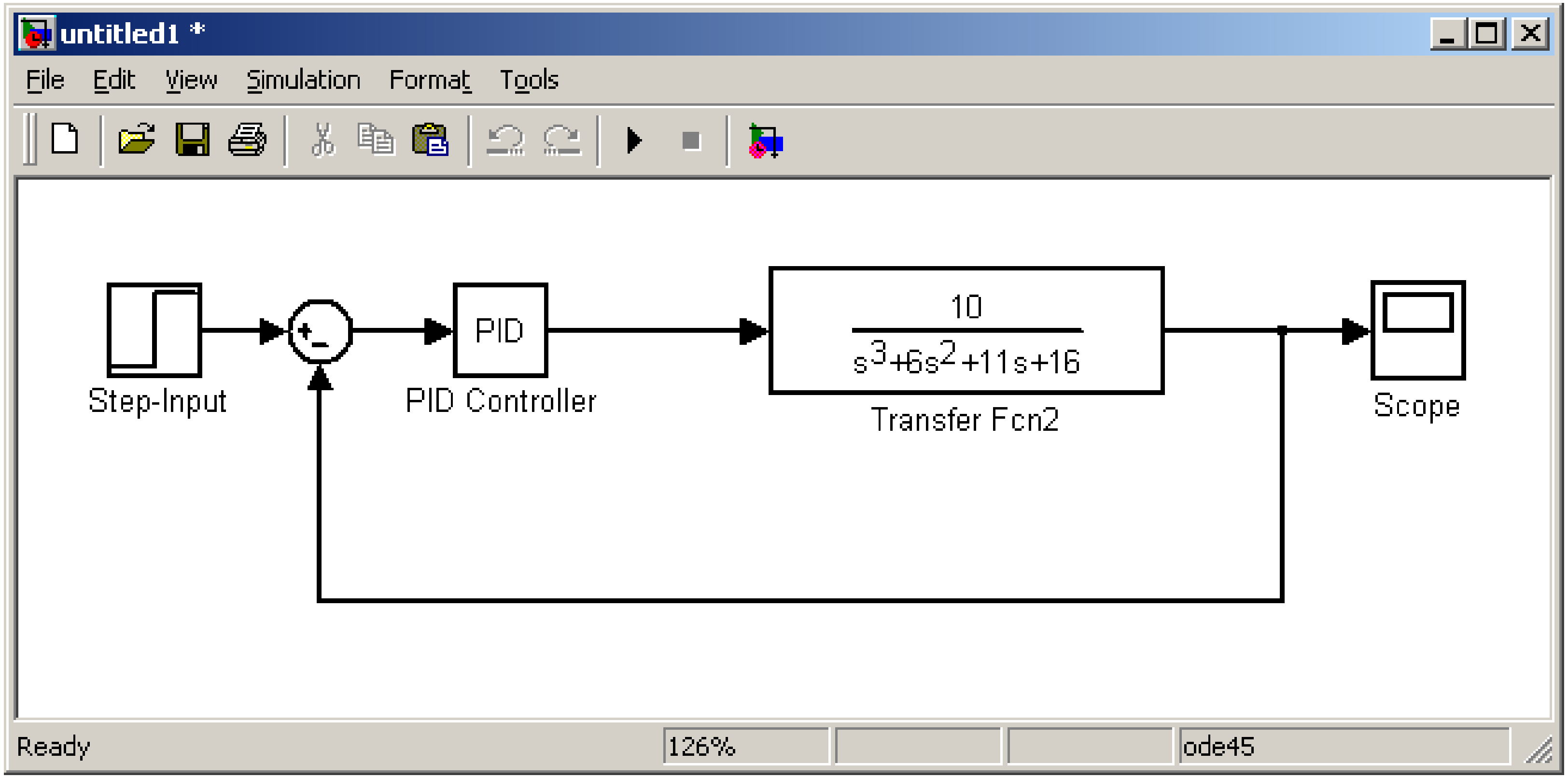
RLTOOL



LTI Viewer for Root Locus Design: sys
File Tools Help
Nichols Charts
From: U(1)
Open-Loop Gain (dB)
To: Y(1)
Open-Loop Phase (deg)
Nyquist Diagrams
From: U(1)
Imaginary Axis
Real Axis
Bode Diagrams
From: U(1)
Phase (deg); Magnitude (dB)
Frequency (rad/sec)
Impulse Response
From: U(1)
Amplitude
Time (sec.)
Step Response
From: U(1)
Right-click on any response plot axes to access the LTI Viewer controls.

Example - Practice

Consider the following configuration:



Example - Practice

The design a system for the following specifications:

- Zero steady state error
- Settling time within 5 seconds
- Rise time within 2 seconds
- Only some overshoot permitted

Lead or Phase-Lead Compensator Using Root Locus

A first-order lead compensator can be designed using the root locus. A lead compensator in root locus form is given by

$$G_c(s) = \frac{(s + z)}{(s + p)}$$

where the magnitude of z is less than the magnitude of p. A phase-lead compensator tends to shift the root locus toward the left half plane. This results in an improvement in the system's stability and an increase in the response speed.

When a lead compensator is added to a system, the value of this intersection will be a larger negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a larger negative number than the added zero. Thus, the result of a lead compensator is that the asymptotes' intersection is moved further into the left half plane, and the entire root locus will be shifted to the left. This can increase the region of stability as well as the response speed.

Lead or Phase-Lead Compensator Using Root Locus

In Matlab a phase lead compensator in root locus form is implemented by using the transfer function in the form

```
numlead=kc*[1 z];
denlead=[1 p];
```

and using the `conv()` function to implement it with the numerator and denominator of the plant

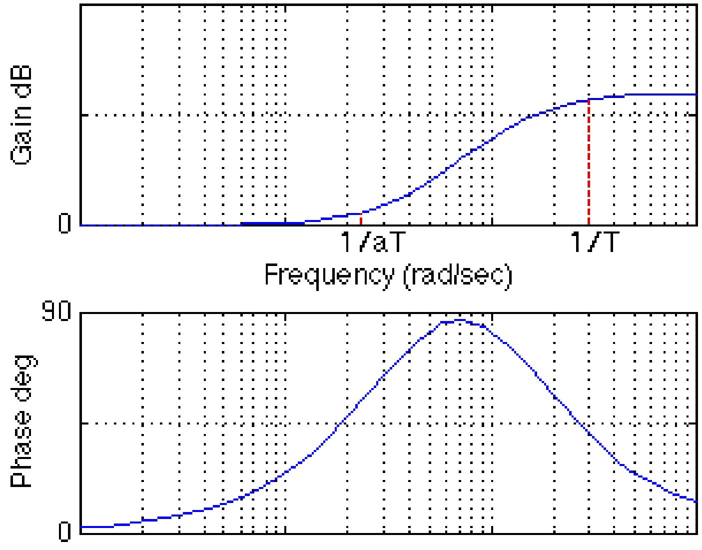
```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lead or Phase-Lead Compensator Using Frequency Response

A first-order phase-lead compensator can be designed using the frequency response. A lead compensator in frequency response form is given by

$$G_C(s) = \frac{(1 + \alpha\cdot\tau\cdot s)}{\alpha\cdot(1 + \tau\cdot s)} \qquad p = \frac{1}{\tau} \qquad z = \frac{1}{\alpha\tau} \qquad \omega_m = \sqrt{z\cdot p} \qquad \sin(\phi_m) = \frac{\alpha - 1}{\alpha + 1}$$

In frequency response design, the phase-lead compensator adds positive phase to the system over the frequency range. A bode plot of a phase-lead compensator looks like the following



Lead or Phase-Lead Compensator Using Frequency Response

Additional positive phase increases the phase margin and thus increases the stability of the system. This type of compensator is designed by determining **alfa** from the amount of phase needed to satisfy the phase margin requirements, and determining **tal** to place the added phase at the new gain-crossover frequency.

Another effect of the lead compensator can be seen in the magnitude plot. The lead compensator increases the gain of the system at high frequencies (the amount of this gain is equal to alfa. This can increase the crossover frequency, which will help to decrease the rise time and settling time of the system.

Lead or Phase-Lead Compensator Using Frequency Response

In Matlab, a phase lead compensator in frequency response form is implemented by using the transfer function in the form

```
numlead=[aT 1];
denlead=[T 1];
```

and using the `conv()` function to multiply it by the numerator and denominator of the plant

```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lag or Phase-Lag Compensator Using Root Locus

A first-order lag compensator can be designed using the root locus. A lag compensator in root locus form is given by

$$G_c(s) = \frac{(s + z)}{(s + p)}$$

where the magnitude of z is greater than the magnitude of p. A phase-lag compensator tends to shift the root locus to the right, which is undesirable. For this reason, the pole and zero of a lag compensator must be placed close together (usually near the origin) so they do not appreciably change the transient response or stability characteristics of the system.

When a lag compensator is added to a system, the value of this intersection will be a smaller negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a smaller negative number than the added zero. Thus, the result of a lag compensator is that the asymptotes' intersection is moved closer to the right half plane, and the entire root locus will be shifted to the right.

Lag or Phase-Lag Compensator Using Root Locus

It was previously stated that that lag controller should only minimally change the transient response because of its negative effect. If the phase-lag compensator is not supposed to change the transient response noticeably, what is it good for? The answer is that a phase-lag compensator can improve the system's steady-state response. It works in the following manner. At high frequencies, the lag controller will have unity gain. At low frequencies, the gain will be z0/p0 which is greater than 1. This factor z/p will multiply the position, velocity, or acceleration constant (Kp, Kv, or Ka), and the steady-state error will thus decrease by the factor z0/p0. In Matlab, a phase lead compensator in root locus form is implemented by using the transfer function in the form

```
numlag=[1 z];
denlag=[1 p];
```

and using the `conv()` function to implement it with the numerator and denominator of the plant

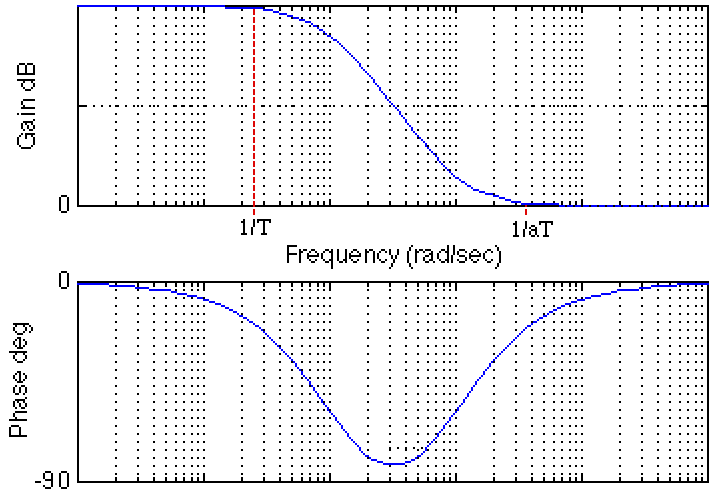
```
newnum=conv(num,numlag);
newden=conv(den,denlag);
```

Lag or Phase-Lag Compensator using Frequency Response

A first-order phase-lag compensator can be designed using the frequency response. A lag compensator in frequency response form is given by

$$G_c(s) = \frac{(1 + \alpha \cdot \tau \cdot s)}{\alpha \cdot (1 + \tau \cdot s)}$$

The phase-lag compensator looks similar to a phase-lead compensator, except that a is now less than 1. The main difference is that the lag compensator adds negative phase to the system over the specified frequency range, while a lead compensator adds positive phase over the specified frequency. A bode plot of a phase-lag compensator looks like the following



Lag or Phase-Lag Compensator using Frequency Response

In Matlab, a phase-lag compensator in frequency response form is implemented by using the transfer function in the form

```
numlead=[a*T 1];
denlead=a*[T 1];
```

and using the `conv()` function to implement it with the numerator and denominator of the plant

```
newnum=conv(num,numlead);
newden=conv(den,denlead);
```

Lead-lag Compensator using either Root Locus or Frequency Response

A lead-lag compensator combines the effects of a lead compensator with those of a lag compensator. The result is a system with improved transient response, stability and steady-state error. To implement a lead-lag compensator, first design the lead compensator to achieve the desired transient response and stability, and then add on a lag compensator to improve the steady-state response

Exercise - Dominant Pole-Zero Approximations and Compensations

The influence of a particular pole (or pair of complex poles) on the response is mainly determir
by two factors: the real part of the pole and the relative magnitude of the residue at the pole. T
real part determines the rate at which the transient term due to the pole decays; the larger the r
part, the faster the decay. The relative magnitude of the residue determines the percentage of
total response due to a particular pole.

Investigate (using Simulink) the impact of a closed-loop negative real pole on the overshoot of
system having complex poles.

$$T(s) = \frac{pr\cdot\omega n^2}{(s + pr)\cdot\left[s^2 + (2\cdot\zeta\cdot\omega n\cdot s) + \omega n^2\right]}$$

Make pr to vary (2, 3, 5) times the real part of the complex pole for different values of ζ (0.3, 0.5, 0.7).

Investigate (using Simulink) the impact of a closed-loop negative real zero on the overshoot of
system having complex poles.

$$T(s) = \frac{(s + zr)}{\left[s^2 + (2\cdot\zeta\cdot\omega n\cdot s) + \omega n^2\right]}$$

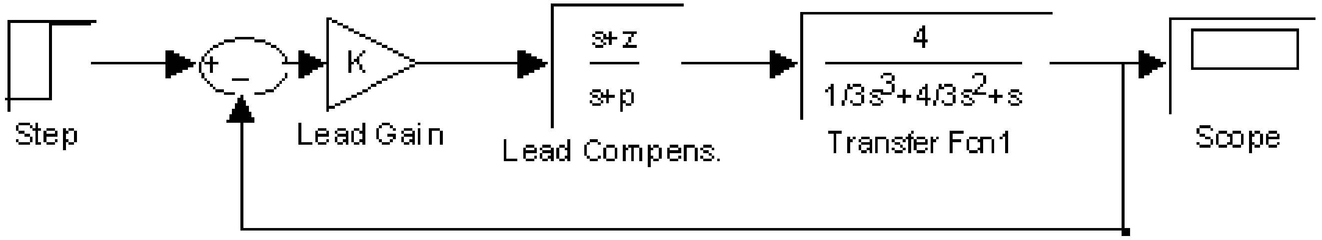
Make zr to vary (2, 3, 5) times the real part of the complex pole for different values of ζ (0.3, 0.5, 0.7).

Exercise - Lead and Lag Compensation

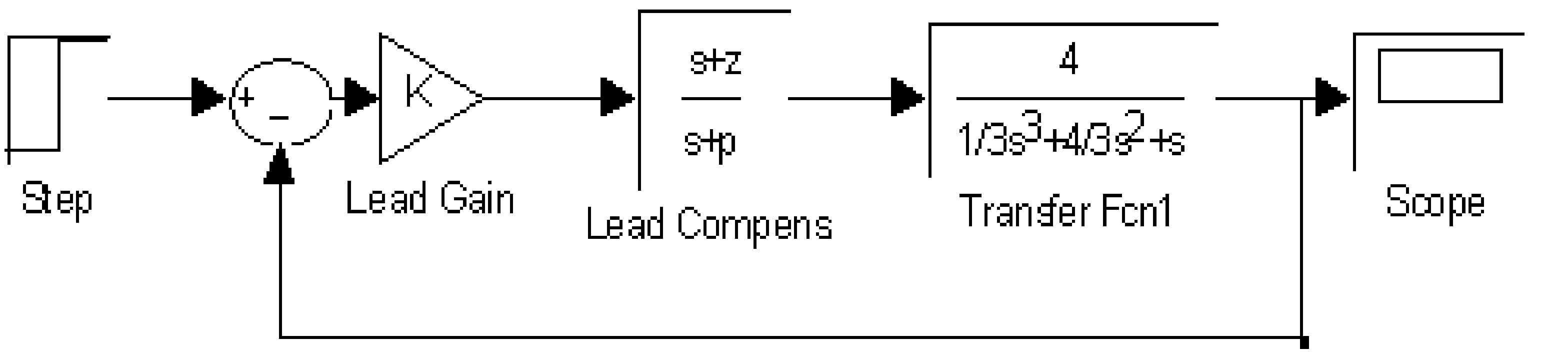
Investigate (using Matlab and Simulink) the effect of lead and lag compensations on the systems indicated below. Summarize your observations. Plot the root-locus, bode diagra and output for a step input before and after the compensations.

Remember
lead compensation: z<p (place zero below the desired root location or to the left of the fi real poles)
lag compensation: z>p (locate the pole and zero near the origin of the s-plane)

Lead Compensation (use z=1.33, p=20 and K =15).



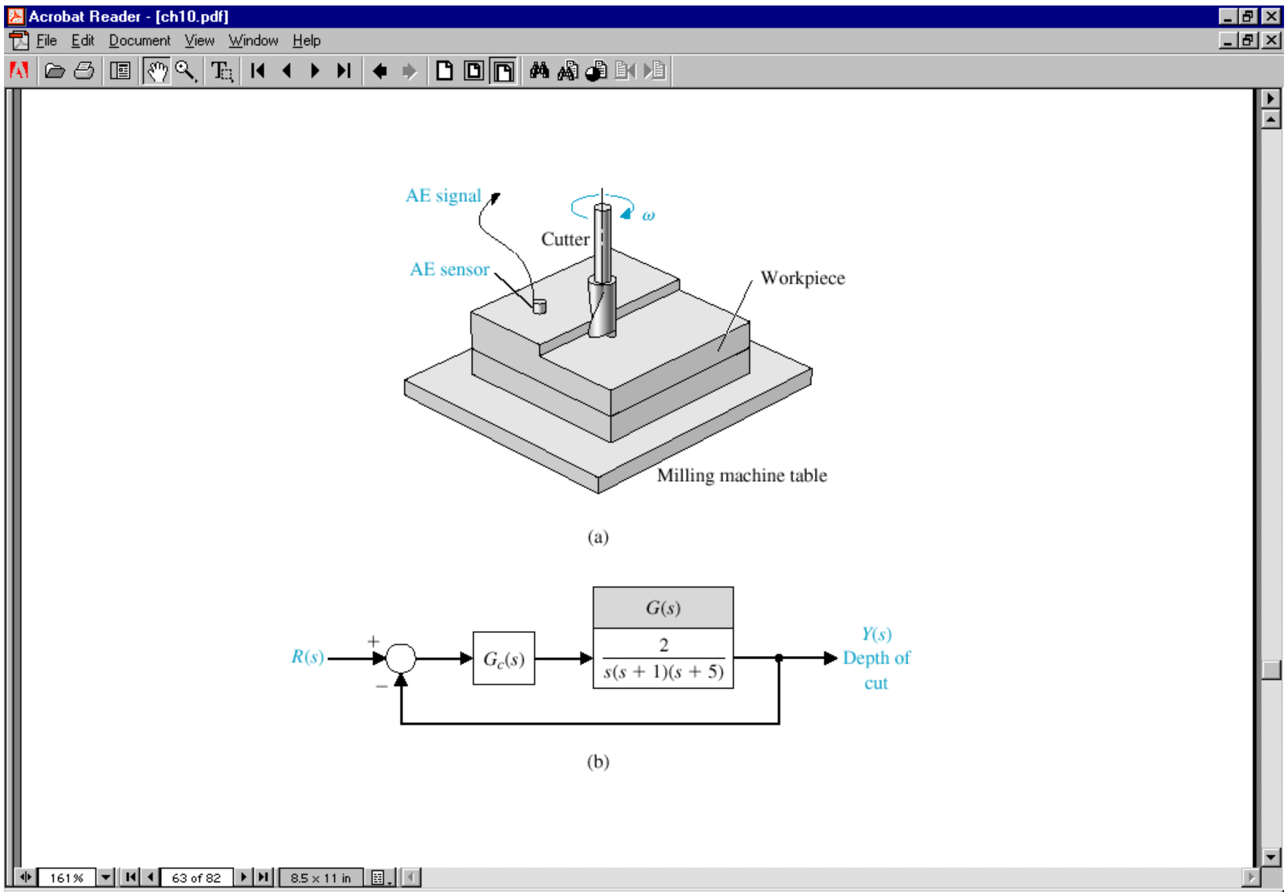
Lag Compensation (use z=0.09 , and p=0.015, K=1/6)

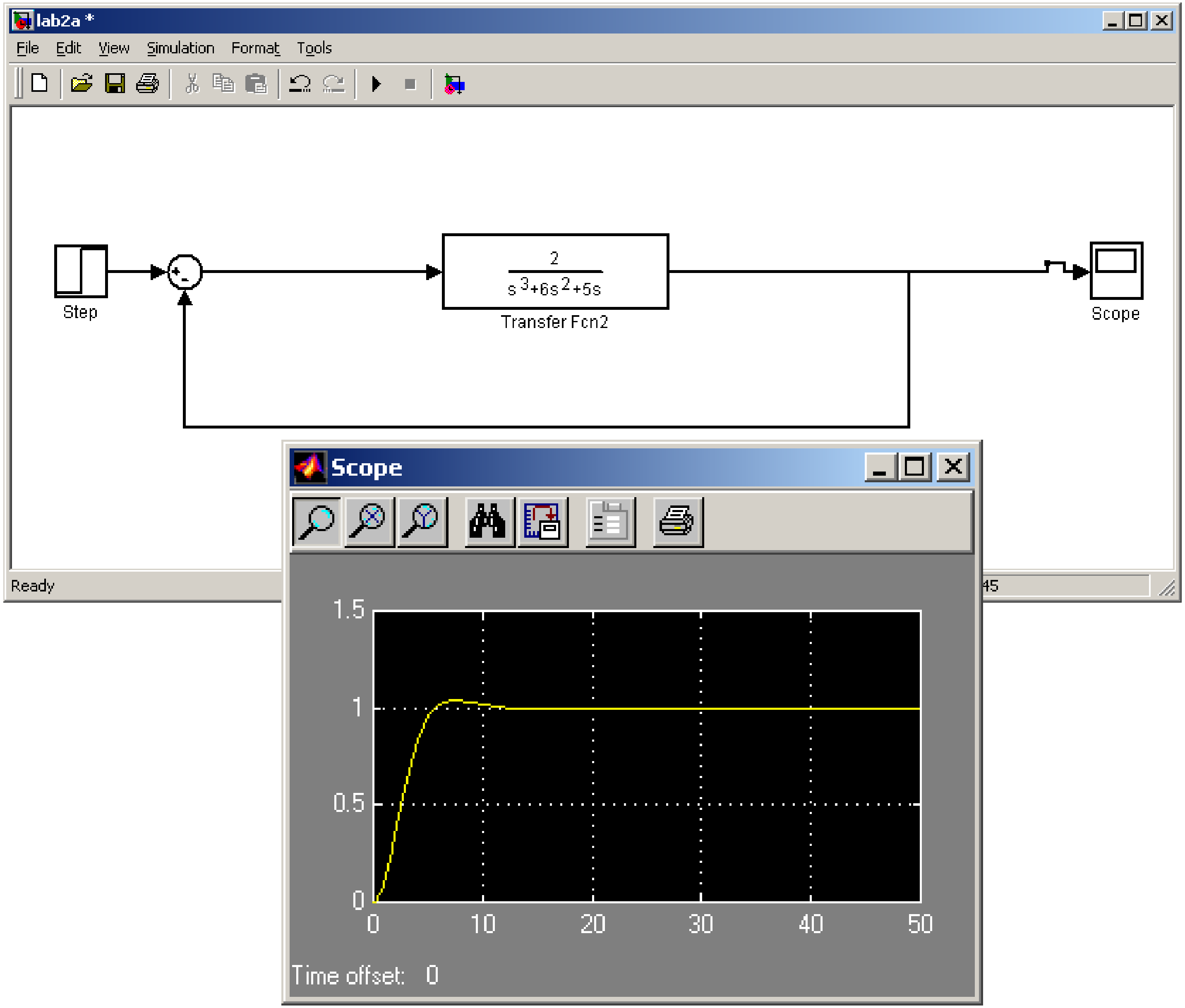


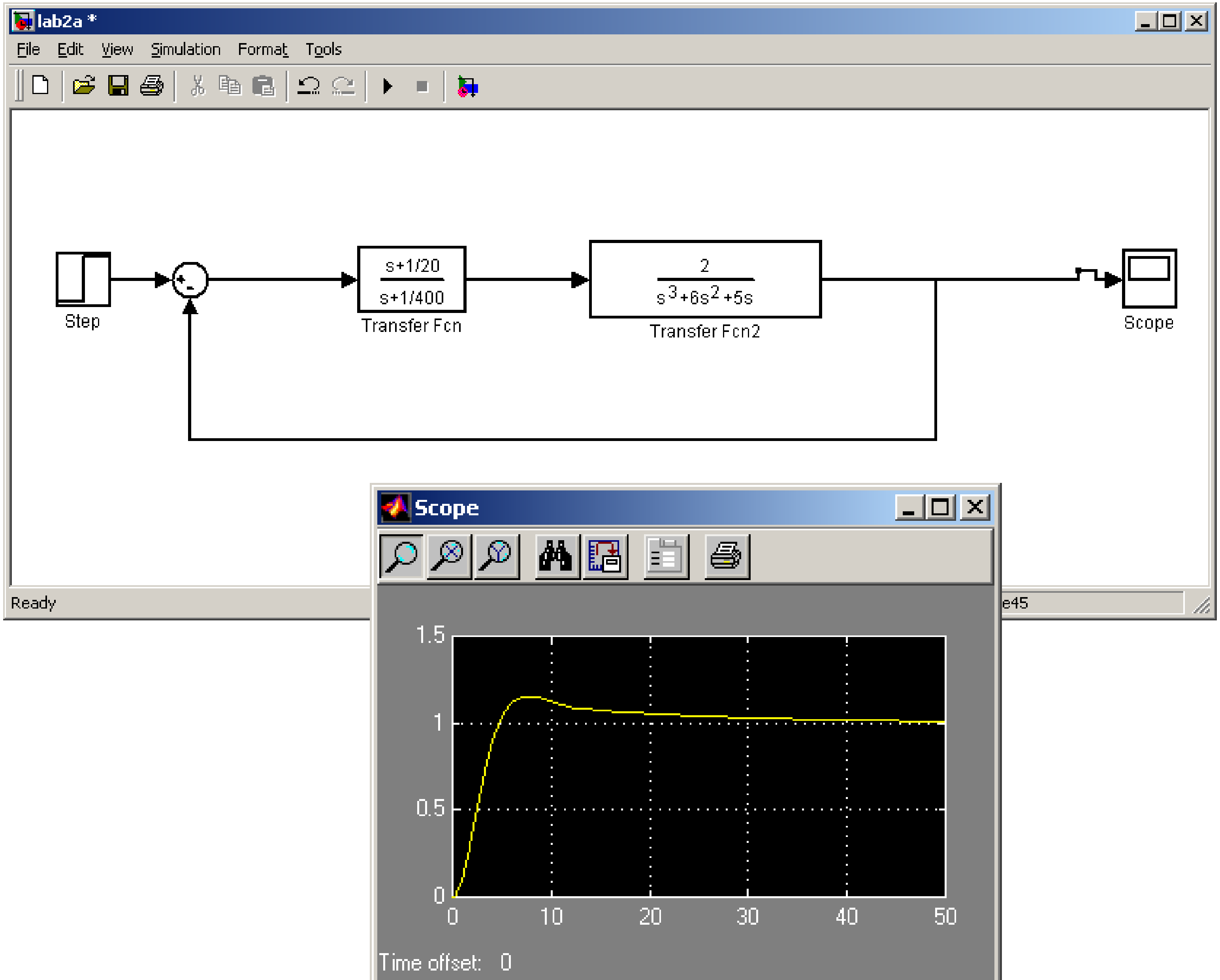
Summarize your findings

Problem 10.36

Determine a compensator so that the percent overshoot is less than 20% and Kv (velocity constant) is greater than 8.




lab2a *
File Edit View Simulation Format Tools
Step
2
s 3+6s 2+5s
Transfer Fcn2
Scope
Ready
Scope
1.5
1
0.5
0
0
10
20
30
40
50
Time offset: 0


lab2a *
File Edit View Simulation Format Tools
s+1/20
s+1/400
Transfer Fcn
2
s3+6s2+5s
Transfer Fcn2
Step
Scope
Ready
e45
Scope
1.5
1
0.5
0
0 10 20 30 40 50
Time offset: 0

Poles and Zeros and Transfer Functions

Transfer Function:

A transfer function is defined as the ratio of the Laplace transform of the output to the input with all initial conditions equal to zero. Transfer functions are defined only for linear time invariant systems.

Considerations:

Transfer functions can usually be expressed as the ratio of two polynomials in the complex variable, s.

Factorization:

A transfer function can be factored into the following form.

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ \ldots\ (s+p_n)}$$

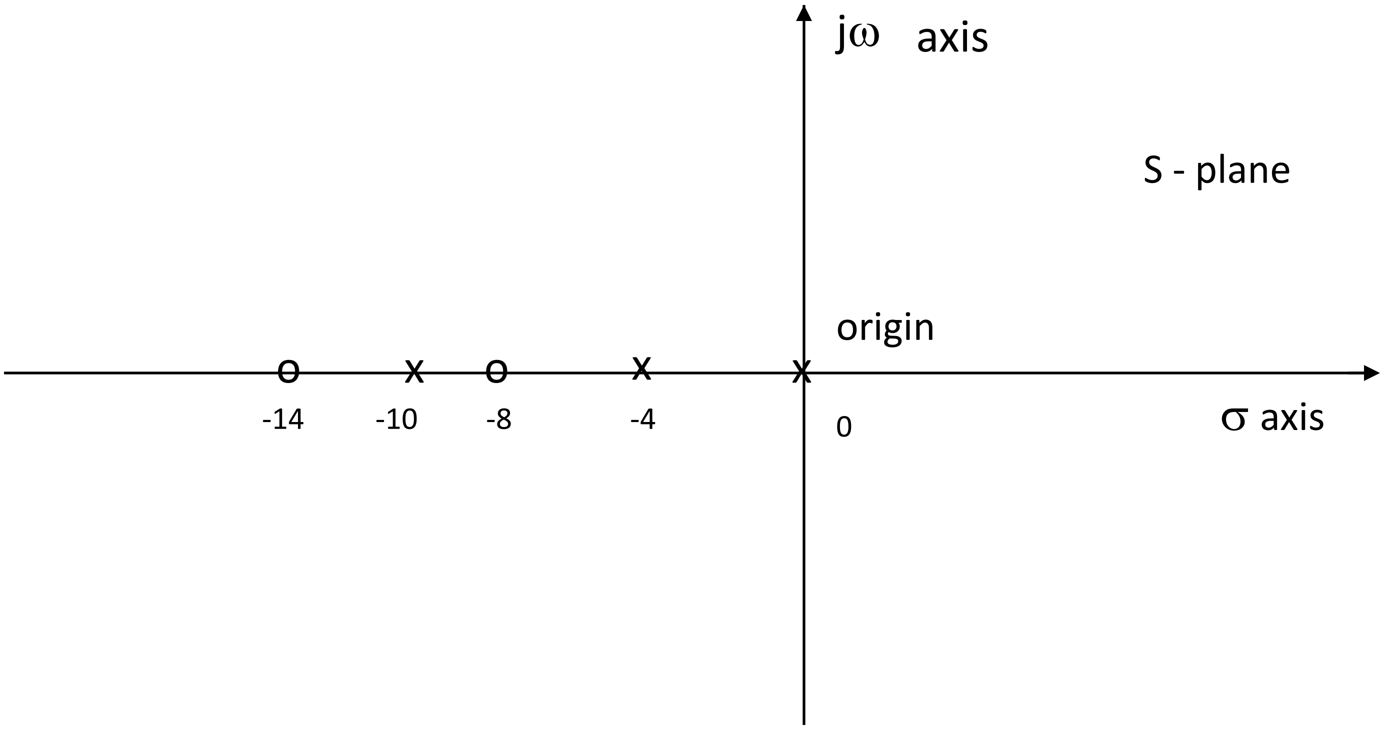
The roots of the numerator polynomial are called zeros.

The roots of the denominator polynomial are called poles.

Poles, Zeros and the S-Plane

An Example: You are given the following transfer function. Show the poles and zeros in the s-plane.

$$G(s) = \frac{(s+8)(s+14)}{s(s+4)(s+10)}$$



wlg

Poles, Zeros and Bode Plots

Characterization: Considering the transfer function of the previous slide. We note that we have 4 different types of terms in the previous general form: These are:

$$K_B, \quad \frac{1}{s}, \quad \frac{1}{(s/p + 1)}, \quad (s/z + 1)$$

Expressing in dB: Given the tranfer function:

$$G(jw) = \frac{K_B(jw/z+1)}{(jw)(jw/p+1)}$$

$$20\log|G(jw| = 20\log K_B + 20\log|(jw/z+1)| - 20\log|jw| - 20\log|jw/p+1|$$

Poles, Zeros and Bode Plots

Mechanics: We have 4 distinct terms to consider:

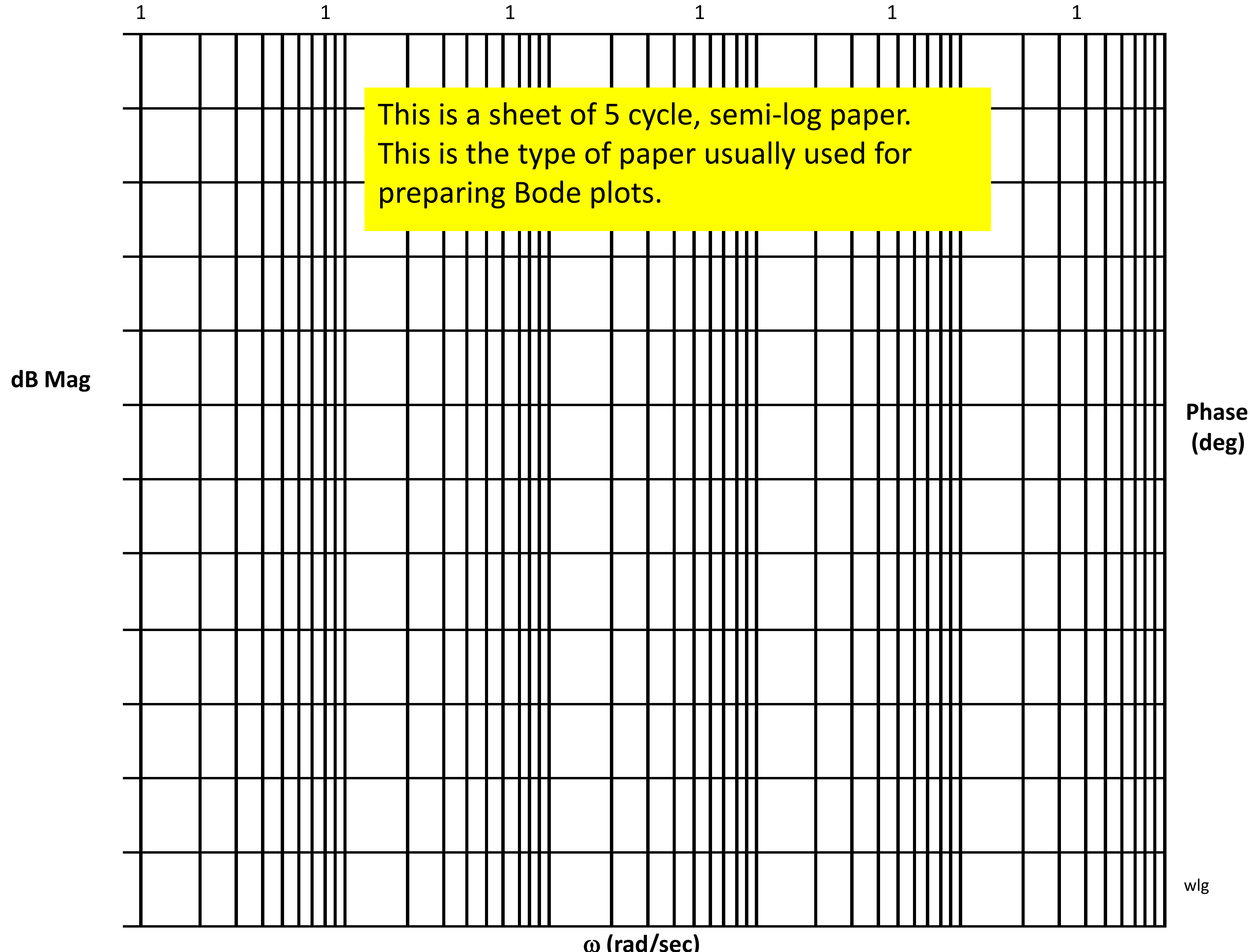
$$20\log K_B$$

$$20\log|(jw/z + 1)|$$

$$-20\log|jw|$$

$$-20\log|(jw/p + 1)|$$

wlg



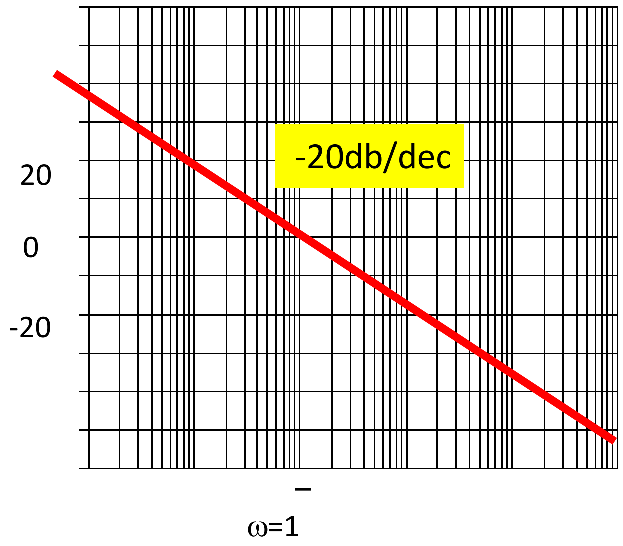
This is a sheet of 5 cycle, semi-log paper.
This is the type of paper usually used for
preparing Bode plots.

Poles, Zeros and Bode Plots

Mechanics:

The gain term, $20\log K_{B,}$ is just so many dB and this is a straight line on Bode paper, independent of omega (radian frequency).

The term, $-20\log|jw| = -20\log w$, when plotted on semi-log paper is a straight line sloping at -20dB/decade. It has a magnitude of 0 at w = 1.



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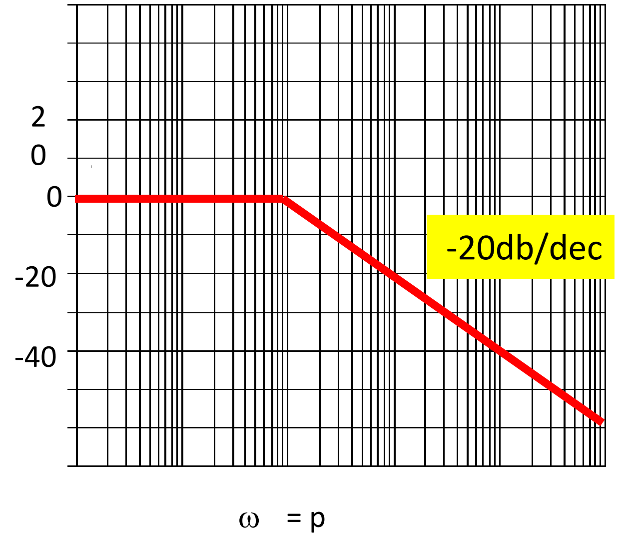
Poles, Zeros and Bode Plots

Mechanics:

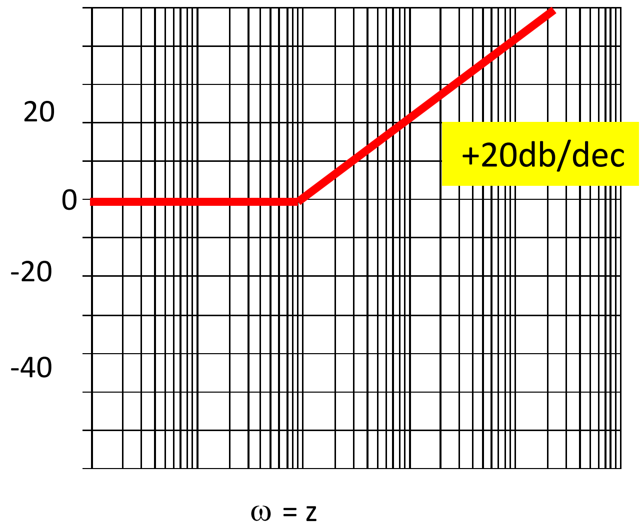
The term, - 20log|(jw/p + 1), is drawn with the following approximation: If w < p we use the approximation that –20log|(jw/p + 1)| = 0 dB, a flat line on the Bode. If w > p we use the approximation of –20log(w/p), which slopes at -20dB/dec starting at w = p. Illustrated below. It is easy to show that the plot has an error of -3dB at w = p and – 1 dB at w = p/2 and w = 2p. One can easily make these corrections if it is appropriate.



Poles, Zeros and Bode Plots

Mechanics:

When we have a term of 20log|(jw/z + 1)| we approximate it be a straight line of slop 0 dB/dec when w < z. We approximate it as 20log(w/z) when w > z, which is a straight line on Bode paper with a slope of + 20dB/dec. Illustrated below.



Example 1:

Given:

$$G(jw) = \frac{50,000(jw+10)}{(jw+1)(jw+500)}$$

First: Always, always, always get the poles and zeros in a form such that the constants are associated with the jw terms. In the above example we do this by factoring out the 10 in the numerator and the 500 in the denominator.

$$G(jw) = \frac{50,000x10(jw/10+1)}{500(jw+1)(jw/500+1)} = \frac{100(jw/10+1)}{(jw+1)(jw/500+1)}$$

Second: When you have neither poles nor zeros at 0, start the Bode at $20\log_{10}K = 20\log_{10}100 = 40$ dB in this case.

Example 1: (continued)

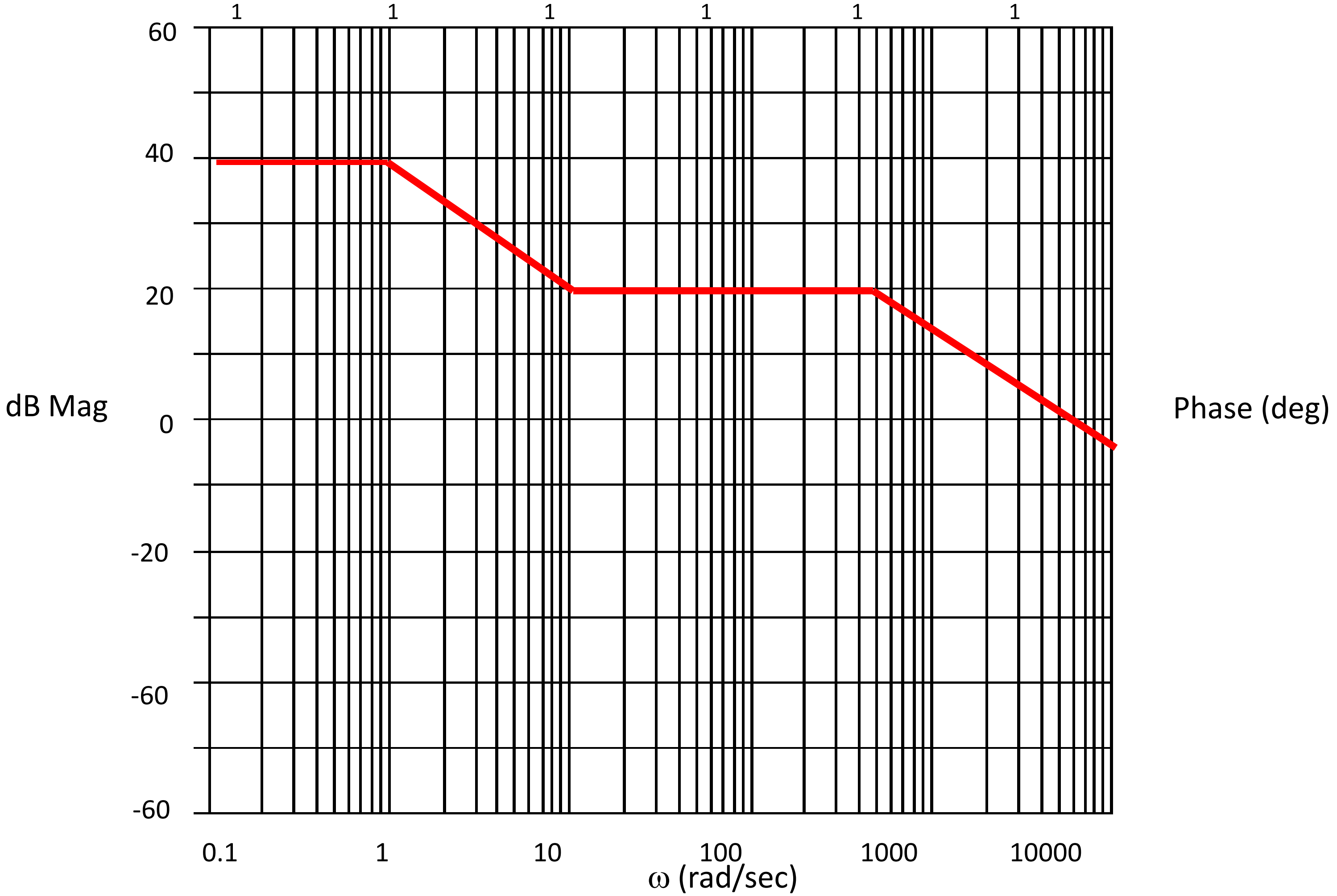
Third: Observe the order in which the poles and zeros occur.
This is the secret of being able to quickly sketch the Bode.
In this example we first have a pole occurring at 1 which causes the Bode to break at 1 and slope – 20 dB/dec. Next, we see a zero occurs at 10 and this causes a slope of +20 dB/dec which cancels out the – 20 dB/dec, resulting in a flat line (0 db/dec). Finally, we have a pole that occurs at w = 500 which causes the Bode to slope down at – 20 dB/dec.

We are now ready to draw the Bode.

Before we draw the Bode we should observe the range over which the transfer function has active poles and zeros. This determines the scale we pick for the w (rad/sec) at the bottom of the Bode.

The dB scale depends on the magnitude of the plot and experience is the best teacher here.

Bode Plot Magnitude for 100(1 + jw/10)/(1 + jw/1)(1 + jw/500)



wlg

Using Matlab For Frequency Response

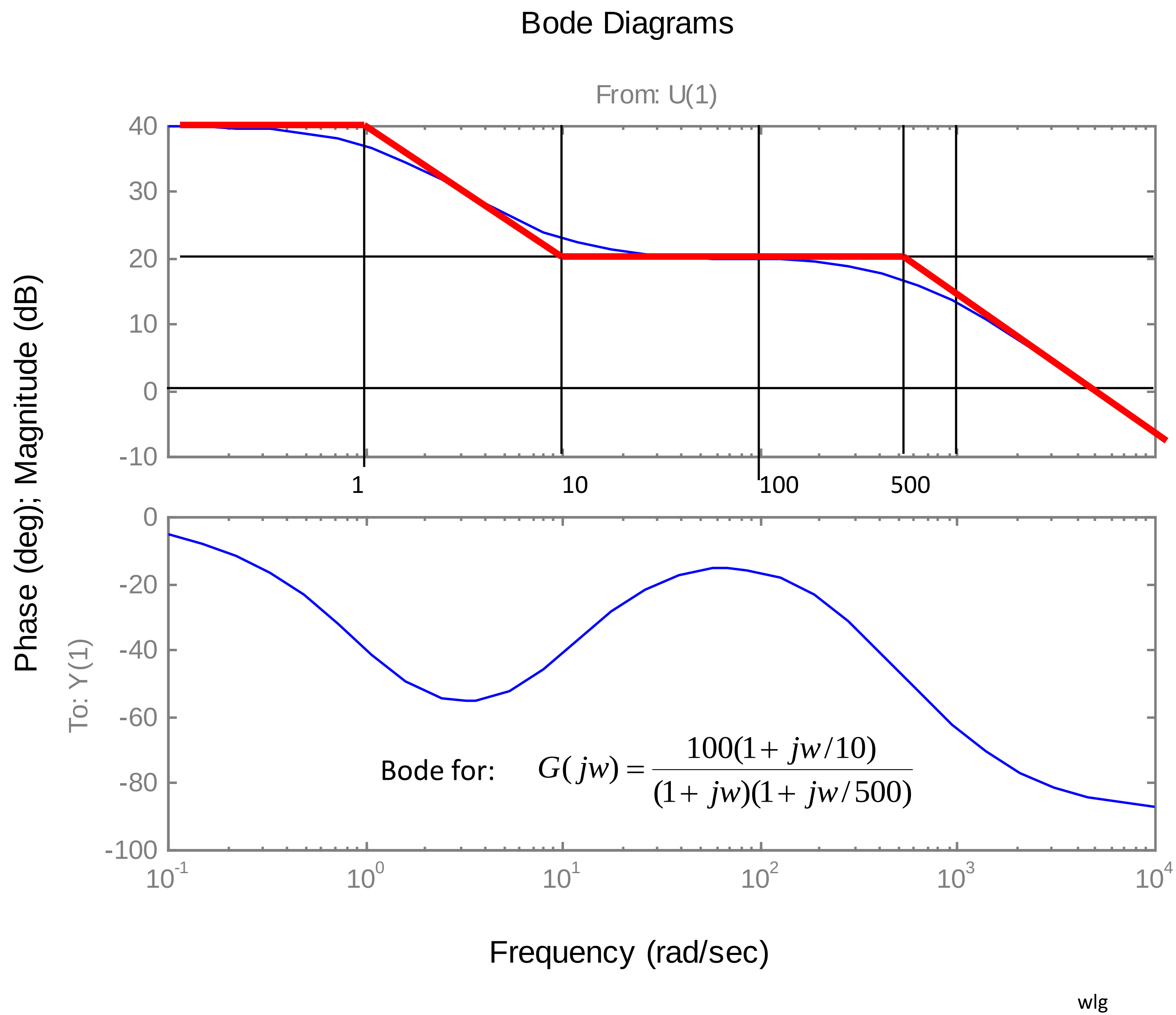
Instruction: We can use Matlab to run the frequency response for the previous example. We place the transfer function in the form:

$$\frac{5000(s+10)}{(s+1)(s+500)} = \frac{[5000s+50000]}{[s^2+501s+500]}$$

The Matlab Program

```
num = [5000 50000];
den = [1 501 500];
Bode (num,den)
```

In the following slide, the resulting magnitude and phase plots (exact) are shown in light color (blue). The approximate plot for the magnitude (Bode) is shown in heavy lines (red). We see the 3 dB errors at the corner frequencies.


Bode Diagrams
From: U(1)
To: Y(1)
Phase (deg); Magnitude (dB)
40
30
20
10
0
-10
1
10
100
500
0
-20
-40
-60
-80
-100
Bode for: $G(jw) = \frac{100(1 + jw/10)}{(1 + jw)(1 + jw/500)}$
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/sec)
wlg

Phase for Bode Plots

Comment: Generally, the phase for a Bode plot is not as easy to draw or approximate as the magnitude. In this course we will use an analytical method for determining the phase if we want to make a sketch of the phase.

Illustration: Consider the transfer function of the previous example. We express the angle as follows:

$$\angle G(jw) = \tan^{-1}(w/10) - \tan^{-1}(w/1) - \tan^{-1}(w/500)$$

We are essentially taking the angle of each pole and zero. Each of these are expressed as the $\tan^{-1}$(j part/real part)

Usually, about 10 to 15 calculations are sufficient to determine a good idea of what is happening to the phase.

Bode Plots

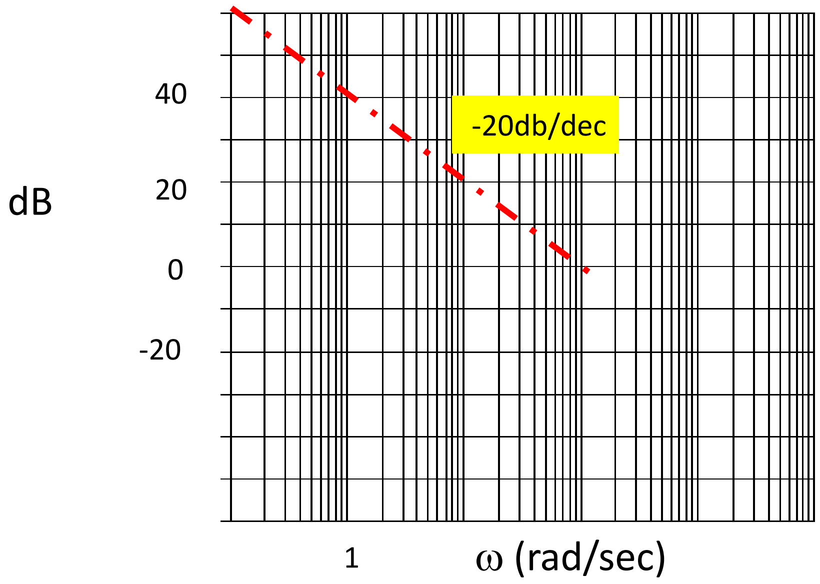
Example 2: Given the transfer function. Plot the Bode magnitude.

$$G(s) = \frac{100(1 + s/10)}{s(1 + s/100)^2}$$

Consider first only the two terms of

$$\frac{100}{jw}$$

Which, when expressed in dB, are; 20log100 – 20 logw.
This is plotted below.



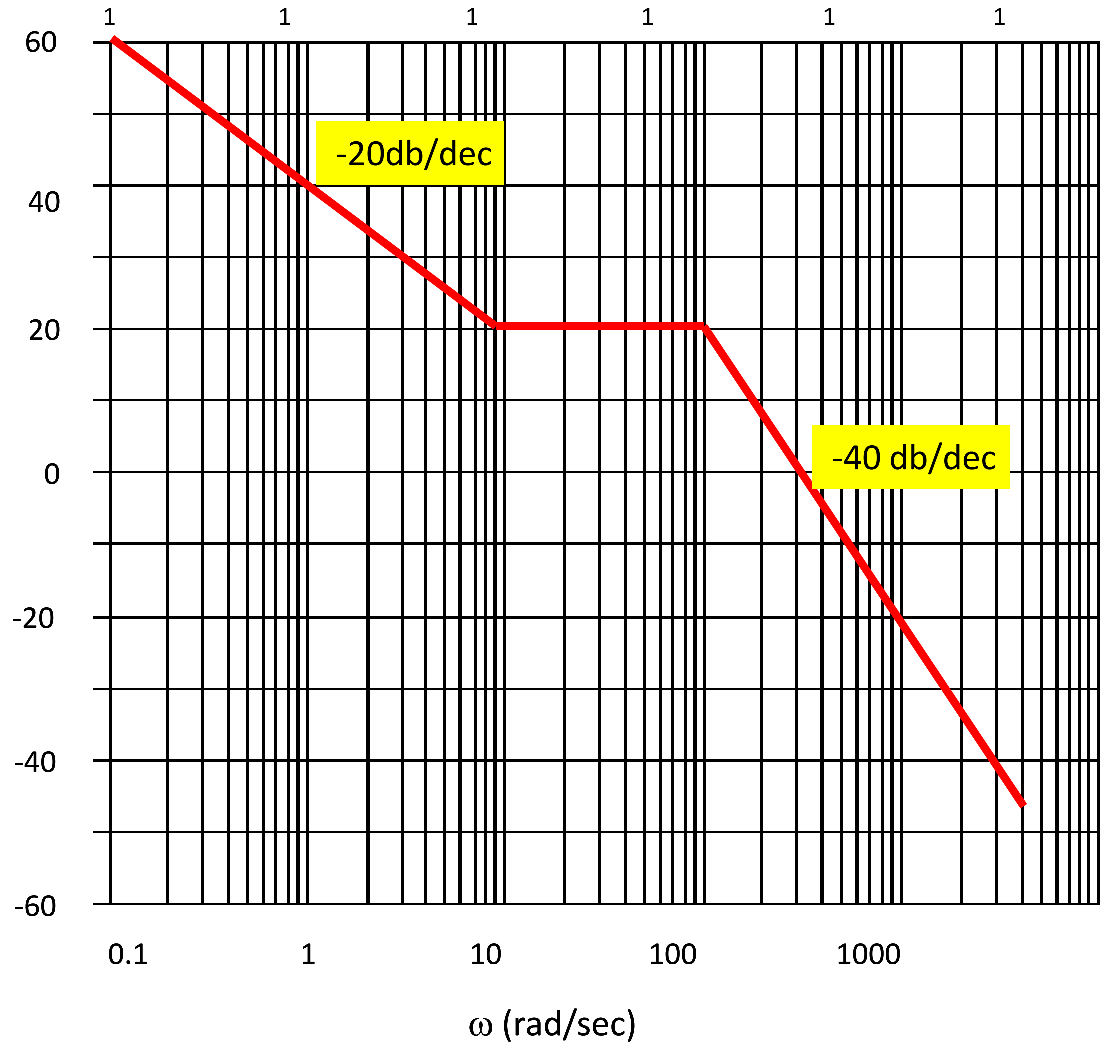
The .... is a tentative line we use until we encounter the first pole(s) or zero(s) not at the origin.

wlg

Bode Plots

Example 2: (continued) The completed plot is shown below.

$$G(s)=\frac{100(1+s/10)}{s(1+s/100)^{2}}$$



$$G(s)=\frac{100(1+s/10)}{s(1+s/100)^{2}}$$

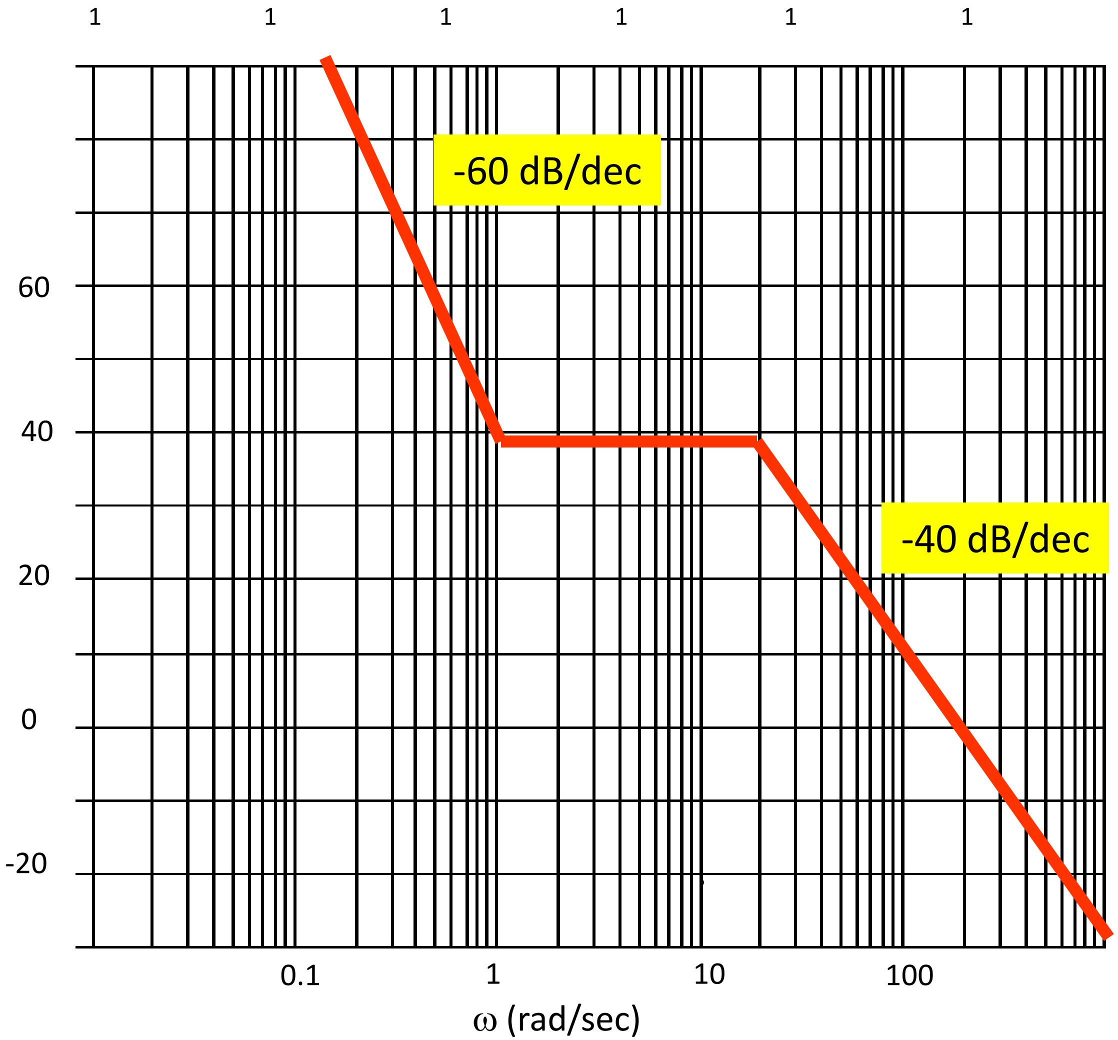
wlg

Bode Plots

Example 3:

Given: $$G(s) = \frac{80(1 + jw)^3}{(jw)^3(1 + jw/20)^2}$$

20log80 = 38 dB

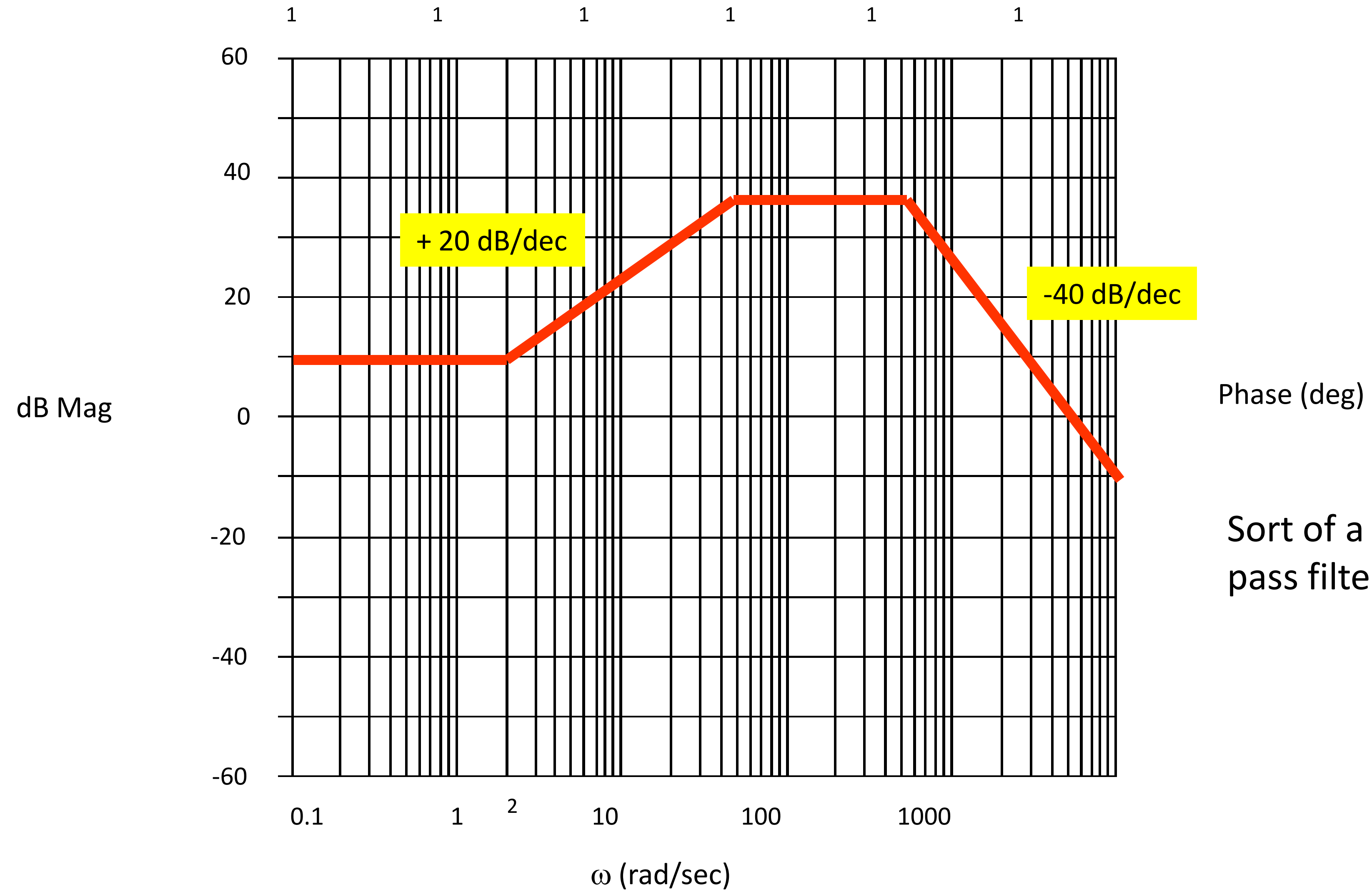


wlg

Bode Plots

Example 4:

Given: $$G(jw) = \frac{10(1 - jw/2)}{(1 + j0.025w)(1 + jw/500)^2}$$



Sort of a low pass filter

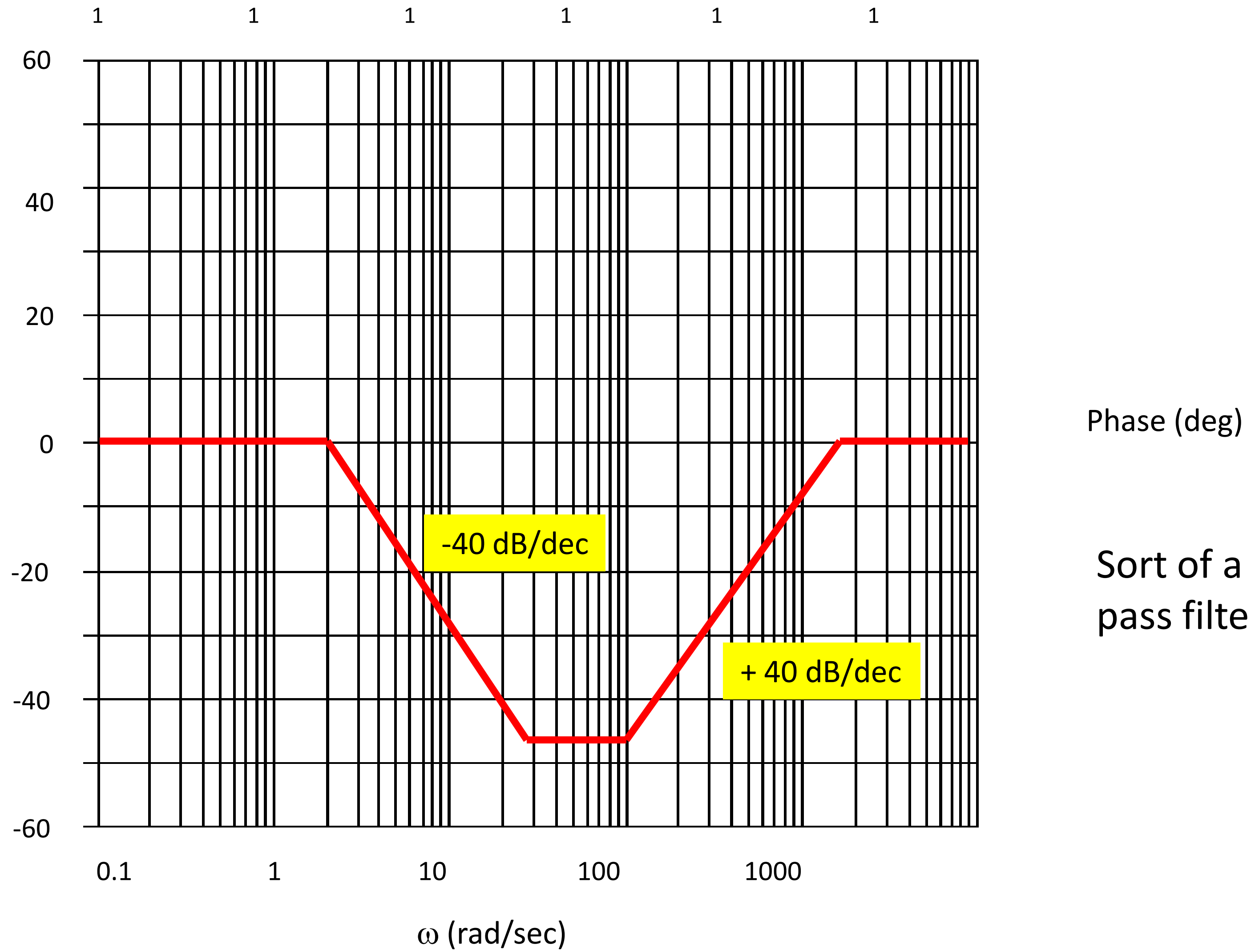
wlg

Bode Plots

Given:

$$G(jw) = \frac{(1 + jw/30)^2 (1 + jw/100)^2}{(1 + jw/2)^2 (1 + jw/1700)^2}$$

Example 5



Sort of a low pass filter

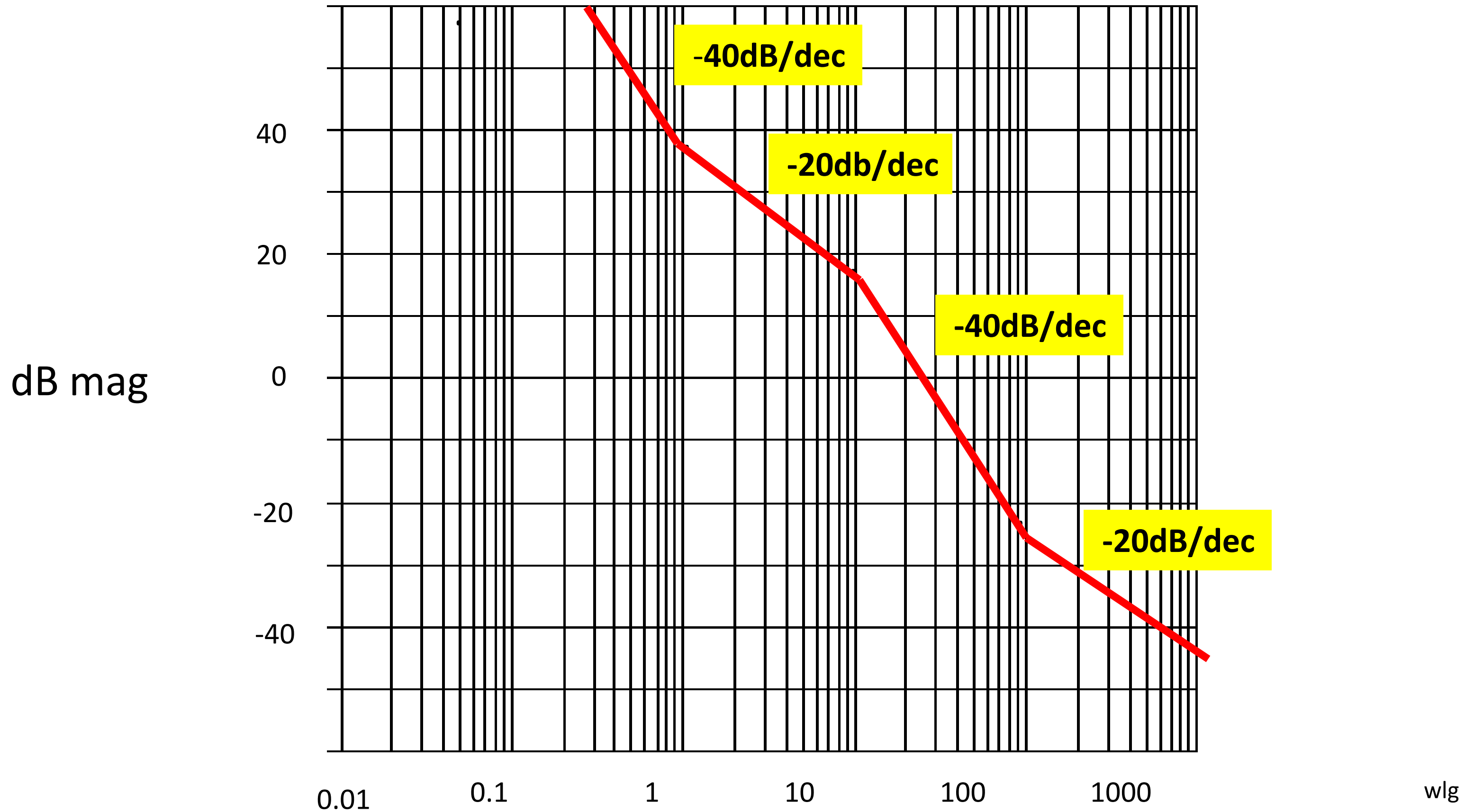
wlg

Bode Plots

Given: **problem 11.15 text**

Example 6

$$H(jw) = \frac{640(jw+1)(0.01jw+1)}{(jw)^2(jw+10)} = \frac{64(jw+1)(0.01jw+1)}{(jw)^2(0.1jw+1)}$$

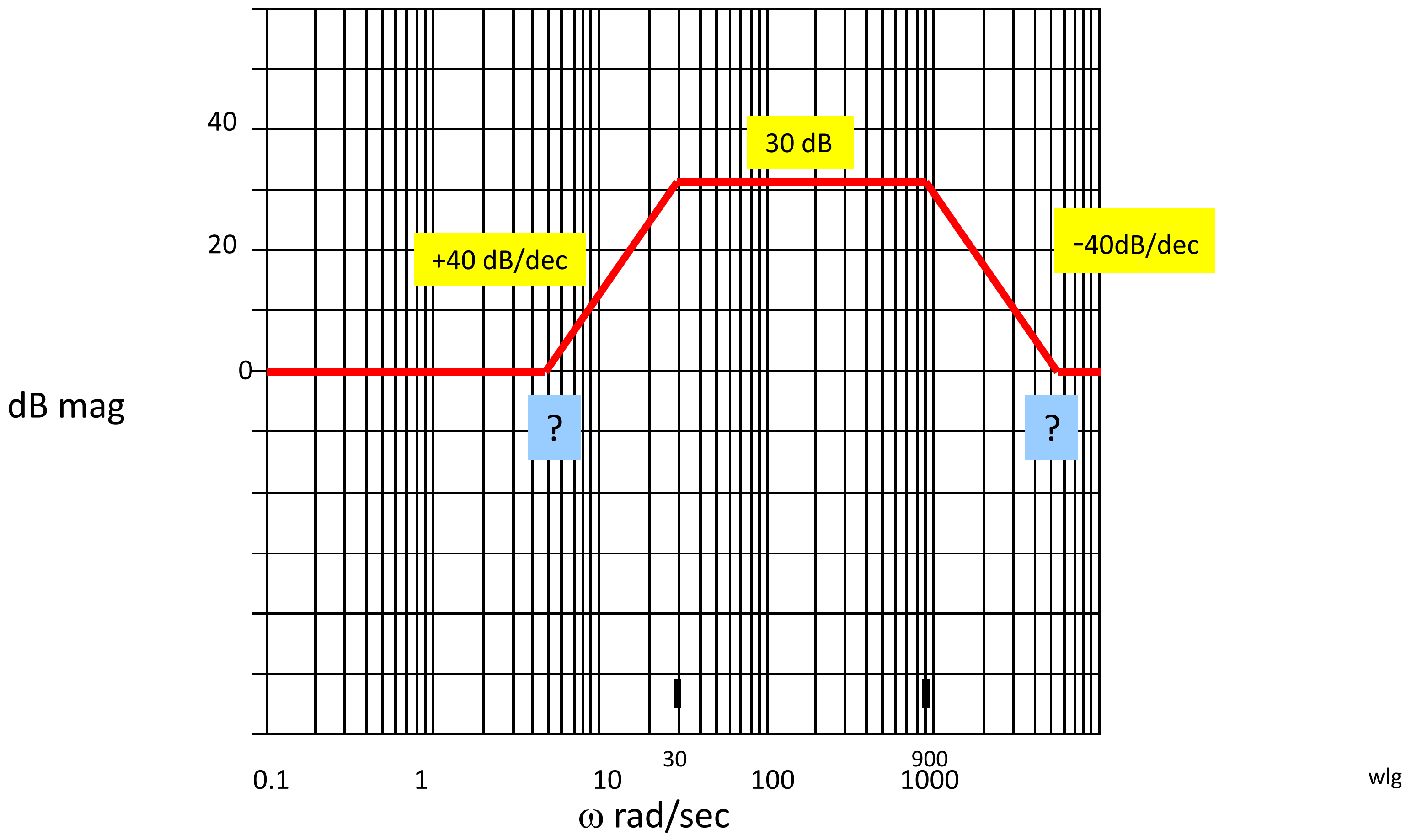


Bode Plots

Design Problem:

Design a G(s) that has the following Bode plot.

Example 7



Bode Plots

Procedure: The two break frequencies need to be found.
Recall:

$\#dec = \log_{10}[w_2/w_1]$

Then we have:

$(\#dec)(40dB/dec) = 30\ dB$

$\log_{10}[w_1/30] = 0.75$ ⟶ $\underline{w_1 = 5.33\ rad/sec}$

Also:

$\log_{10}[w_2/900]\ (-40dB/dec) = -30dB$

This gives $\underline{w_2 = 5060\ rad/sec}$

wlg

Bode Plots

Procedure:

$$G(s) = \frac{(1+s/5.3)^2(1+s/5060)^2}{(1+s/30)^2(1+s/900)^2}$$

Clearing:

$$G(s) = \frac{(s+5.3)^2(s+5060)^2}{(s+30)^2(s+900)^2}$$

Use Matlab and conv:

$$N1 = (s^2 + 10.6s + 28.1) \qquad N2 = (s^2 + 10120s + 2.56xe^7)$$

N1 = [1 10.6 28.1]   N2 = [1 10120 2.56e+7]

N = conv(N1,N2)

1	1.86e+3	2.58e+7	2.73e+8	7.222e+8
s^4	s^3	s^2	s^1	s^0

Bode Plots

Procedure:

The final G(s) is given by;

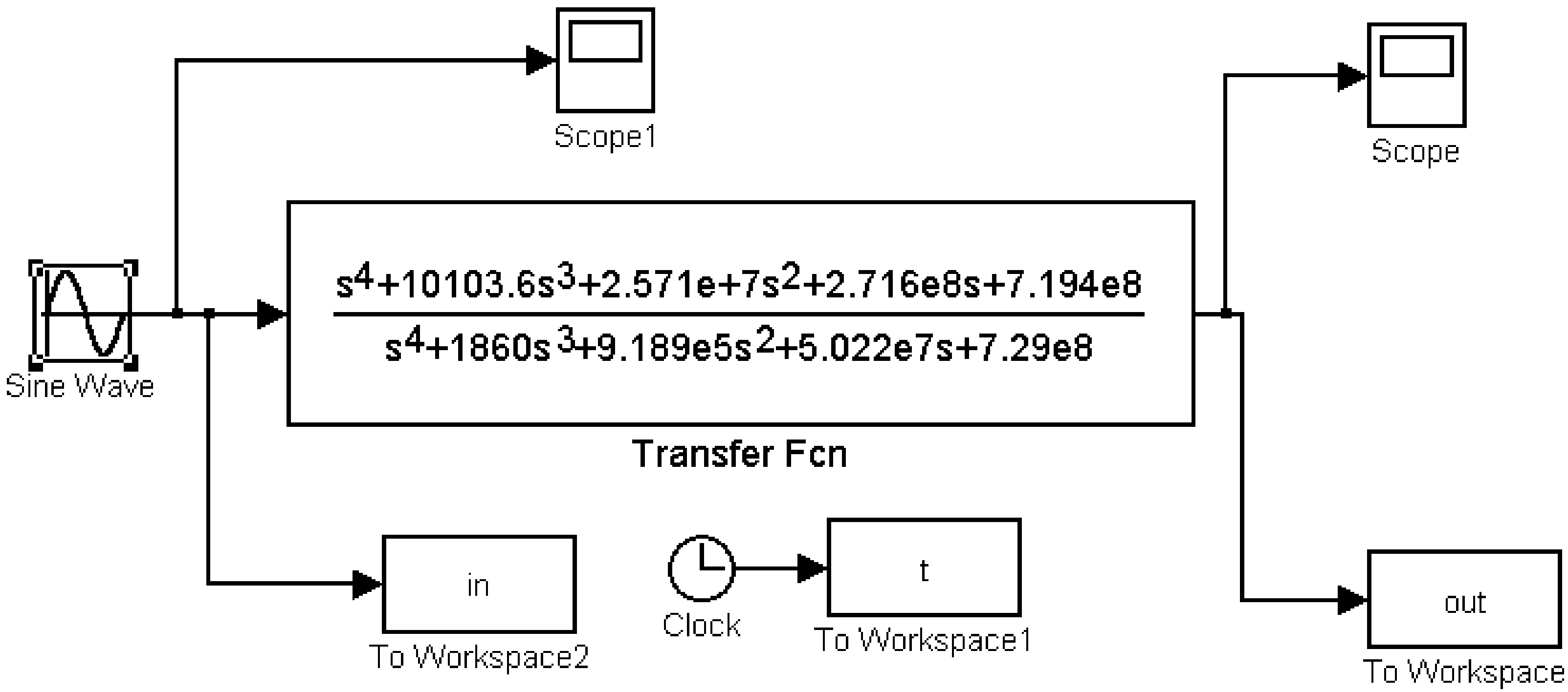
$$G(s)=\frac{(s^4+10130.6s^3+2.571e^8s^2+2.716e^8s+7.194e^8)}{(s^4+1860s^3+9.189e^2s^2+5.022e^7s+7.29e^8)}$$

Testing:

We now want to test the filter. We will check it at ω = 5.3 rad/sec And ω = 164. At ω = 5.3 the filter has a gain of 6 dB or about 2. At ω = 164 the filter has a gain of 30 dB or about 31.6.

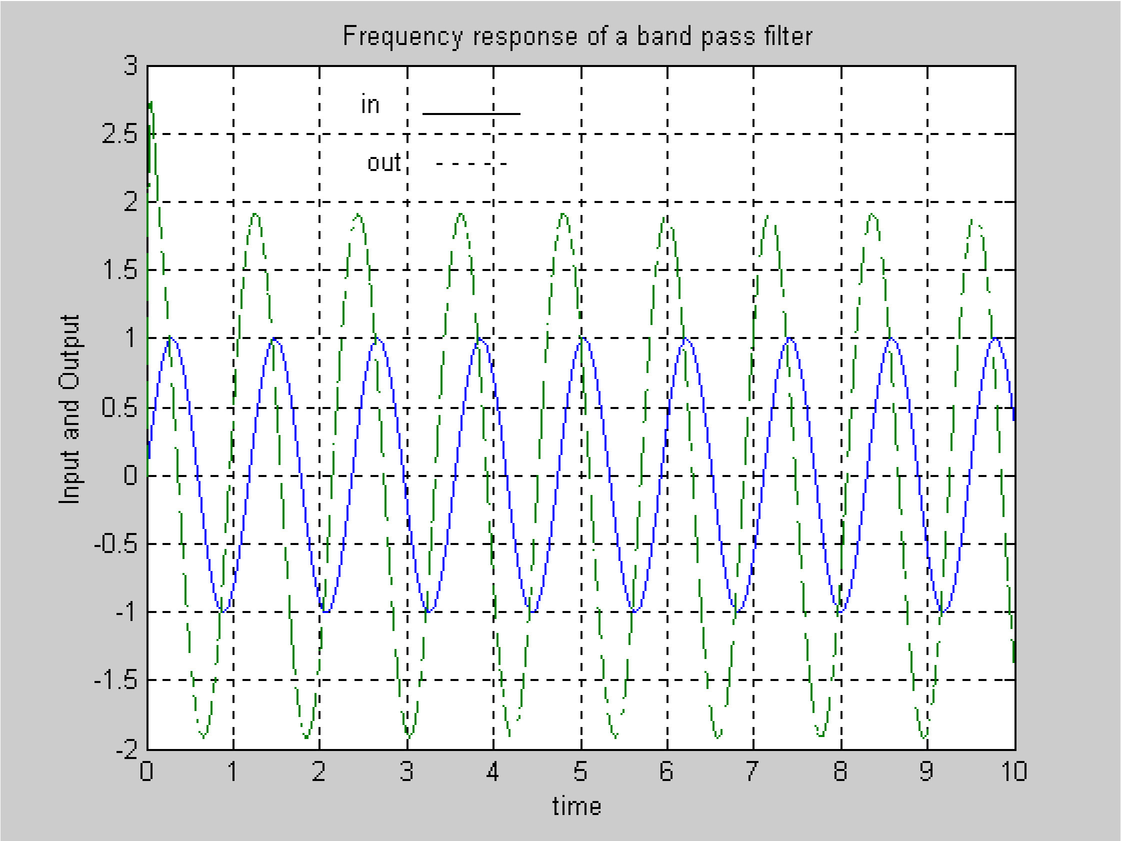
We will check this out using MATLAB and particularly, Simulink.

Matlab (Simulink) Model:



wlg

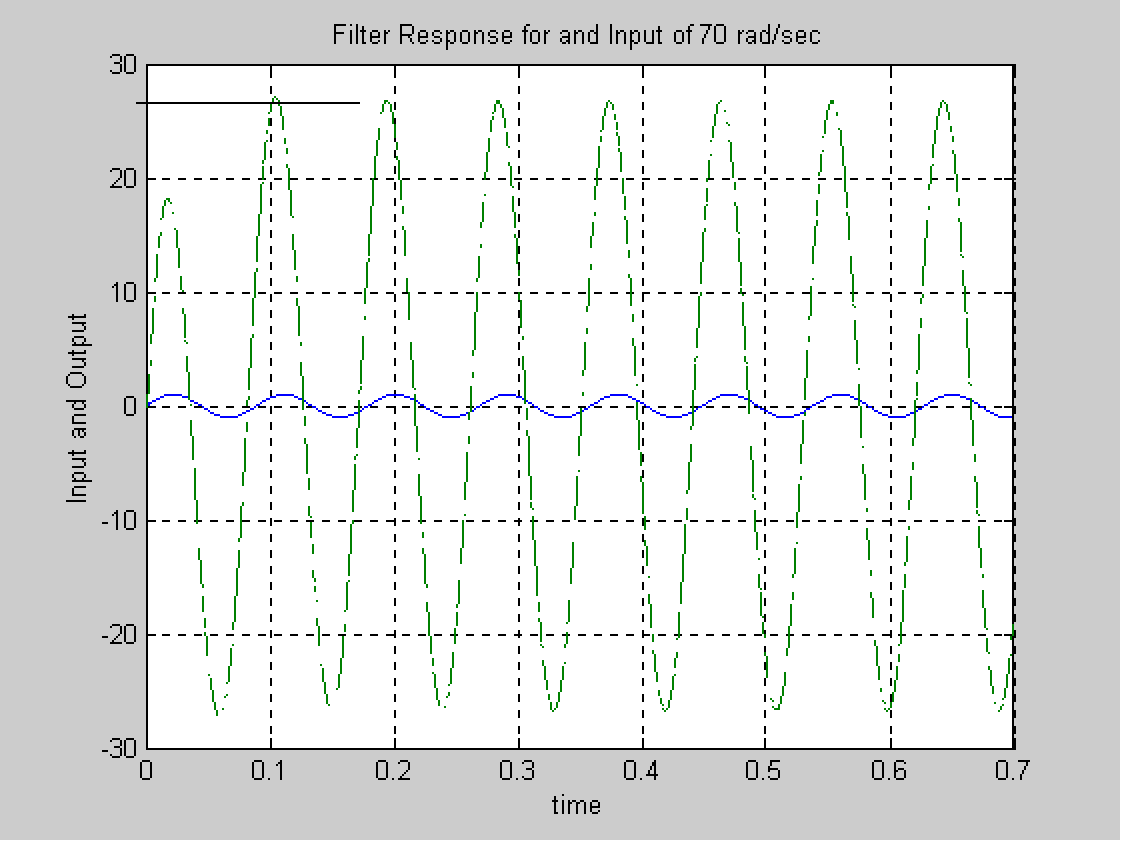
Filter Output at ω = 5.3 rad/sec



Produced from Matlab Simulink

wlg

Filter Output at ω = 70 rad/sec



Produced from Matlab Simulink

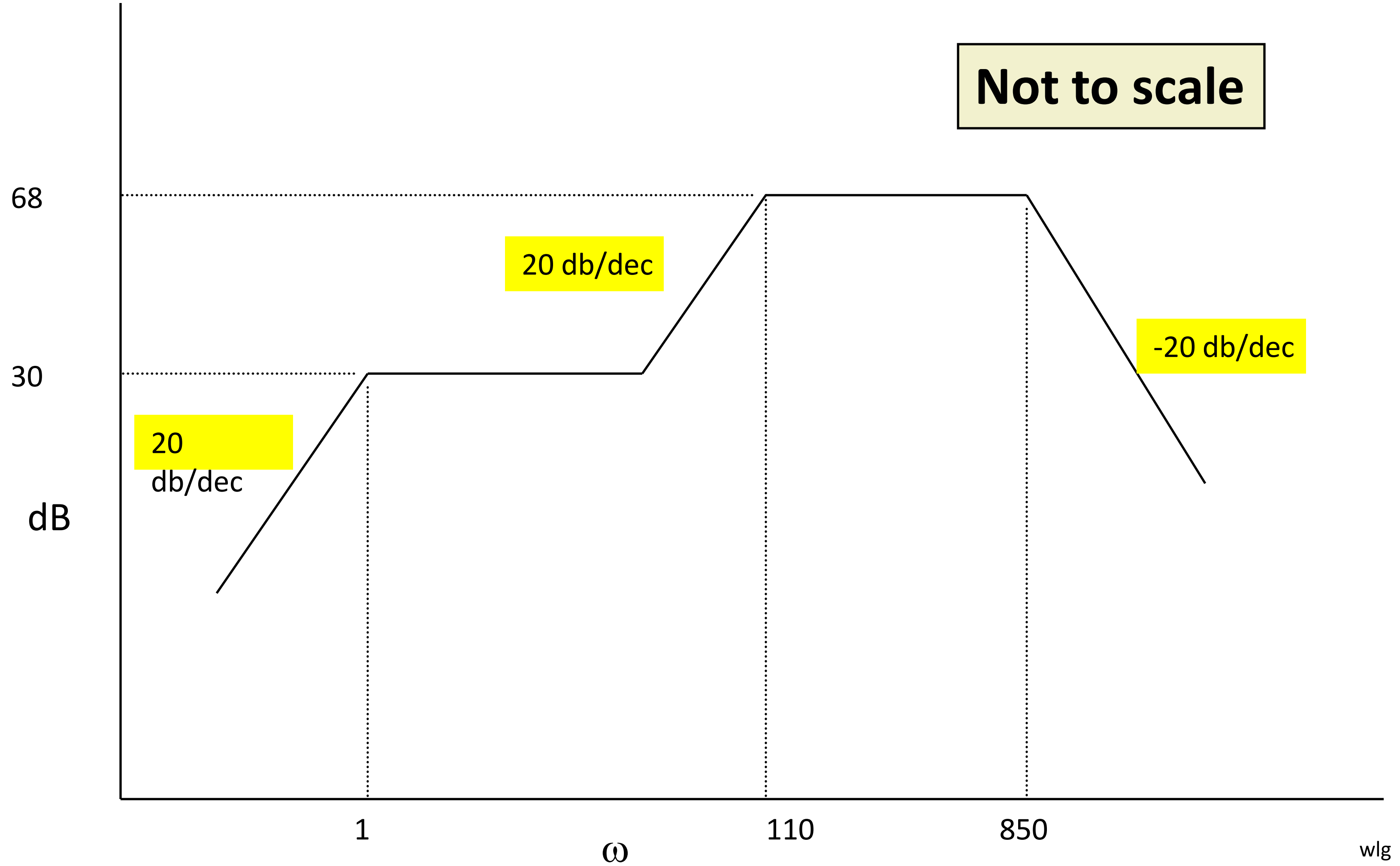
Reverse Bode Plot

Required:

Example 8

From the partial Bode diagram, determine the transfer function

(Assume a minimum phase system)



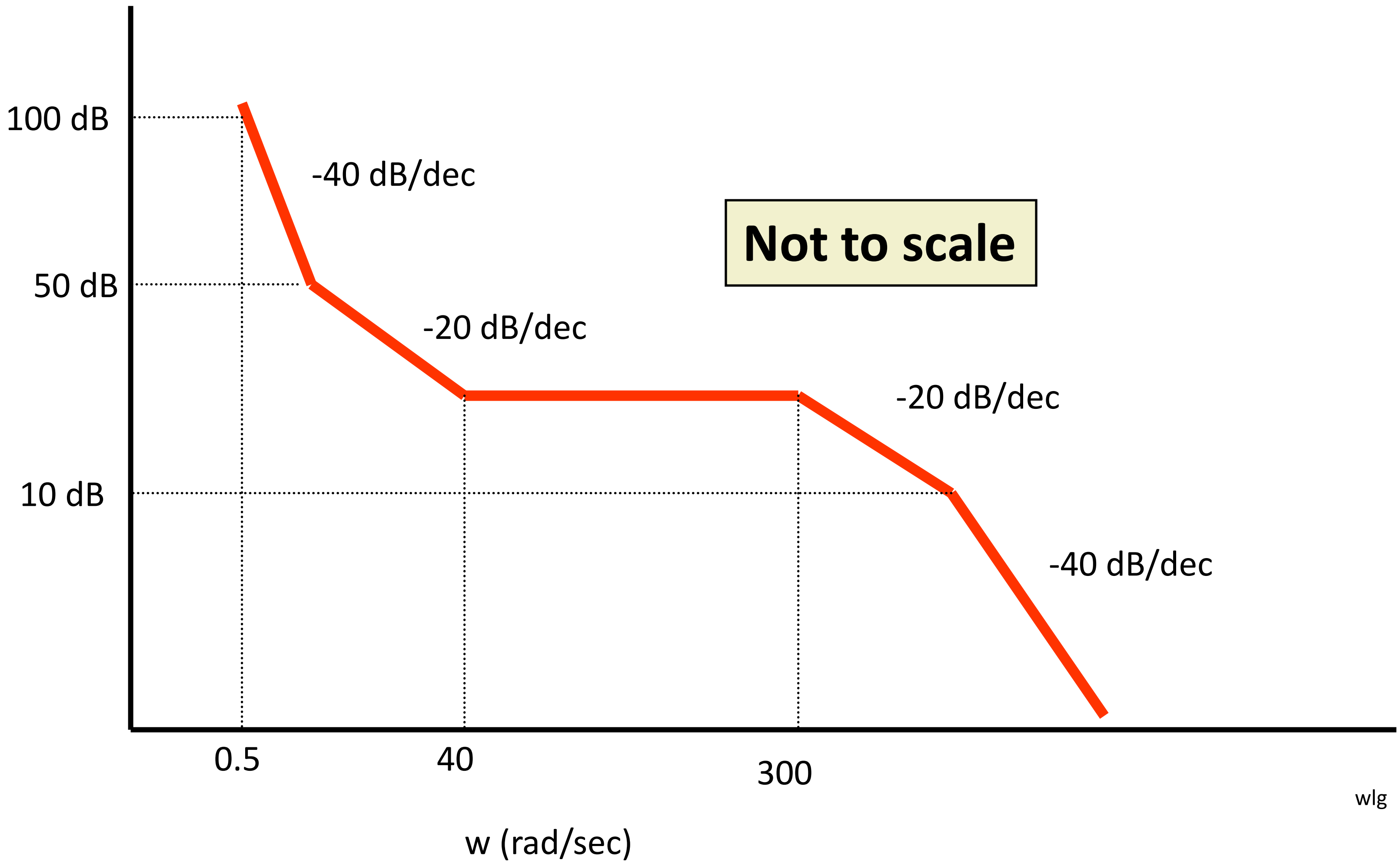
Reverse Bode Plot

Required:

Example 9

From the partial Bode diagram, determine the transfer function

(Assume a minimum phase system)



wlg

Polar Plot

Introduction

The polar plot of sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) verses the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity.

Therefore it is the locus of $|G(j\omega)|\angle G(j\omega)$ as ω is varied from zero to infinity.

As

$$|G(j\omega)|\angle G(j\omega) = Me^{j\phi(\omega)}$$

So it is the plot of vector $Me^{j\phi(\omega)}$ as ω is varied from zero to infinity

Introduction conti…

In the polar plot the magnitude of G(jω) is plotted as the distance from the origin while phase angle is measured from positive real axis.
+ angle is taken for anticlockwise direction.
Polar plot is also known as Nyquist Plot.

Steps to draw Polar Plot

Step 1: Determine the T.F G(s)

Step 2: Put s=jω in the G(s)

Step 3: At ω=0 & ω=∞ find $\left|G(j\omega)\right|$ by $\lim_{\omega\to 0}\left|G(j\omega)\right|$ & $\lim_{\omega\to\infty}\left|G(j\omega)\right|$

Step 4: At ω=0 & ω=∞ find $\angle G(j\omega)$ by $\lim_{\omega\to 0}\angle G(j\omega)$ & $\lim_{\omega\to\infty}\angle G(j\omega)$

Step 5: Rationalize the function G(jω) and separate the real and imaginary parts

Step 6: Put Re [G(jω)]=0, determine the frequency at which plot intersects the Im axis and calculate intersection value by putting the above calculated frequency in G(jω)

Steps to draw Polar Plot conti…

Step 7: Put Im [G(jω)]=0, determine the frequency at which plot intersects the real axis and calculate intersection value by putting the above calculated frequency in G(jω)

Step 8: Sketch the Polar Plot with the help of above information

Polar Plot for Type 0 System

Let $G(s) = \dfrac{K}{(1+sT_1)(1+sT_2)}$

Step 1: Put s=jω

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

Step 2: Taking the limit for magnitude of G(jω)

$$= \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Type 0 system conti…

$$\lim_{\omega\to 0}|G(j\omega)| = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = K$$

$$\lim_{\omega\to \infty}|G(j\omega)| = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = 0$$

Step 3: Taking the limit of the Phase Angle of G(jω)

$$\lim_{\omega\to 0}\angle G(j\omega) = \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = 0$$

$$\lim_{\omega\to \infty}\angle G(j\omega) = \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -180$$

Type 0 system conti…

Step 4: Separate the real and Im part of G(jω)

$$G(j\omega)=\frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}-j\frac{K\omega(T_1+T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}$$

Step 5: Put Re[G(jω)]=0

$$\frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}=0\Rightarrow\omega=\frac{1}{\sqrt{T_1T_2}}\ \&\ \omega=\infty$$

So When

$$\omega=\frac{1}{\sqrt{T_1T_2}}\Rightarrow G(j\omega)=\frac{K\sqrt{T_1T_2}}{T_1+T_2}\angle-90^0$$

$$\&\ \omega=\infty\qquad\Rightarrow G(j\omega)=0\angle-180^0$$

Type 0 system conti…

Step 6: Put Im [G(jω)]=0

$$\frac{K\omega(T_1+T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1T_2}=0\Rightarrow\omega=0\ \&\ \pm\infty$$

$$So\ \ When$$

$$\omega=0\Rightarrow G(j\omega)=K\angle 0^0$$

$$\omega=\infty\Rightarrow G(j\omega)=0\angle 180^0$$

Type 0 system conti…

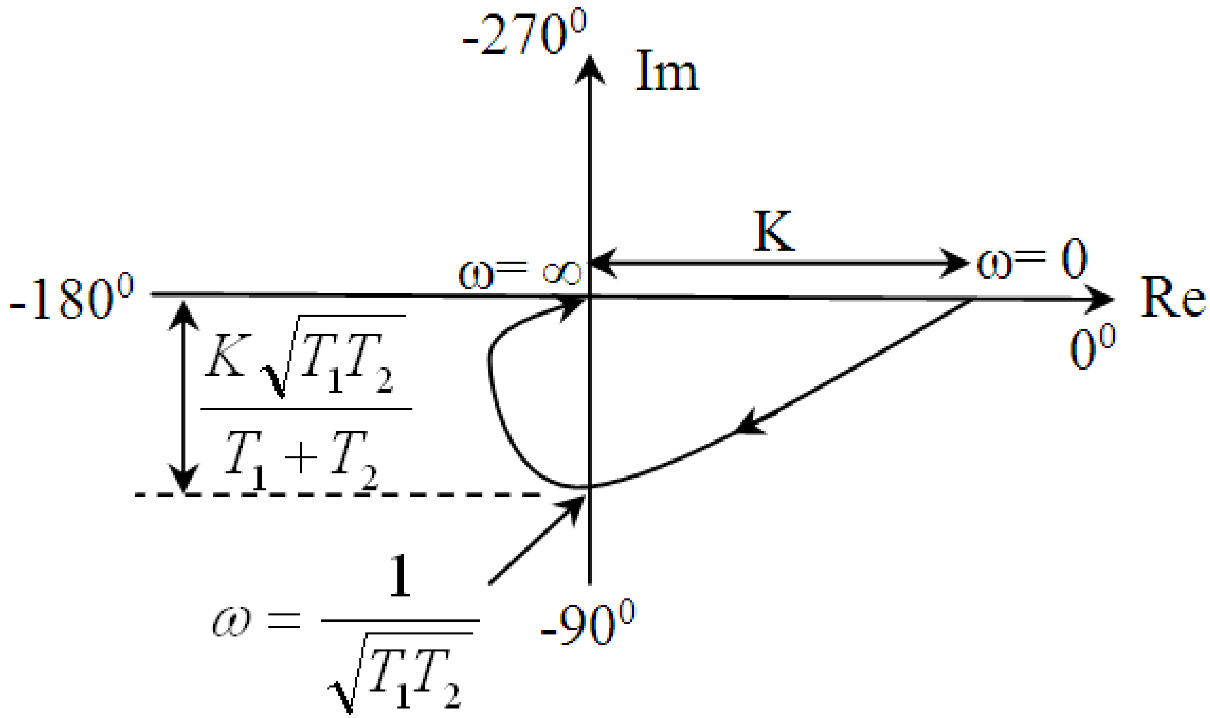


Figure 1: Polar Plot of Type 0 System

Polar Plot for Type 1 System

Let $G(s) = \dfrac{K}{s(1+sT_1)(1+sT_2)}$

Step 1: Put s=jω

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$= \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\,\sqrt{1+j(\omega T_2)^2}} \angle -90^0 - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Type 1 system conti…

Step 2: Taking the limit for magnitude of G(jω)

$$\lim_{\omega\to 0}\left|G(j\omega)\right| = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = \infty$$

$$\lim_{\omega\to \infty}\left|G(j\omega)\right| = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+j(\omega T_2)^2}} = 0$$

Step 3: Taking the limit of the Phase Angle of G(jω)

$$\lim_{\omega\to 0}\angle G(j\omega) = \angle -90^0 - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -90^0$$

$$\lim_{\omega\to \infty}\angle G(j\omega) = \angle -90^0 - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = -270^0$$

Type 1 system conti…

Step 4: Separate the real and Im part of G(jω)

Step 5: Put Re [G(jω)]=0

$$G(j\omega) = \frac{-\omega K(T_1+T_2)}{\omega+\omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)} + j\frac{j(K\omega^2 T_1 T_2 - K)}{\omega+\omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)}$$

$$\frac{-\omega K(T_1+T_2)}{\omega+\omega^3(T_1^2+T_2^2+\omega^2 T_1^2 T_2^2)} = 0 \Rightarrow \omega = \infty$$

$$So\ at$$

$$\omega = \infty \qquad \Rightarrow G(j\omega) = 0\angle -270^0$$

Type 1 system conti…

Step 6: Put Im [G(jω)]=0

$$\frac{j(K\omega^2 T_1 T_2 - K)}{\omega + \omega^3 (T_1^{\,2} + T_2^{\,2} + \omega^2 T_1^{\,2} T_2^{\,2})} = 0 \Rightarrow \omega = \frac{1}{\sqrt{T_1 T_2}} \;\; \& \; \omega = \pm\infty$$

So When

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \Rightarrow \quad G(j\omega) = -\frac{K\sqrt{T_1 T_2}}{T_1 + T_2} \angle 0^0$$

$$\omega = \infty \Rightarrow \qquad G(j\omega) = \infty \angle 0^0$$

Type 1 system conti…

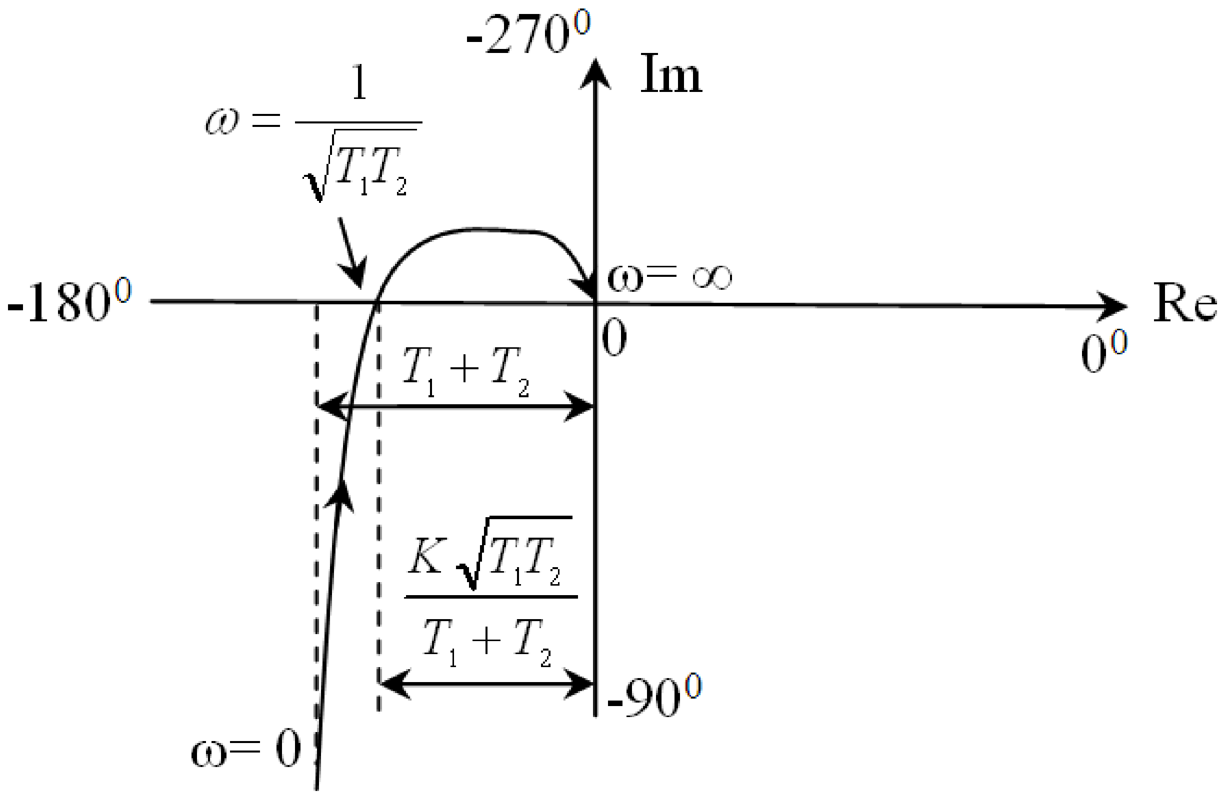


Figure 2: Polar Plot of Type 1 System

Polar Plot for Type 2 System

Let

Similar to above

$$G(s) = \frac{K}{s^2(1+sT_1)(1+sT_2)}$$

Type 2 system conti…

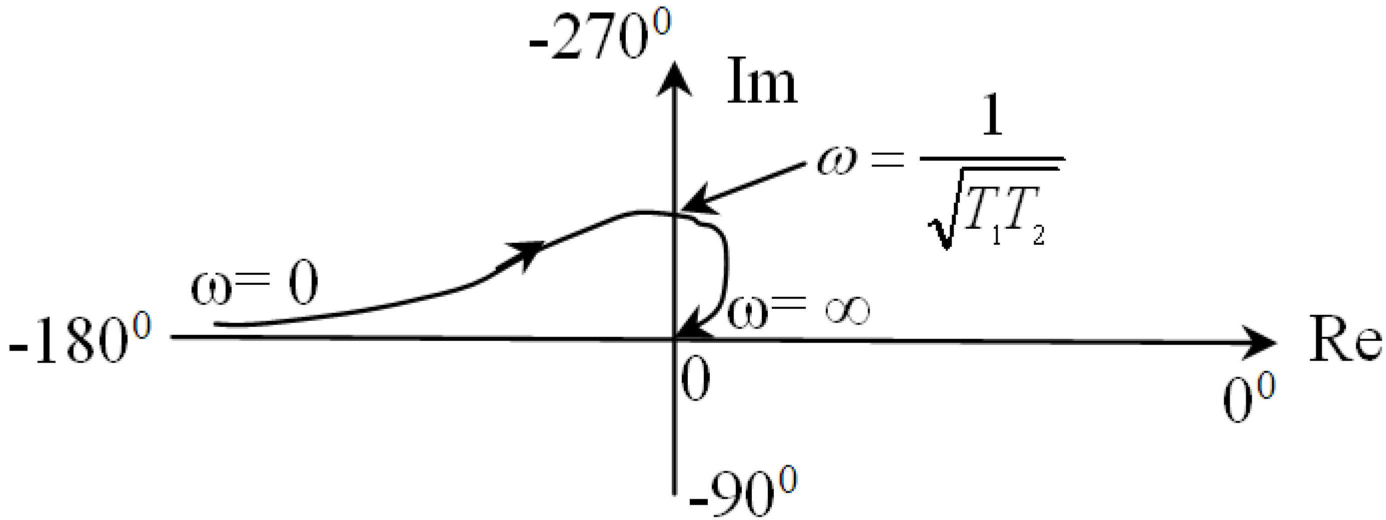


Figure 3: Polar Plot of Type 2 System

Note: Introduction of additional pole in denominator contributes a constant -180^0 to the angle of $G(j\omega)$ for all frequencies. See the figure 1, 2 & 3

Figure 1+(-180^0 Rotation)=figure 2

Figure 2+(-180^0 Rotation)=figure 3

Ex: Sketch the polar plot for G(s)=20/s(s+1)(s+2)

Solution:

Step 1: Put s=jω

$$G(j\omega) = \frac{20}{j\omega(j\omega+1)(j\omega+2)}$$

$$= \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}} \angle -90^0 - \tan^{-1}\omega - \tan^{-1}\omega/2$$

Step 2: Taking the limit for magnitude of G(jω)

$$\lim_{\omega \to 0} |G(j\omega)| = \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}} = \infty$$

$$\lim_{\omega \to \infty} |G(j\omega)| = \frac{20}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}} = 0$$

Step 3: Taking the limit of the Phase Angle of G(jω)

$$\lim_{\omega \to 0} \angle G(j\omega) = \angle -90^0 - \tan^{-1}\omega - \tan^{-1}\omega/2 = -90^0$$

$$\lim_{\omega \to \infty} \angle G(j\omega) = \angle -90^0 - \tan^{-1}\omega - \tan^{-1}\omega/2 = -270^0$$

Step 4: Separate the real and Im part of G(jω)

$$G(j\omega) = \frac{-60\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} + j\frac{j20(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$

$$\frac{-60\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = 0 \Rightarrow \omega = \infty$$

$$So\ at$$

$$\omega = \infty \qquad \Rightarrow G(j\omega) = 0\angle -270^0$$

Step 6: Put Im [G(jω)]=0

$$\frac{j20(\omega^3 - 2\omega)}{(\omega^4 + \omega^2)(4 + \omega^2)} = 0 \Rightarrow \omega = \pm\sqrt{2} \;\; \& \;\omega = \pm\infty$$

$$So \;\; for \;\; positive\, value\, of \;\; \omega$$

$$\omega = \sqrt{2} \Rightarrow \quad G(j\omega) = -\frac{10}{3}\angle 0^0$$

$$\omega = \infty \Rightarrow \qquad G(j\omega) = 0\angle 0^0$$

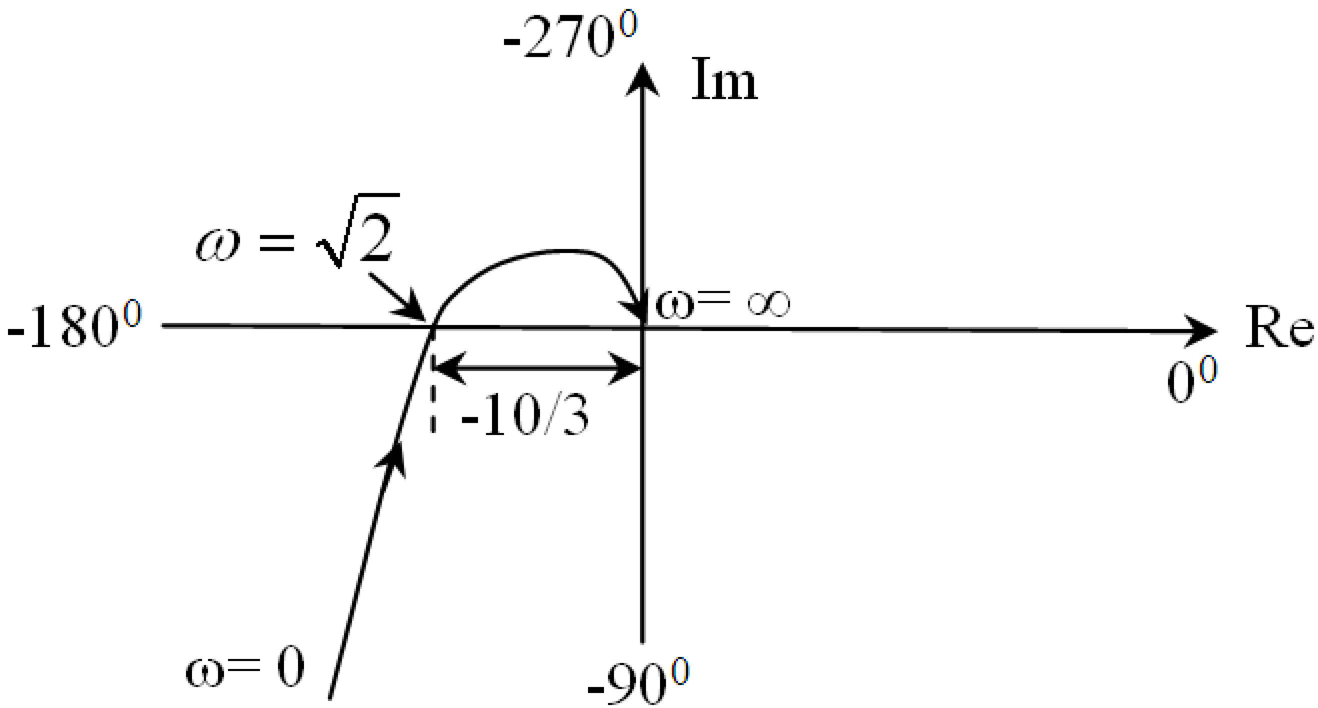
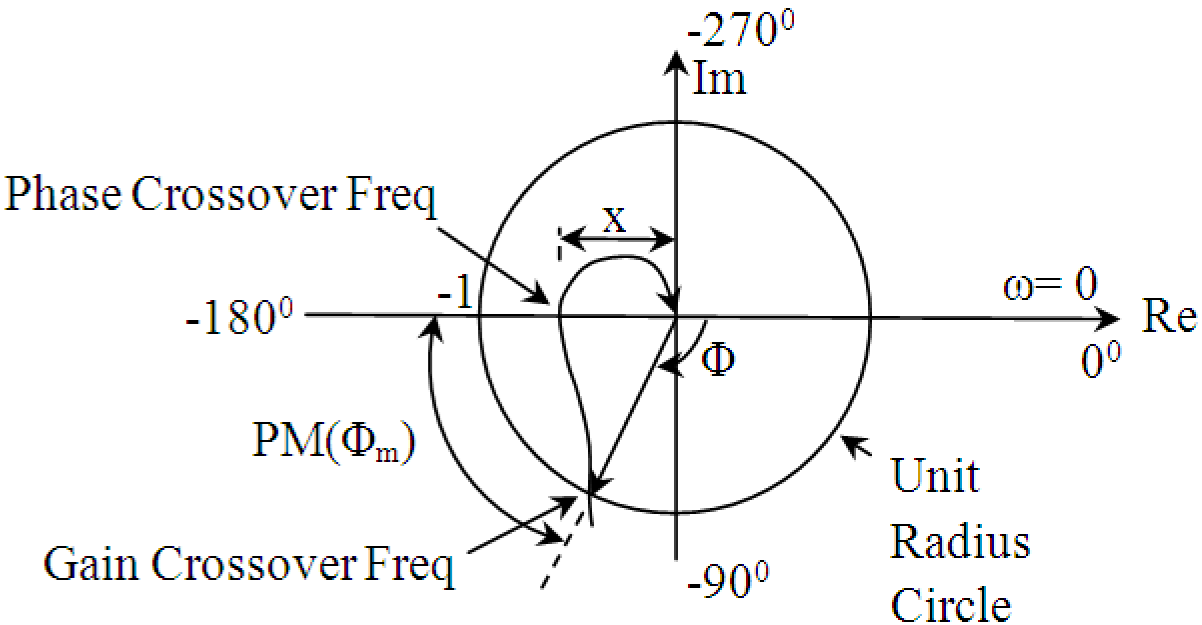


Figure 4: Polar Plot G(s)=20/s(s+1)(s+2)

Gain Margin, Phase Margin & Stability



Phase Crossover Frequency (ω_p) : The frequency where a polar plot intersects the –ve real axis is called phase crossover frequency

Gain Crossover Frequency (ω_g) : The frequency where a polar plot intersects the unit circle is called gain crossover frequency

So at ω_g

$$\left|G(j\omega)\right| = Unity$$

Phase Margin (PM):

Phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability (marginally stabile)

$$\Phi_m = 180^0 + \Phi$$

Where

$$\Phi = \angle G(j\omega_g)$$

if $\Phi_m > 0 \Rightarrow +PM$ (Stable System)

if $\Phi_m < 0 \Rightarrow -PM$ (Unstable System)

Gain Margin (GM):

The gain margin is the reciprocal of magnitude $|G(j\omega)|$ at the frequency at which the phase angle is -180^0.

In terms of dB

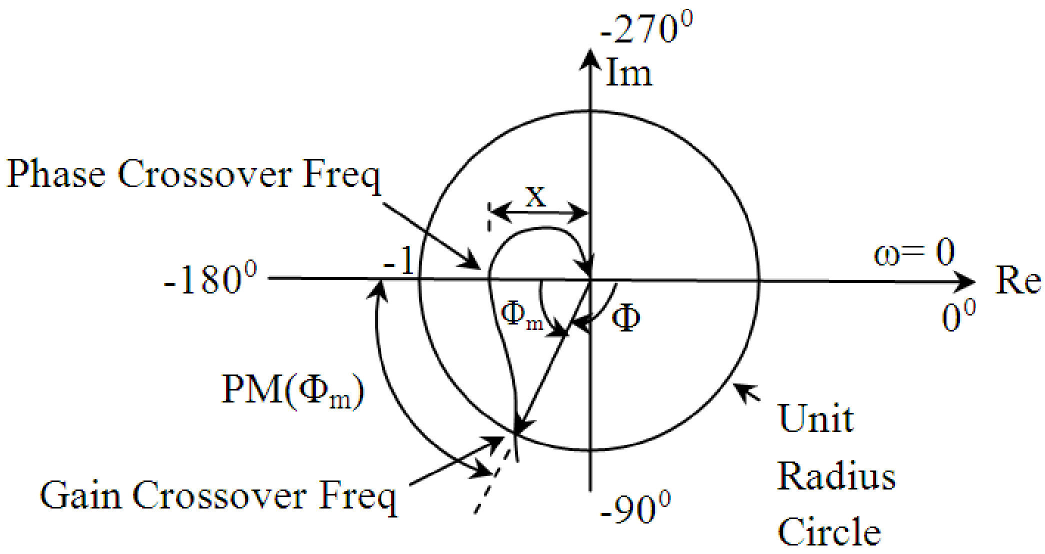
$$GM = \frac{1}{|G(jwc)|} = \frac{1}{x}$$

$$GM\ in\ dB = 20\log_{10}\frac{1}{|G(jwc)|} = -20\log_{10}|G(jwc)| = -20\log_{10}(x)$$

Stability

Stable: If critical point (-1+j0) is within the plot as shown, Both GM & PM are +ve

$GM=20\log_{10}(1$

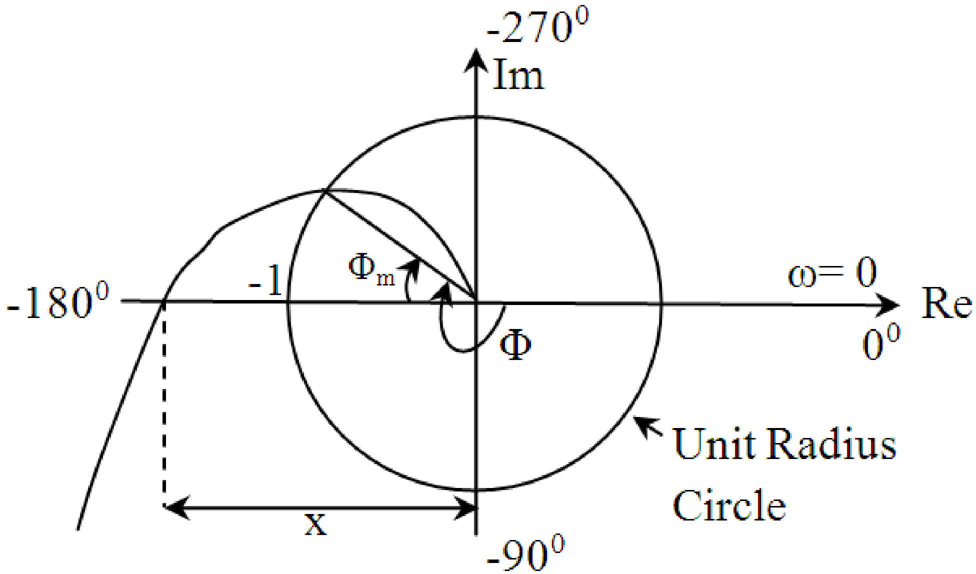


Stable system: +GM, +PM

Unstable: If critical point (-1+j0) is outside the plot as shown, Both GM & PM are -ve

GM=$20\log_{10}$(1

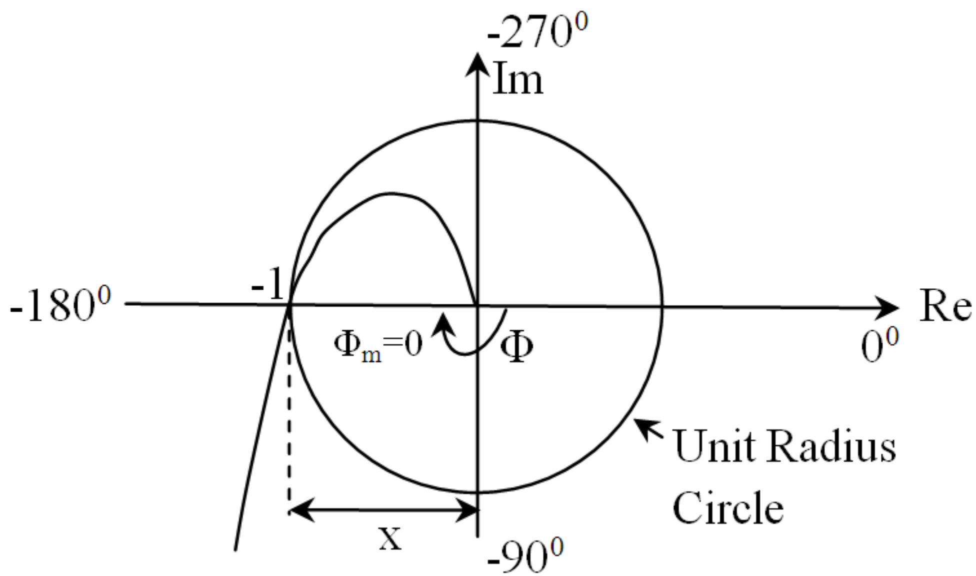
/



Unstable system: -GM, -PM

Marginally Stable System: If critical point (-1+j0) is on the plot as shown, Both GM & PM are ZERO

$GM=20\log_{10}(1$



Marginally stable system: GM=0 dB, PM=0^0

MATLAB Margin

```
>> [Gm,Pm,Wcg,Wcp] = margin(g)
Gm =
    2.5000
Pm =
   22.5359
Wcg =
   10.0000 (phase crossover freq.)
Wcp =
    6.0783 (gain crossover freq.)
```

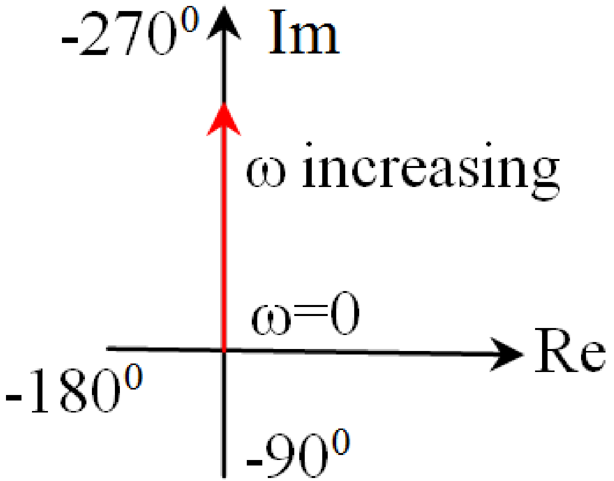
$$G(s) = \frac{10}{s(s/5+1)(s/20+1)}$$

Inverse Polar Plot

The inverse polar plot of G(jω) is a graph of 1/G(jω) as a function of ω.
Ex: if G(jω) =1/jω then 1/G(jω)=jω

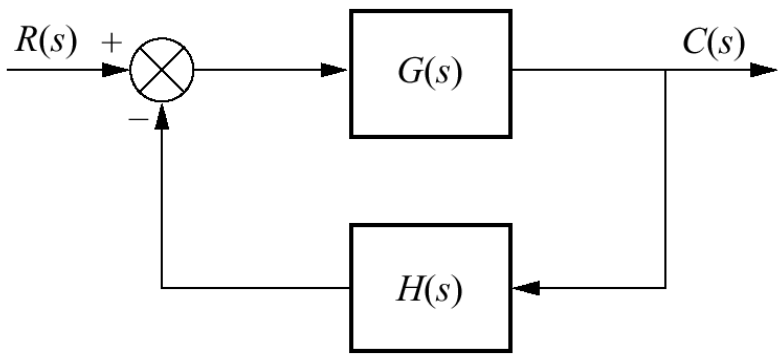
$$\lim_{\omega \to 0} \left| G(j\omega)^{-1} \right| = 0$$

$$\lim_{\omega \to \infty} \left| G(j\omega)^{-1} \right| = \infty$$



Inverse polar Plot of 1/jω

Knowledge Before Studying Nyquist Criterion



$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

unstable if there is any pole on RHP (right half plane)

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

Open-loop system:

$$G(s)H(s) = \frac{N_G(s)}{D_G(s)} \frac{N_H(s)}{D_H(s)}$$

Characteristic equation:

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

poles of G(s)H(s) and 1+G(s)H(s) are the same

Closed-loop system:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G(s) D_H(s)}{D_G(s) D_H(s) + N_G(s) N_H(s)}$$

zero of 1+G(s)H(s) is pole of T(s)

$$G(s)H(s) = \frac{(s-1)(s-2)(s-3)(s-4)}{(s-5)(s-6)(s-7)(s-8)}$$

$$G(s)H(s)$$

Zero – 1,2,3,4

Poles – 5,6,7,8

$$1+G(s)H(s)$$

Zero – a,b,c,d

Poles – 5,6,7,8

$$\frac{G(s)}{1+G(s)H(s)}$$

Zero – ?,?,?,?

Poles – a,b,c,d

To know stability, we have to know a,b,c,d

Stability from Nyquist plot

From a Nyquist plot, we can tell a number of closed-loop poles on the right half plane.

- If there is any closed-loop pole on the right half plane, the system goes unstable.
- If there is no closed-loop pole on the right half plane, the system is stable.

Nyquist Criterion

Nyquist plot is a plot used to verify stability of the system.

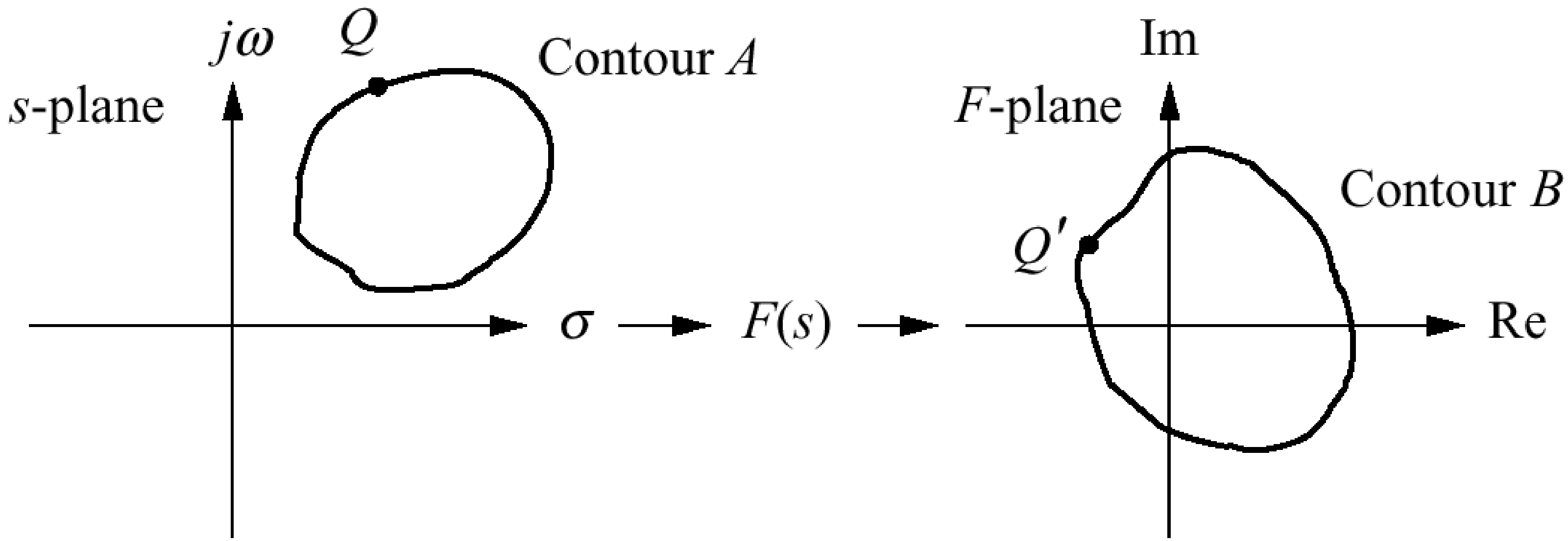
mapping contour

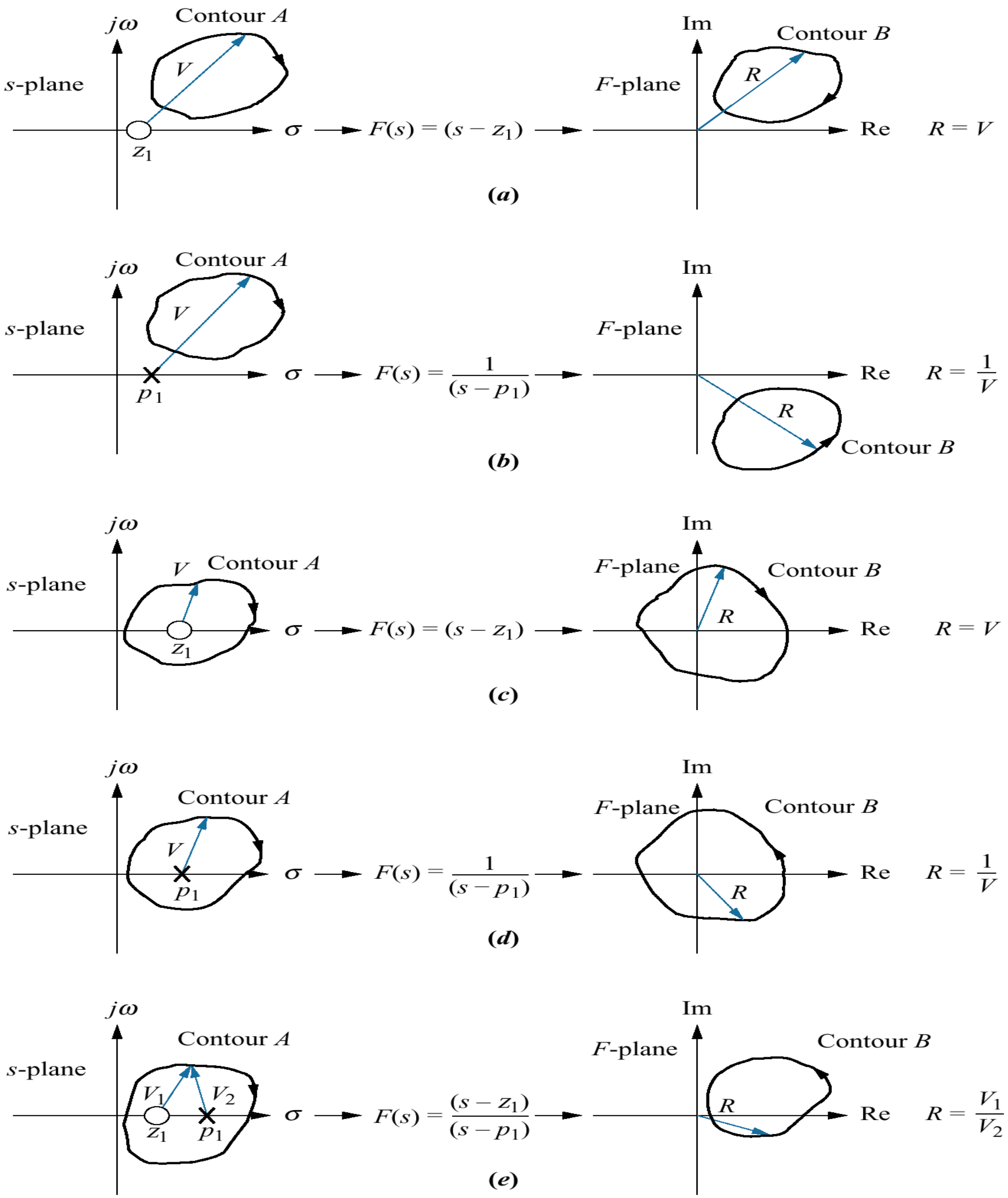
function

$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

mapping all points (contour) from one plane to another by function F(s).

$$F(s)=\frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}$$





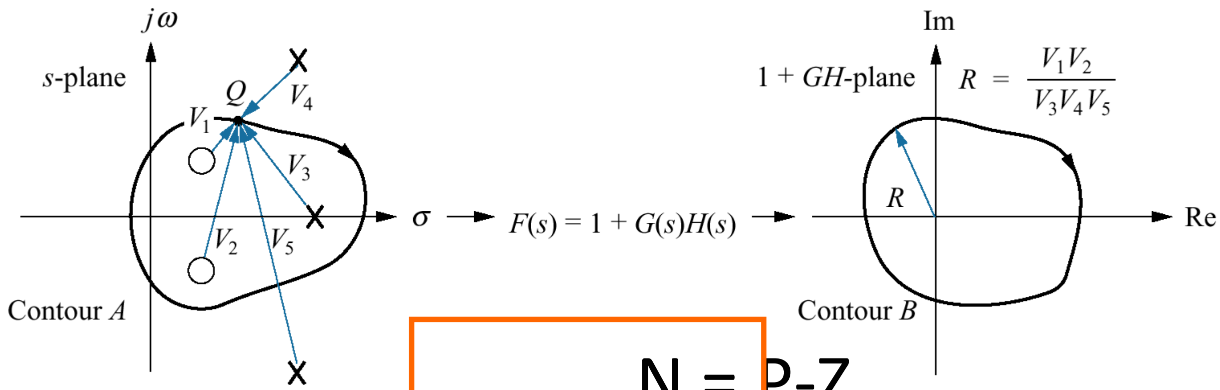
Pole/zero inside the contour has 360 deg. angular change.

Pole/zero outside contour has 0 deg. angular change.

Move clockwise around contour, zero inside yields rotation in clockwise, pole inside yields rotation in counterclockwise

Characteristic equation

$$F(s) = 1 + G(s)H(s)$$



N = P-Z

N = # of counterclockwise direction about the origin

P = # of poles of characteristic equation inside contour

= # of poles of open-loop system

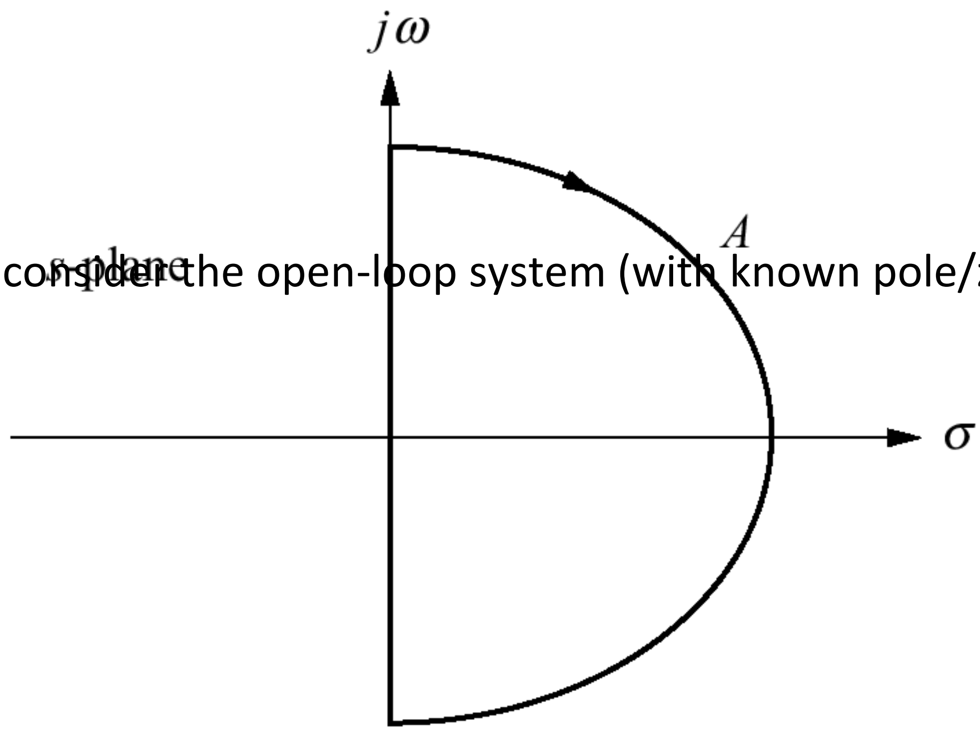
z = # of zeros of characteristic equation inside contour

= # of poles of closed-loop system

Z = P-N

Characteristic equation

Increase size of the contour to cover the right half plane



More convenient to consider the open-loop system (with known pole/zero)

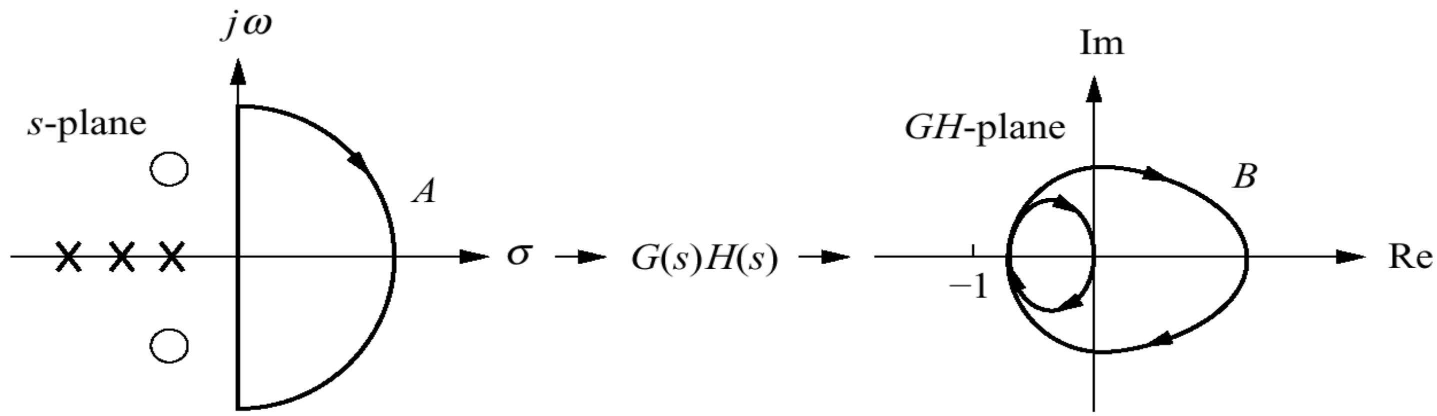
Nyquist diagram of $G(s)H(s)$

'Open-loop system'

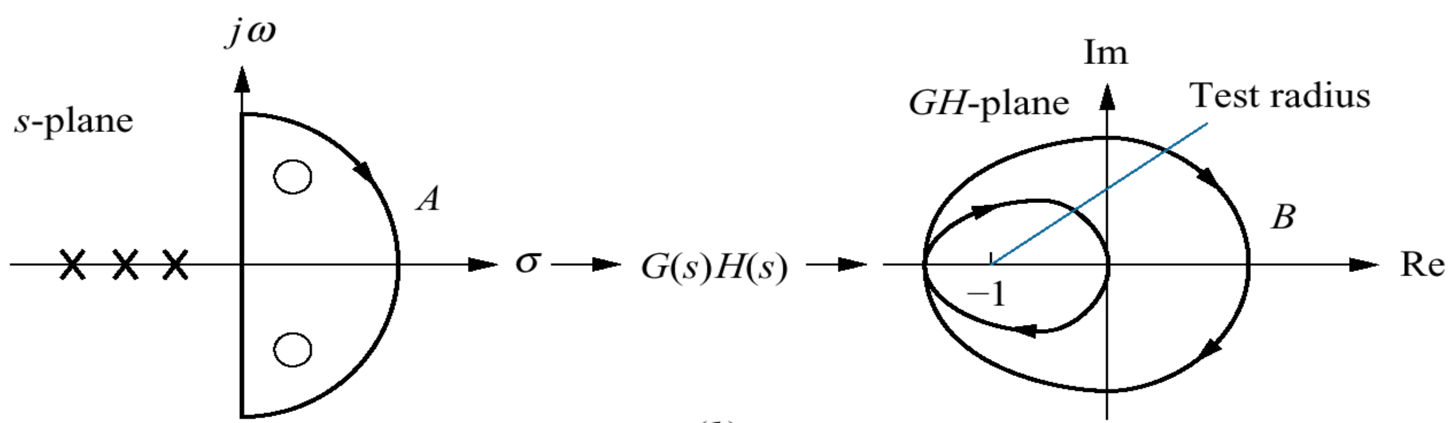
Mapping from characteristic equ. to open-loop system by shifting to the left one step

Z = P-N

Z = # of closed-loop poles inside the right half plane
P = # of open-loop poles inside the right half plane
N = # of counterclockwise revolutions around -1



(*a*)



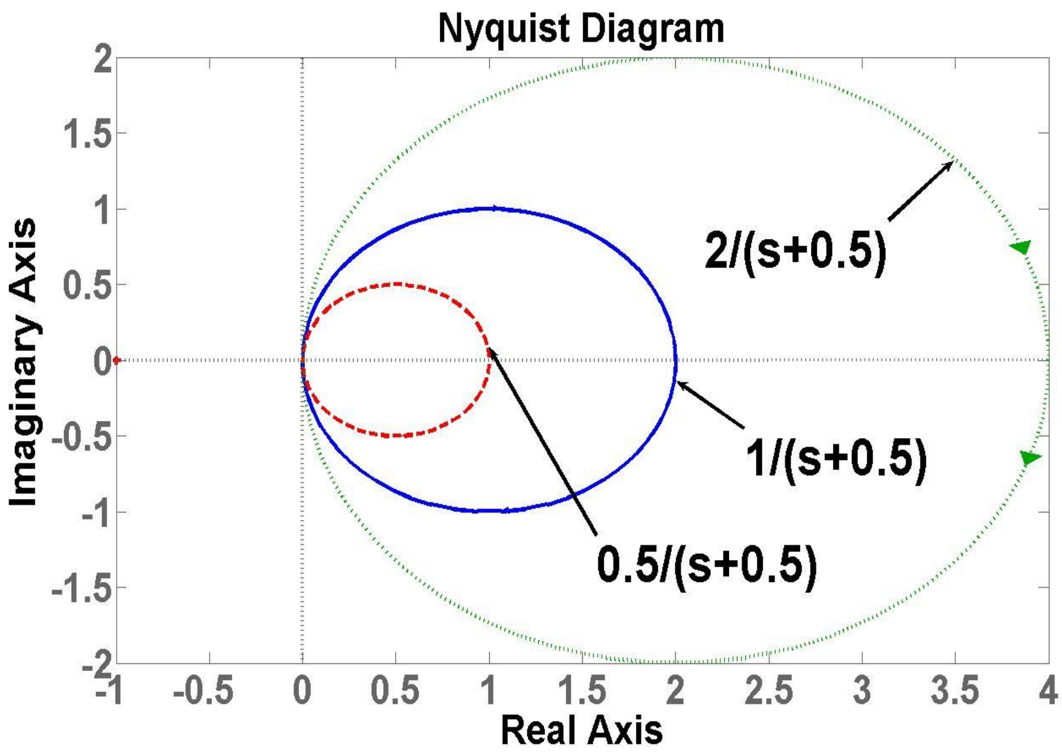
(*b*)

○ = zeros of $1 + G(s)H(s)$
= poles of closed-loop system
Location not known

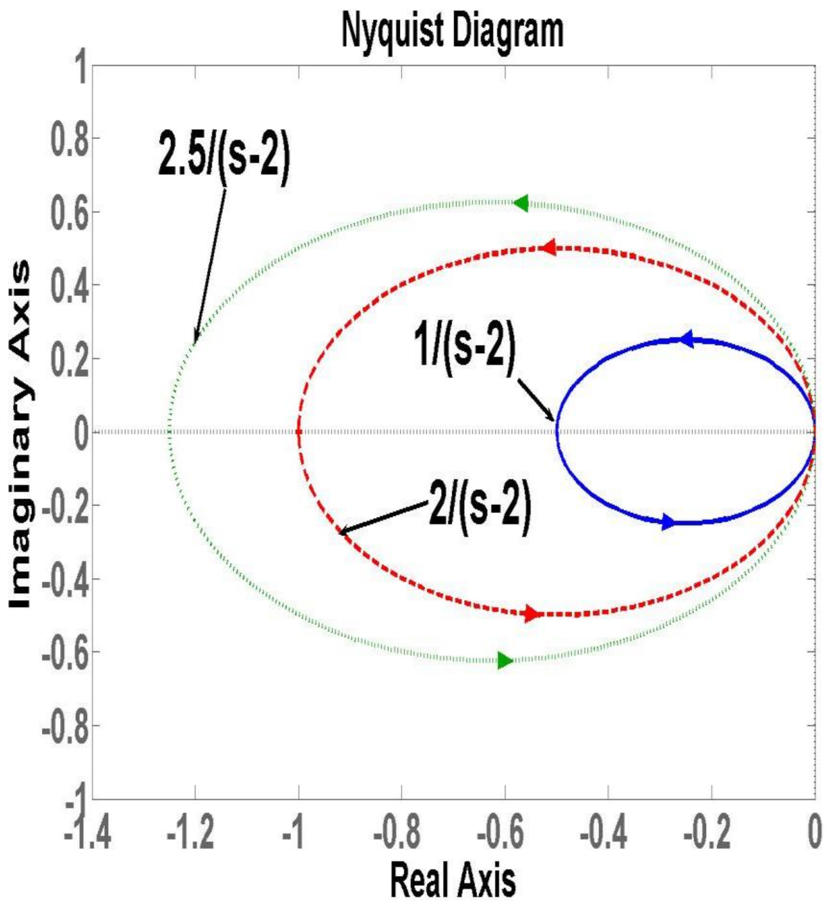
✕ = poles of $1 + G(s)H(s)$
= poles of $G(s)H(s)$
Location is known

Properties of Nyquist plot

If there is a gain, K, in front of open-loop transfer function, the Nyquist plot will expand by a factor of K.



Nyquist plot example



Open loop system has pole at 2

$$G(s) = \frac{1}{s-2}$$

Closed-loop system has pole at 1

$$\frac{G(s)}{1+G(S)} = \frac{1}{(s-1)}$$

If we multiply the open-loop with a gain, K, then we can move the closed-loop pole's position to the left-half plane

Nyquist plot example (cont.)

New look of open-loop system:

$$G(s) = \frac{K}{s-2}$$

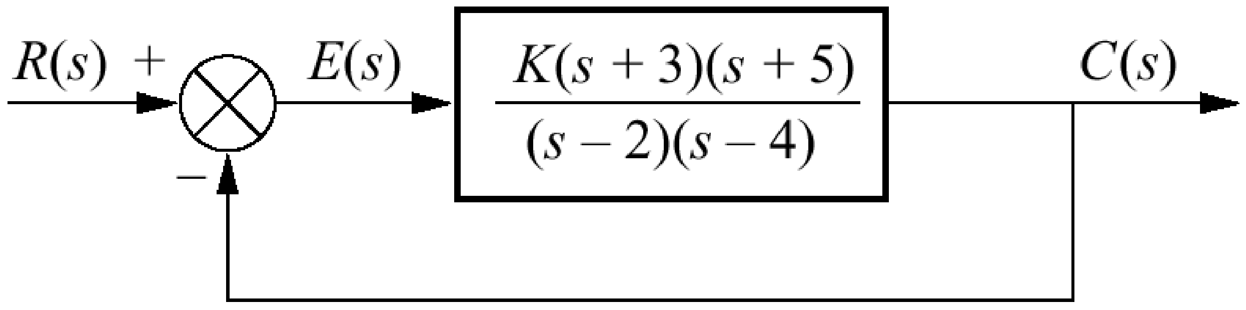
Corresponding closed-loop system:

$$\frac{G(s)}{1+G(s)} = \frac{K}{s+(K-2)}$$

Evaluate value of K for stability

$$\boxed{K \geq 2}$$

Adjusting an open-loop gain to guarantee stability



(a)

Step I: sketch a Nyquist Diagram
Step II: find a range of K that makes the system stable!

How to make a Nyquist plot?

Easy way by Matlab

Nyquist: ‘nyquist’

Bode: ‘bode’

Step I: make a Nyquist plot

Starts from an open-loop transfer function (set K=1)

Set $s = j\omega$ and find frequency response

At dc, $\omega = 0 \rightarrow s = 0$

Find ω at which the imaginary part equals zero

$$G(s)H(s) = \frac{(s+3)(s+5)}{(s-2)(s-4)} = \frac{s^2+8s+15}{s^2-6s+8}$$

$$G(j\omega)H(j\omega) = \frac{-\omega^2+8j\omega+15}{-\omega^2-6j\omega+8} = \frac{(15-\omega^2)+8j\omega}{(8-\omega^2)-6j\omega}$$

$$= \frac{(15-\omega^2)+8j\omega}{(8-\omega^2)-6j\omega} \times \frac{(8-\omega^2)+6j\omega}{(8-\omega^2)+6j\omega}$$

$$= \frac{(15-\omega^2)(8-\omega^2)-48\omega^2 + j(154\omega-14\omega^3)}{(8-\omega^2)^2+6^2\omega^2}$$

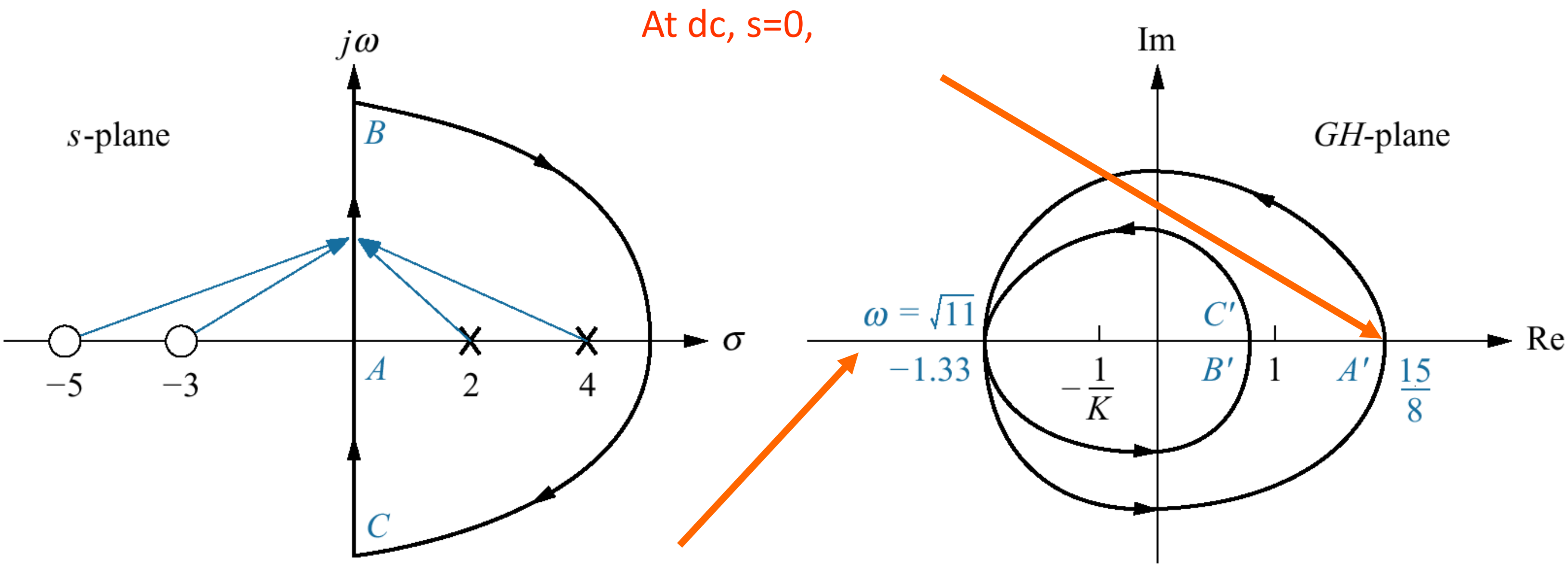
$$\omega = 0, \sqrt{11}$$

Need the imaginary term = 0,

Substitute $\omega = \sqrt{11}$ back in to the transfer function
And get

$$G(s) = -1.33$$

$$\frac{(15-11)(8-11)-48(11)}{(8-11)^2+6^2(11)} = \frac{-540}{412} = -1.31$$


At dc, s=0,
jω
s-plane
B
A
C
−5
−3
2
4
σ
Im
GH-plane
ω = √11
−1.33
−1/K
C′
B′
1
A′
15/8
Re
At imaginary part=0

(b)

(c)

Step II: satisfying stability condition

P = 2, N has to be 2 to guarantee stability
Marginally stable if the plot intersects -1
For stability, 1.33K has to be greater than 1

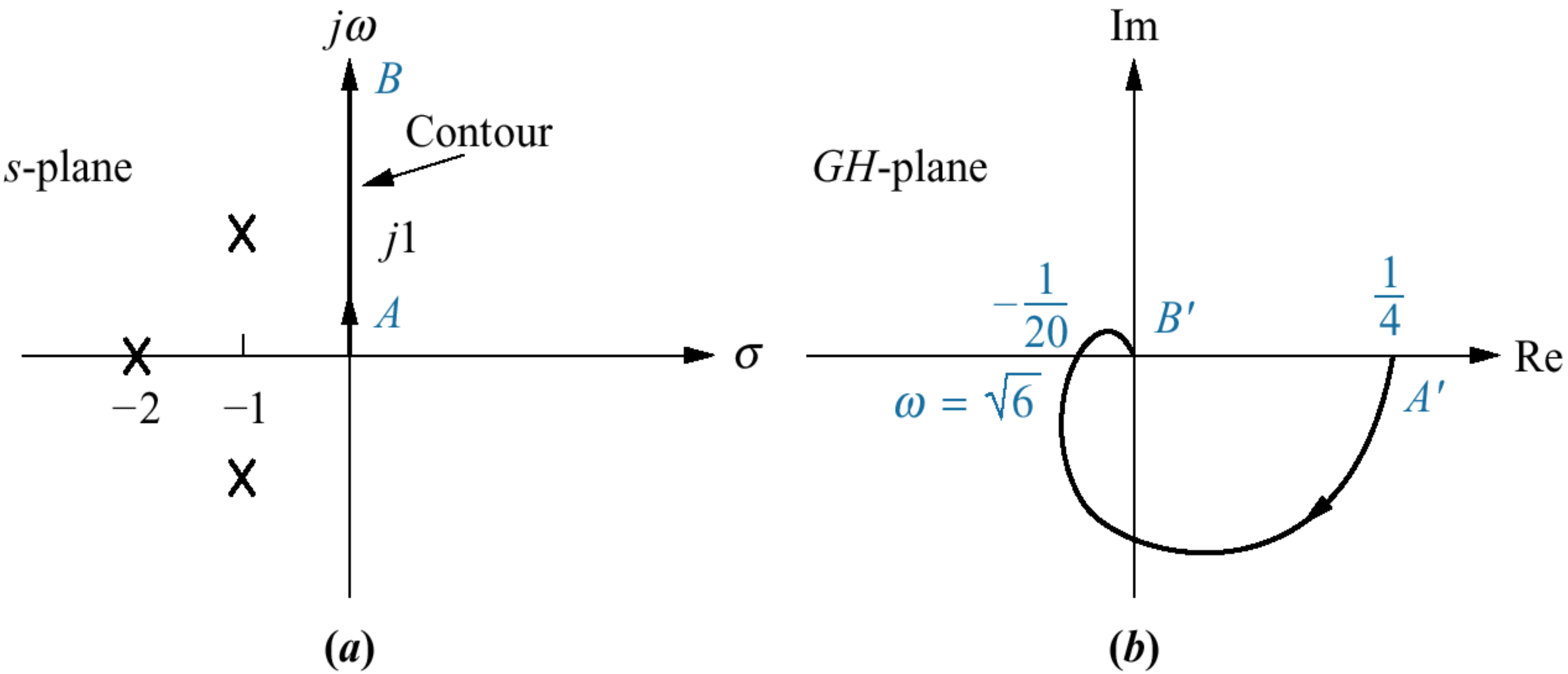
$$K > 1/1.33$$

$$\text{or} \quad K > 0.75$$

Example

Evaluate a range of K that makes the system stable

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s + 2)}$$



Step I: find frequency at which imaginary part = 0

Set $s = j\omega$

$$G(j\omega) = \frac{K}{((j\omega)^2 + 2j\omega + 2)(j\omega + 2)}$$

$$= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

At $\omega = 0, \sqrt{6}$ the imaginary part = 0

Plug $\omega = \sqrt{6}$ back in the transfer function and get G = -0.05

Step II: consider stability condition

P = 0, N has to be 0 to guarantee stability
Marginally stable if the plot intersects -1
For stability, 0.05K has to be less than 1

$$K < 1/0.05$$

or $$K < 20$$

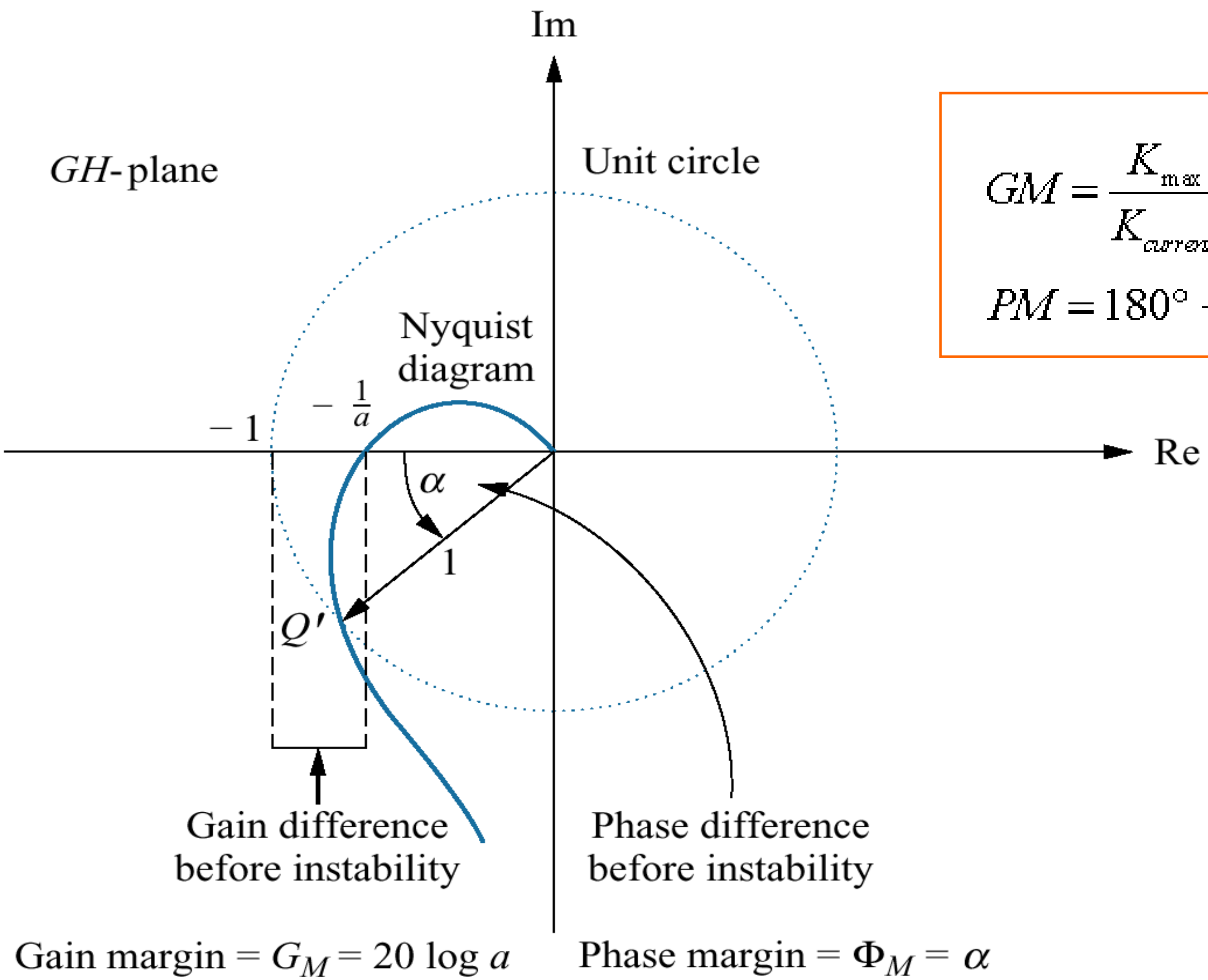
Gain Margin and Phase Margin

Gain margin is the change in open-loop gain (in dB), required at 180 of phase shift to make the closed-loop system unstable.

Phase margin is the change in open-loop phase shift, required at unity gain to make the closed-loop system unstable.

GM/PM tells how much system can tolerate before going unstable!!!

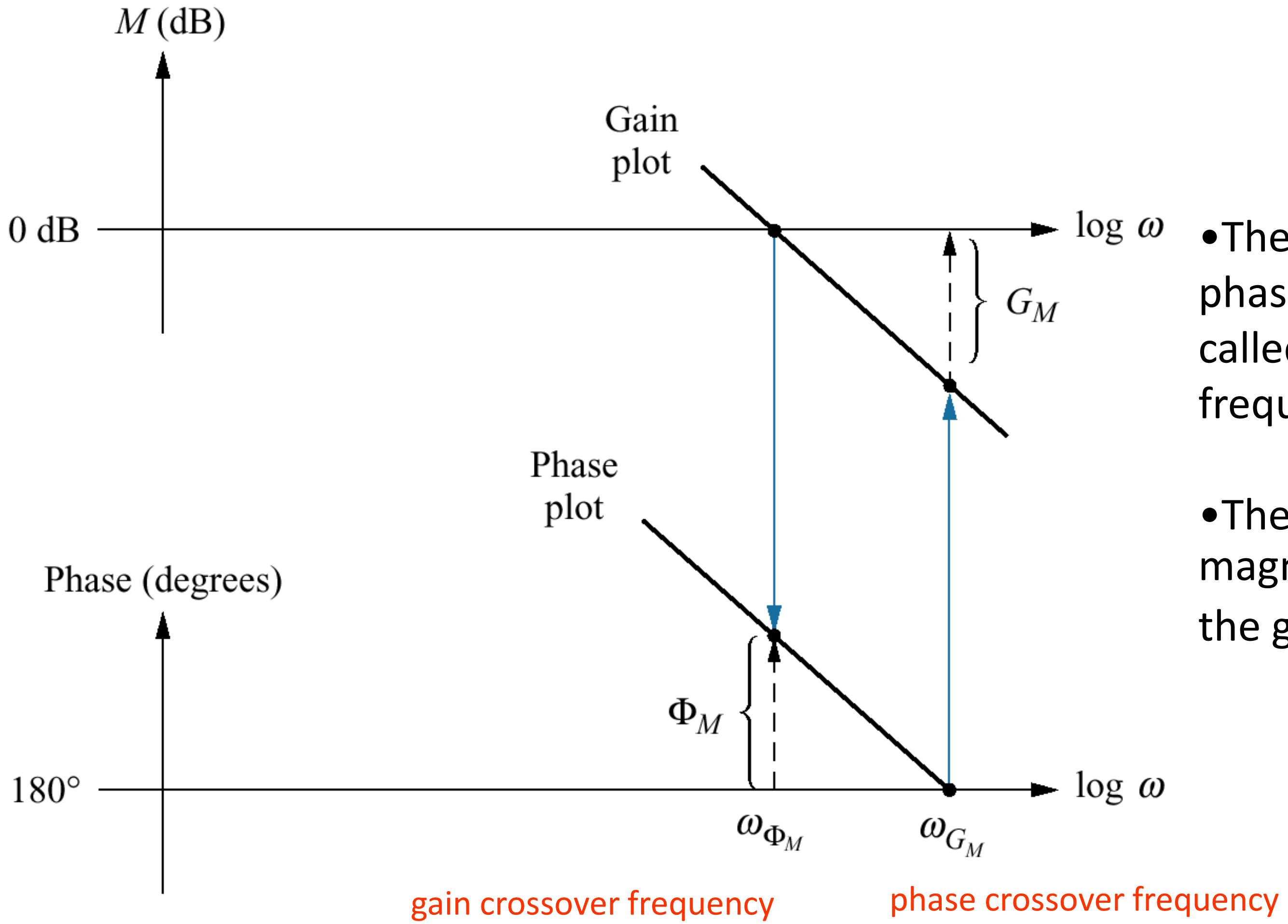
GM and PM via Nyquist plot



$$GM = \frac{K_{max}}{K_{current}} = \frac{1}{|G(j\omega_\phi)H(j\omega_\phi)|}$$

$$PM = 180^\circ + \angle G(j\omega_x)H(j\omega_x)$$

GM and PM via Bode Plot



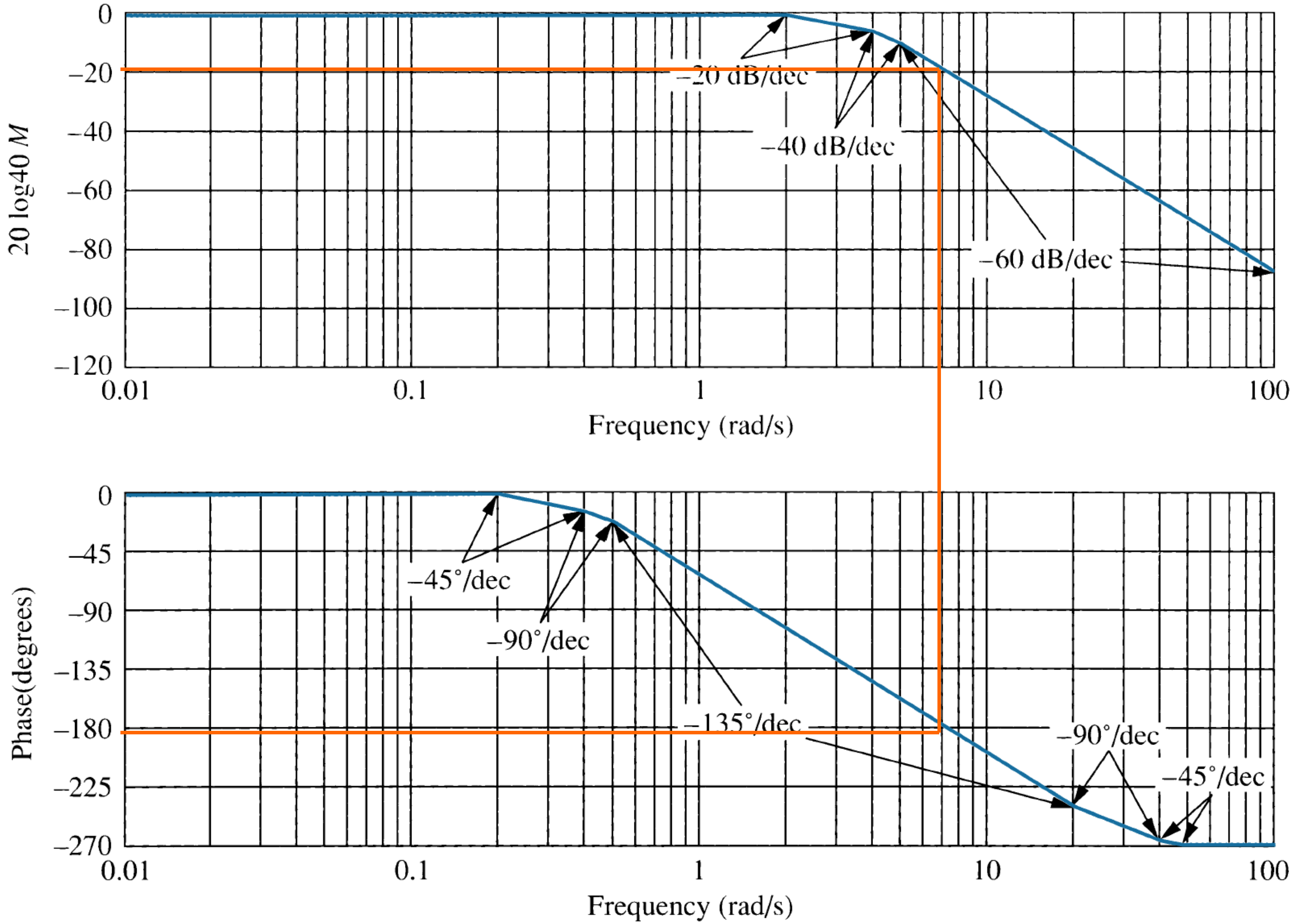
- The frequency at which the phase equals 180 degrees is called the phase crossover frequency ω_{G_M}
- The frequency at which the magnitude equals 1 is called the gain crossover frequency ω_{Φ_M}

Example

Find Bode Plot and evaluate a value of K
that makes the system stable
The system has a unity feedback
with an open-loop transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

First, let's find Bode Plot of G(s) by assuming
that K=40 (the value at which magnitude plot
starts from 0 dB)

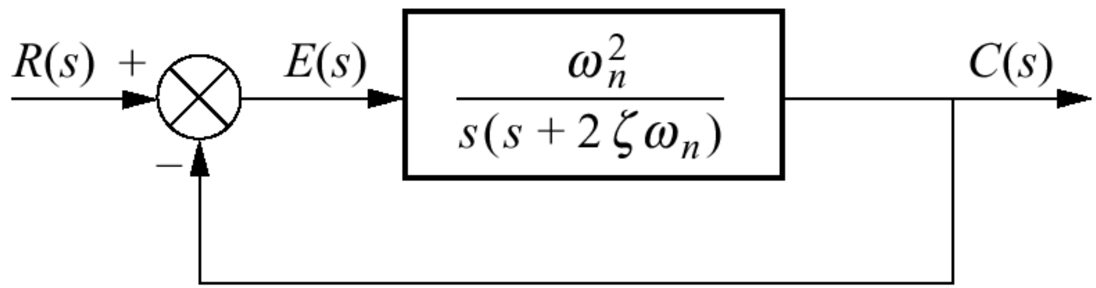


At phase = -180, ω = 7 rad/sec, magnitude = -20 dB

GM>0, system is stable!!!
Can increase gain up 20 dB without causing instability (20dB = 10)
Start from K = 40
with K < 400, system is stable

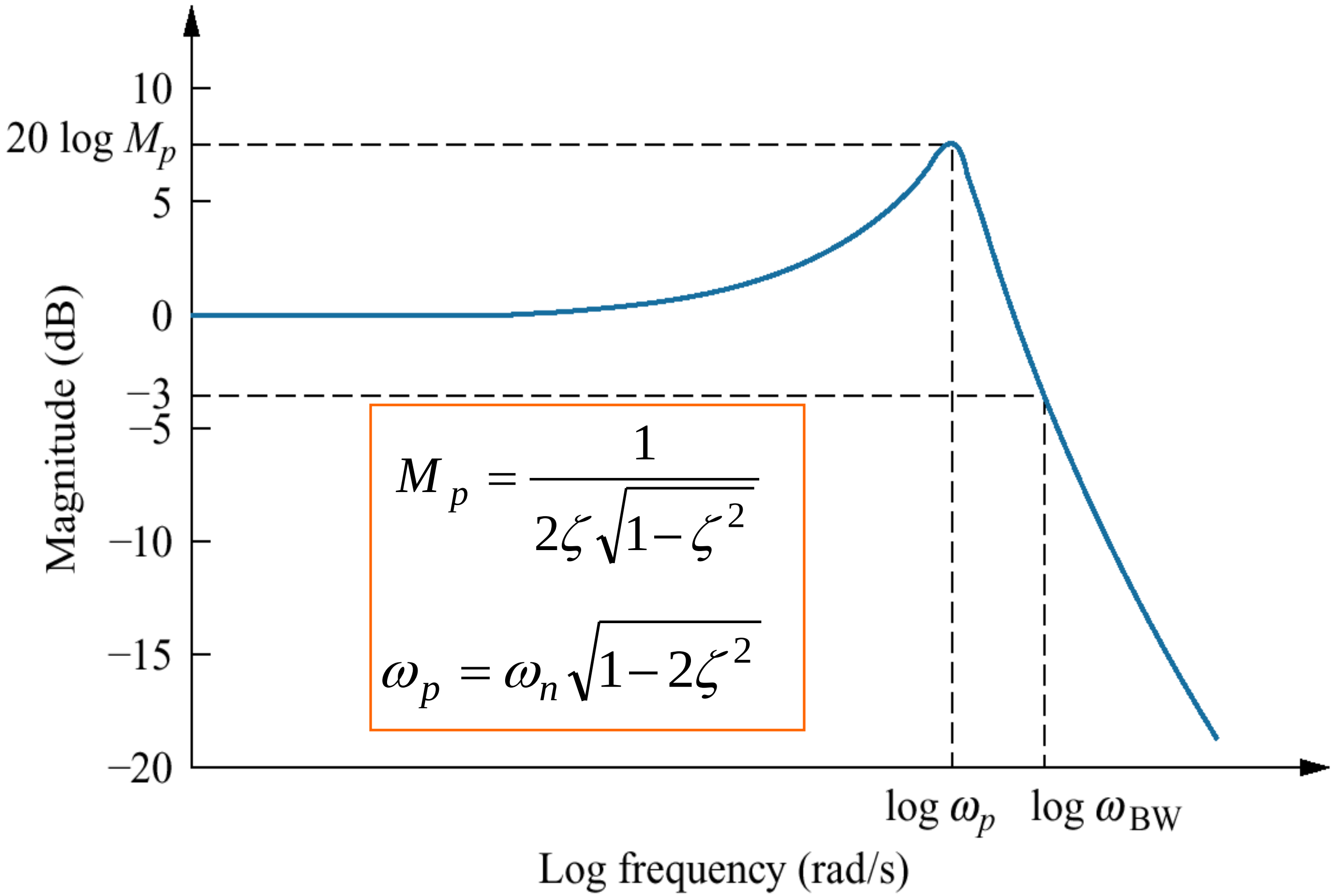
Closed-loop transient and closed-loop frequency responses

'2nd system'



$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping ratio and closed-loop frequency response



Magnitude Plot of closed-loop system

Response speed and closed-loop frequency response

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

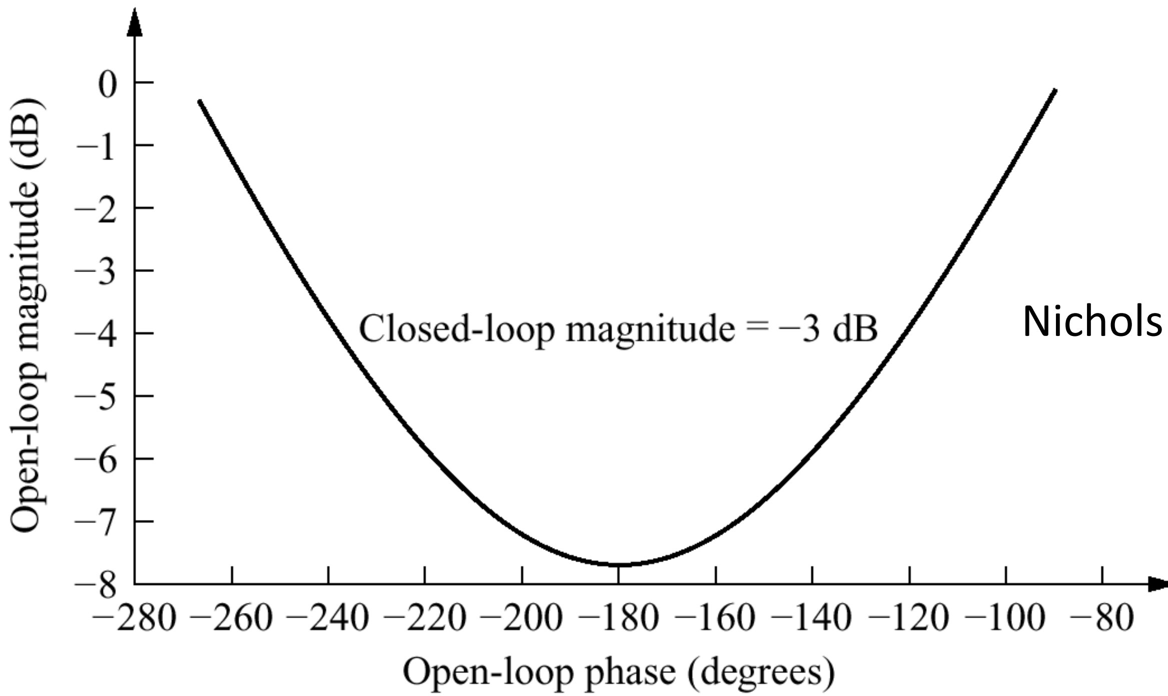
$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

ω_{BW} = frequency at which magnitude is 3dB down

from value at dc (0 rad/sec), or .

$$M = \frac{1}{\sqrt{2}}$$

Find ω_{BW} from Open-loop Frequency Response



Nichols Charts

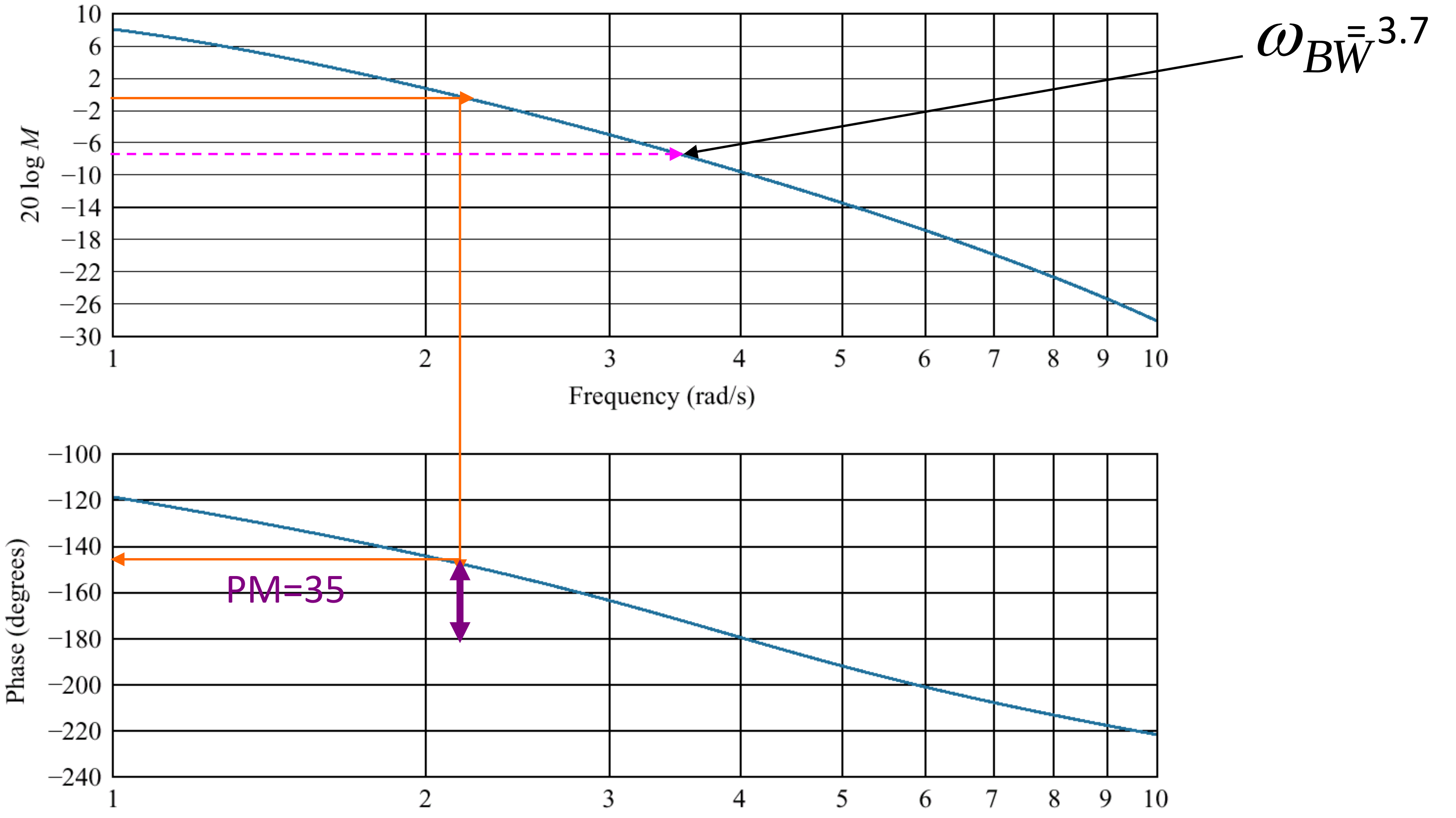
From open-loop frequency response, we can find ω_{BW} at the open-loop frequency that the magnitude lies between -6dB to -7.5dB (phase between -135 to -225)

Relationship between damping ratio and phase margin of open-loop frequency response

Phase margin of open-loop frequency response
Can be written in terms of damping ratio as following

$$\phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

Open-loop system with a unity feedback has a bode plot below, approximate settling time and peak time



$$\phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

Solve for PM = 35

$$\zeta = 0.32$$

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= 5.5$$

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= 1.43$$

UNIT-5

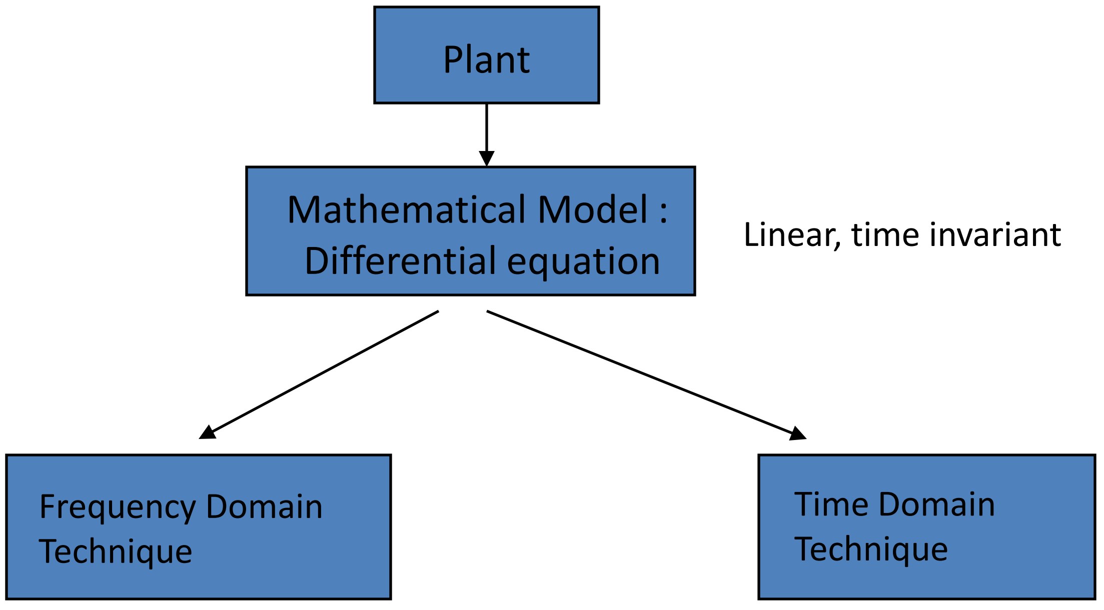
STATE SPACE REPRESENTATION

Objectives

How to find mathematical model, called a state-space representation, for a linear, time-invariant system

How to convert between transfer function and state space models

How to find the solution of state equations for homogeneous &non homogeneous systems


Plant
Mathematical Model :
Differential equation
Linear, time invariant
Frequency Domain
Technique
Time Domain
Technique

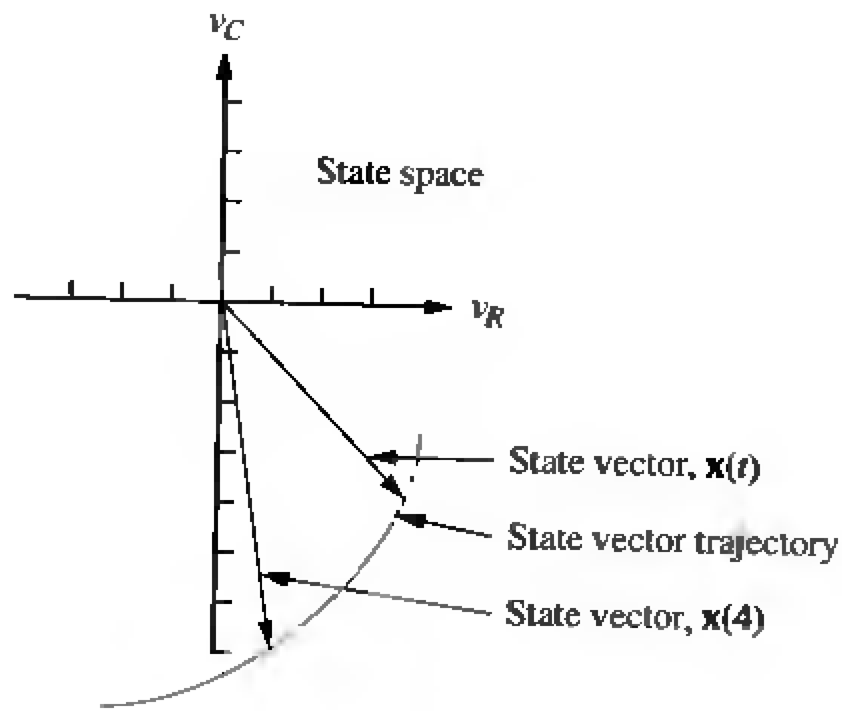
Two approaches for analysis and design of control system

1. Classical Technique or Frequency Domain Technique
2. Modern Technique or Time Domain Technique

Some definitions

- System variable : any variable that responds to an input or initial conditions in a system
- State variables : the smallest set of linearly independent system variables such that the values of the members of the set at time t0 along with known forcing functions completely determine the value of all system variables for all t ≥ t0
- State vector : a vector whose elements are the state variables
- State space : the n-dimensional space whose axes are the state variables
- State equations : a set of first-order differential equations with b variables, where the n variables to be solved are the state variables
- Output equation : the algebraic equation that expresses the output variables of a system as linear combination of the state variables and the inputs.
- For *n*th-order, write *n simultaneous, first-order differential equations* in terms of the state variables (*state equations*).
- If we know the initial condition of all of the state variables at as well as the system input for , we can solve the equations

Graphic representation of state space and a state vector



For a dynamic system, the state of a system is described in terms of a set of state variables

$$[x_1(t) \quad x_2(t) \quad \ldots \quad x_n(t)]$$

The state variables are those variables that determine the future behavior of a system when the present state of the system and the excitation signals are known. Consider the system shown in Figure 1, where $y_1(t)$ and $y_2(t)$ are the output signals and $u_1(t)$ and $u_2(t)$ are the input signals. A set of state variables $[x_1\ x_2\ \ldots\ x_n]$ for the system shown in the figure is a set such that knowledge of the initial values of the state variables $[x_1(t_0)\ x_2(t_0)\ \ldots\ x_n(t_0)]$ at the initial time t_0, and of the input signals $u_1(t)$ and $u_2(t)$ for $t>=t_0$, suffices to determine the future values of the outputs and state variables.

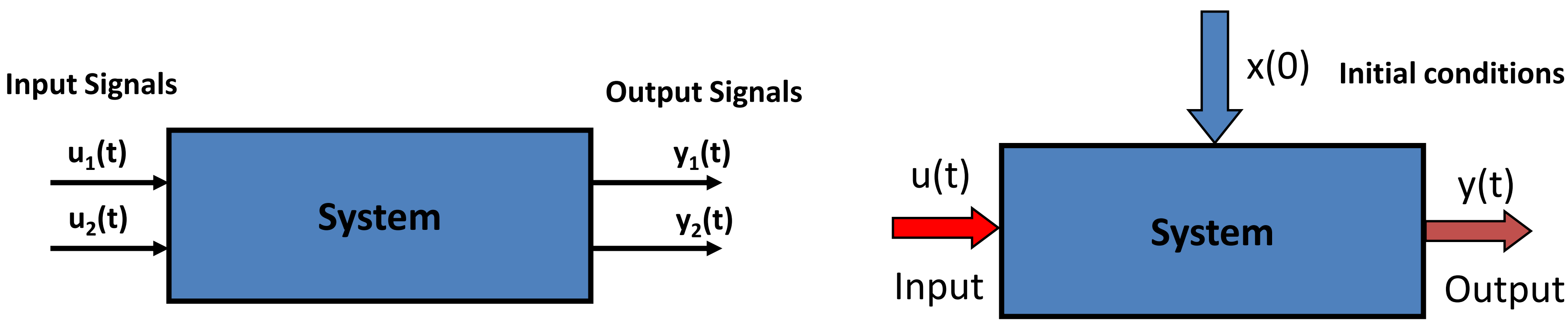


Figure 1. Dynamic system.

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer's interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the **state vector** and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{x} = A x + B u$$

This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = C x + D u$$

Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

General State Representation

$$\dot{x} = Ax + Bu$$ State equation

$$y = Cx + Du$$ output equation

x = state vector

$\dot{x}$ = derivative of the state vector with respect to time

y = output vector

u = input or control vector

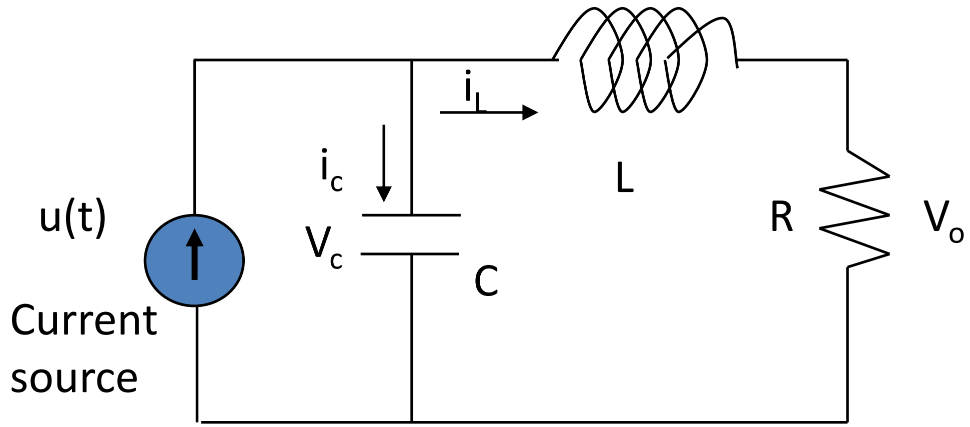
A = system matrix

B = input matrix

C = output matrix

D = feedforward matrix

AN EXAMPLE OF THE STATE VARİABLE CHARACTERİZATİON OF A SYSTEM



- The state of the system can be described in terms of a set of variables $[x_1\ x_2]$, where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff's current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff's voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_L}{dt} = -R\,i_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = R\,i_L(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables x_1 [$v_C(t)$] and x_2 [$i_L(t)$] as follows:

$$C\frac{dv_c}{dt} = u(t) - i_L \quad \Longrightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$L\frac{di_L}{dt} = -R\,i_L + v_c \quad \Longrightarrow \quad \frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then $y_1(t) = v_0(t) = R\,x_2$

Utilizing the first-order differential equations and the initial conditions of the network represented by [$x_1(t_0)$ $x_2(t_0)$], we can determine the system's future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$.

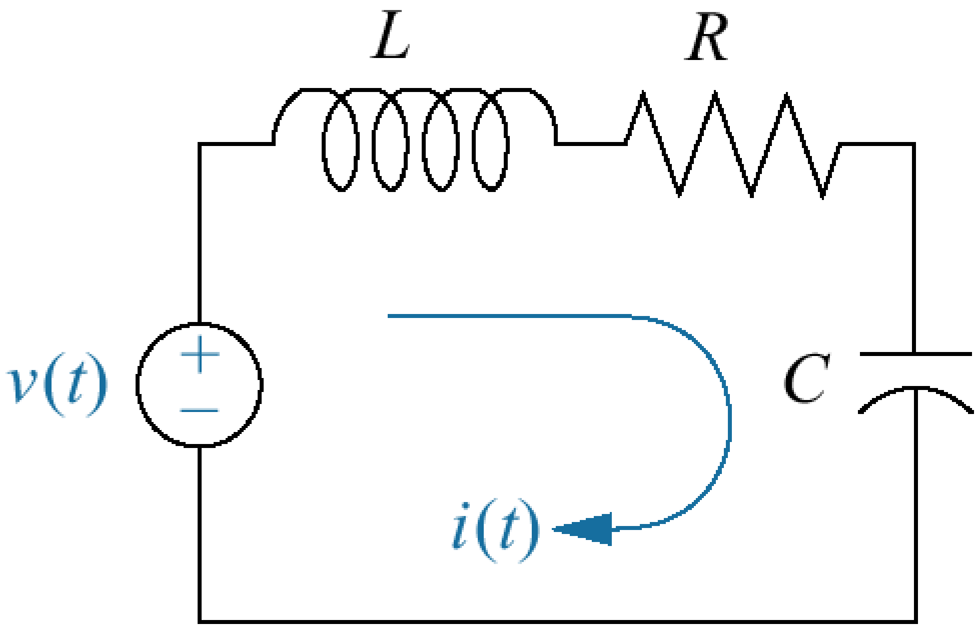
We can write the state variable differential equation for the RLC circuit as

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

and the output as

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

RLC network



1. State variables $i(t)\;\; q(t)$

2. $$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

Using $i(t) = dq/dt$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

$$\frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t) \qquad (1)$$

3. $q(t)$ $i(t)$ Can be solved using Laplace Transform

4. Other network variables can be obtained

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t) \qquad (2)$$

5. (1),(2) : state-space representation

Other variables $v_R(t)\ v_C(t)$

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t)$$

$$\frac{dv_C}{dt} = \frac{1}{RC}v_R$$

Each variables : linearly independent

In vector-matrix form

$$\dot{x} = Ax + Bu \qquad (1)$$

where

$$\dot{x} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$x = \begin{bmatrix} q \\ i \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \qquad u = v(t)$$

$$y = Cx + Du \qquad (2)$$

where

$$y = v_L(t) \qquad C = \begin{bmatrix} -1/C & -R \end{bmatrix} \qquad D = 1$$

State space representation using phase variable form

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

choose

$$x_1 = y \quad x_2 = \frac{dy}{dt} \quad x_3 = \frac{d^2y}{dt^2} \quad \bullet\bullet\bullet \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

↓ diferensiasikan

$$\dot{x}_1 = \frac{dy}{dt}$$

$$\dot{x}_2 = \frac{d^2y}{dt^2}$$

$$\dot{x}_3 = \frac{d^3y}{dt^3}$$

$$\vdots$$

$$\dot{x}_n = \frac{d^n y}{dt^n}$$

⇒

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Example : TF to State Space

1. Inverse Laplace

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

2. Select state variables

$$x_1 = c \qquad x_2 = \dot{c} \qquad x_3 = \ddot{c}$$

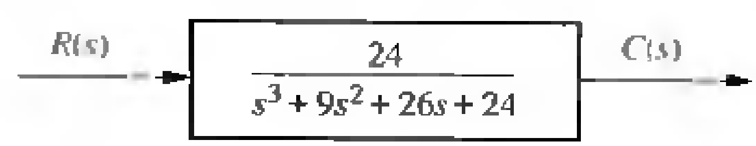
$$\downarrow$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

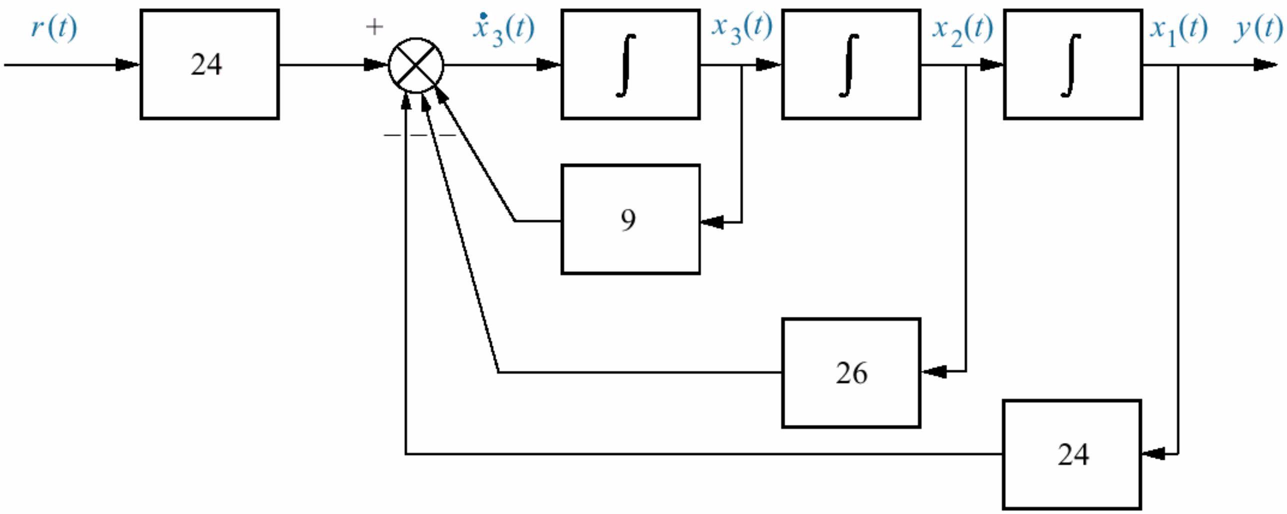


(a)

$$G(s) = \frac{N(s)}{D(s)}$$

$N(s)$ numerator

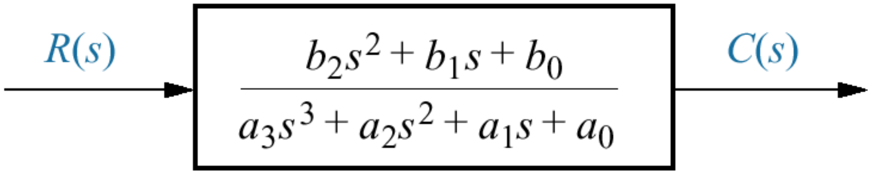
$D(s)$ denominator



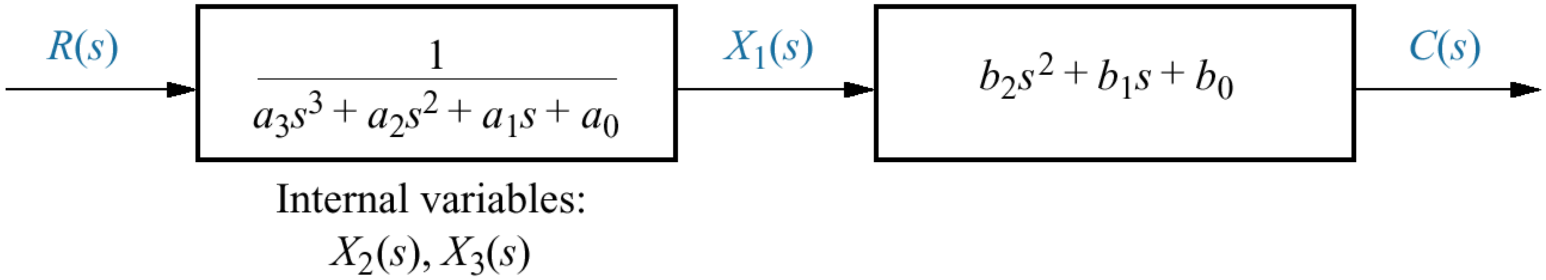
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Decomposing a transfer function



(a)



(b)

$$Y(s) = C(s) = \left(b_2 s^2 + b_1 s + b_0\right) X_1(s)$$

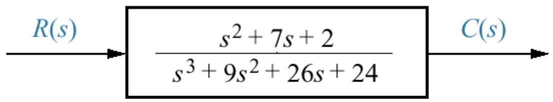
$$\downarrow$$

$$y(t) = b_2 \frac{d^2 x_1}{dt^2} + b_1 \frac{dx_1}{dt} + b_0 x_1$$

$$\downarrow$$

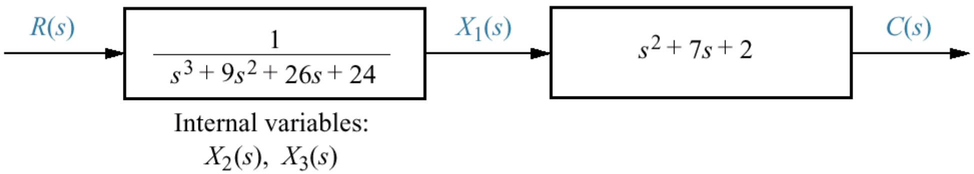
$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3$$

Example



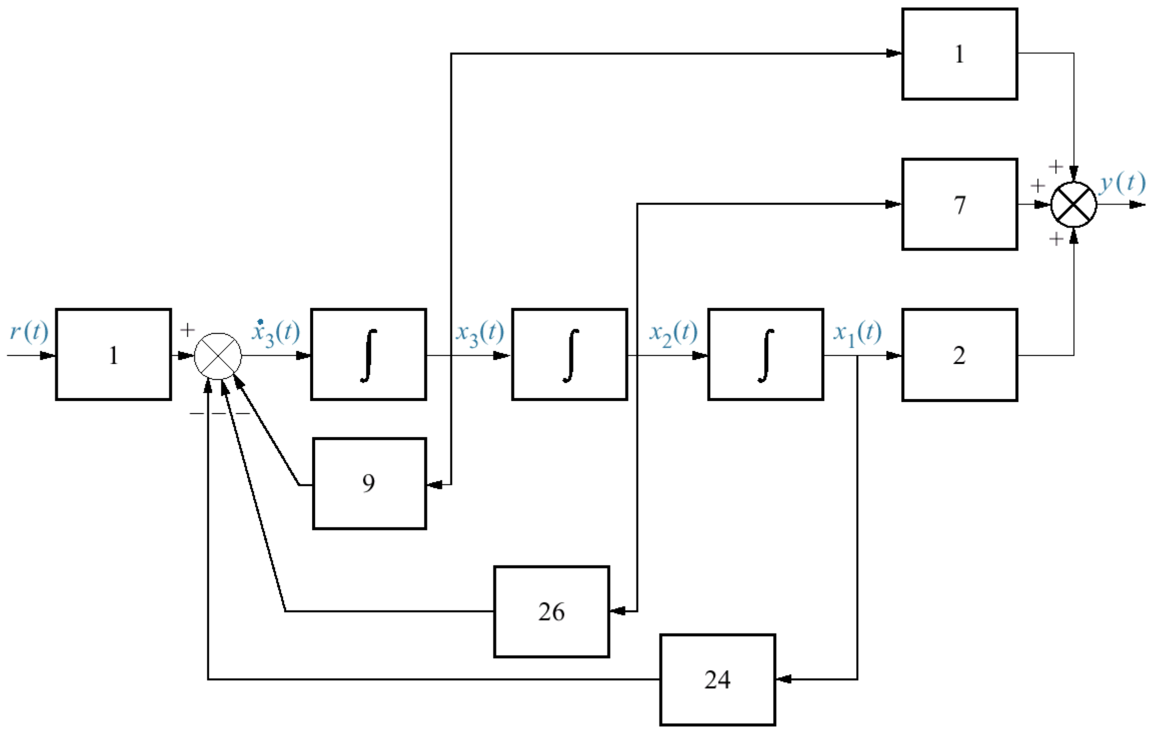
R(s)
s^2 + 7s + 2 / s^3 + 9s^2 + 26s + 24
C(s)

(a)



R(s)
1 / s^3 + 9s^2 + 26s + 24
X1(s)
s^2 + 7s + 2
C(s)
Internal variables:
X2(s), X3(s)

(b)



r(t)
1
+
ẋ3(t)
∫
x3(t)
∫
x2(t)
∫
x1(t)
2
7
1
y(t)
+
+
+
9
26
24
– – –

(c)

$$y(t) = 2x_1 + 7x_2 + x_3$$

$$\downarrow$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space to TF

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

→ Laplace Transform

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

→ $X(s) = (sI - A)^{-1} BU(s)$

→ $Y(s) = \left[C(sI - A)^{-1} B + D\right] U(s)$

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$\longrightarrow \quad (sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

$$= \frac{\begin{bmatrix} (s^2 + 3s + 2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$T(s) = C(sI - A)^{-1}B + D$$

$$\longrightarrow \qquad T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Solution of homogeneous state equation

- The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax; x(0) = 0$$

$$dx = axdt$$

Where x(t) and u(t) are scalar functions of time. We expect an exponential solution of the form $e^{at.}$ Taking the Laplace transform of both sides, we have

$\dot{x}$

on integrating above equation

logx=at+c

X= $e^{at.}$ e^{c}

x=x(0)= e^{c}

on substituting the intial condition ,the solution of homogeneous state equation of first order differential equation is

$$\dot{x} = e^{at} x(0)$$

$$\dot{X} = AX(t), x(o) = 0$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

Solution of homogeneous state equation

- The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu; x(0) = 0$$

Where x(t) and u(t) are scalar functions of time. By taking laplace transform

$$s\,X(s) - x_0 = a\,X(s) + bU(s)$$

The inverse Laplace transform of X(s) results in the solution

$$\mathbf{x(t)} = \mathbf{e}^{\mathbf{at}}\mathbf{x(0)} + \int_0^t \mathbf{e}^{\mathbf{a(t-\tau)}}\mathbf{b\,u(\tau)\,d\tau}$$

We expect the solution of the state differential equation to be similar to x(t) and to be of differential form. The **matrix exponential function** is defined as

$$\mathbf{e}^{\mathbf{At}} = \mathbf{I} + \mathbf{At} + \frac{\mathbf{A}^2\mathbf{t}^2}{2!} + \cdots + \frac{\mathbf{A}^k\mathbf{t}^k}{\mathbf{k!}} + \cdots$$

which converges for all finite t and any A. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

where we note that $[sI-A]^{-1}=\varphi(s)$, which is the Laplace transform of $\varphi(t)=e^{At}$. The matrix exponential function φ(t) describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where y is the single output and u is the single input. The Laplace transform of the equations

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

where B is an nx1 matrix, since u is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation

$$[sI - A]X(s) = BU(s)$$

$$X(s) = [sI - A]^{-1}BU(s) = \phi(s)BU(s)$$

$$Y(s) = C\phi(s)BU(s)$$

Therefore, the transfer function G(s)=Y(s)/U(s) is

$$G(s) = C\phi(s)B$$

Example:

Determine the transfer function G(s)=Y(s)/U(s) for the RLC circuit as described by the state differential function

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

$$[sI - A] = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$$

$$\phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$\Delta(s) = s^2 + \frac{R}{L} s + \frac{1}{LC}$$

Then the transfer function is

$$G(s) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} \frac{s + \frac{R}{L}}{\Delta(s)} & -\frac{1}{C\Delta(s)} \\ \frac{1}{L\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$G(s) = \frac{R/LC}{\Delta(s)} = \frac{R/LC}{s^2 + \frac{R}{L} s + \frac{1}{LC}}$$

CONSIDER THE SYSTEM

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$\left(s^3 + 9s^2 + 26s + 24\right)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

State variables

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

System equations

$$y = c = x_1$$

Output equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_2 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

THANK YOU