STRENGTH OF MATERIALS

COURSE OBJECTIVES

The Course should enable the students to:

I. Relate mechanical properties of a material with its behavior under various load types.

II. Apply the concepts of mechanics to find the stresses at a point in a material of a structural member.

III. Analyze a loaded structural member for deflections and failure strength.

IV. Evaluate the stresses and strains in materials and deflections in beam members.

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COURSE STRUCTURE

UNIT I: STRESSES AND STRAINS(SIMPLE AND PRINCIPAL)

Concept of stress and strain, elasticity and plasticity, Hooke's law, stress-strain diagram for mild steel, Poisson's ratio, volumetric strain, elastic module and the relationship between them bars of varying section, composite bars, temperature stresses; Strain energy, modulus of resilience, modulus of toughness; stresses on an inclined section of a bar under axial loading; compound stresses; Normal and tangential stresses on an inclined plane for biaxial stresses; Two perpendicular normal stresses accompanied by a state of simple shear; Mohr's circle of stresses; Principal stresses and strains; Analytical and graphical solutions. Theories of Failure: Introduction, various theories of failure, maximum principal stress theory, maximum principal strain theory, strain energy and shear strain energy theory.

3

COURSE STRUCTURE

UNIT II: SHEAR FORCE AND BENDING MOMENT

Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – Point of contraflexure – Relation between S.F., B.M and rate of loading at a section of a beam.

UNIT III: FLEXURAL STRESSES AND SHEAR STRESSES IN BEAMS

Flexural Stresses: Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/y = E/R - Neutral axis – Determination of bending stresses – Section modulus of rectangular and circular sections (Solid and Hollow), I,T, Angle and Channel sections – Design of simple beam sections.

Shear Stresses: Derivation of formula – Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T angle sections.

4

COURSE STRUCTURE

UNIT IV: TORSION OF CIRCULAR SHAFTS

Theory of pure torsion- derivation of torsion equations: - assumptions made in the theory of pure torsion - torsional moment of resistance - polar section modulus - power transmitted by shaft - combined bending and torsion and end thrust - design of shafts according to theories of failure. Introduction to springs- types of springs - deflection of close and open coiled helical springs under axial pull and axial couple - springs in series and parallel - carriage or leaf springs.

UNIT V: COLUMNS AND STRUTS: BUCKLING

Types of columns, short, medium and long columns, axially loaded compression members, crushing load, Euler's theorem for long columns, assumptions, derivation of Euler's critical load formulae for various end conditions. Equivalent length of a column, slenderness ratio, Euler's critical stress, limitations of Euler's theory, Rankine's and Gordon formula, long columns subjected to eccentric loading, secant formula, empirical formulae, straight line formula and Prof. Perry's formula. Laterally loaded struts, subjected to uniformly distributed and concentrated loads, maximum bending moment and stress due to transverse and lateral loading.

5

TEXT BOOKS

1. F. Beer, E. R. Johnston, J. De Wolf , "Mechanics of Materials", Tata McGraw-Hill Publishing Company Limited, New Delhi, Indian 1st Edition, 2008.

2. B. C. Punmia, Ashok Kumar Jain, Arun Kumar Jain, "Mechanics of Materials", Laxmi Publications Private Limited, New Delhi, 4th Edition, 2007.

3. R. K. Rajput, "Strength of Materials: Mechanics of Solids", S. Chand & Co Limited, New Delhi, 3rd Edition, 2007.

4. S. S. Rattan, "Strength of Materials", Tata McGraw-Hill Publishers, 4th Edition, 2011

REFERENCE BOOKS

1. J. M. Gere, S.P. Timoshenko, "Mechanics of Materials", CL Engineering, USA, 5th Edition, 2000.

2. D. S. PrakashRao, "Strength of Materials A Practical Approach Vol.1", University Press India Private Limited, India, 1st Edition, 2007.

3. S. S. Bhavikatti, "Strength of Materials", Vikas Publishing House Pvt. Ltd., New Delhi, 3rd Edition, 2013.

6

Teaching Strategies

- The course will be taught via Lectures. Lectures will also involve the solution of tutorial questions. Tutorial questions are designed to complement and enhance both the lectures and the students appreciation of the subject.
- Course work assignments will be reviewed with the students.
- Daily assessment through questioning and class notes.

7

UNITS:

	British	Metric	S.I.
1. Force	Ib, kip, Ton	g, kg,	N, kN
	1 kip = 1000 Ib 1 ton = 2240 Ib	1 kg = 1000 g Ton = 1000 kg	1 kN = 1000 N 1 kg = 10 N
2. Long	in, ft	m, cm, mm	m, cm, mm
	1 f = 12 in	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm
3. Stress	psi, ksi $\frac{p}{in^2}, \frac{kip}{in^2}$	Pa ($\frac{N}{mm^2}$), MPa, GPa	

$$MPa = 10^6\ Pa = 10^6\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}}$$

$$MPa = \frac{N}{mm^2}$$

$$GPa = 10^9\ Pa = 10^9\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}} = 10^3 \frac{N}{mm^2} \times \frac{1}{1000 \frac{N}{kN}}$$

$$GPa = kN/mm^2$$

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UNIT –I
Stresses and Strain
(Simple and Principal)

Concept of elasticity and plasticity

- **Strength** of Material is its ability to withstand and applied load without failure.
- **Elasticity**: Property of material by which it return to its original shape and size after removing the applied load , is called elasticity. And material itself is said to elastic.
- Plasticity: Characteristics of material by which it undergoes inelastic strains (Permanent Deformation) beyond the elastic limit, known as **plasticity**. This property is useful for pressing and forging.

Direct or Normal Stress

- When a force is transmitted through a body, the body tends to change its shape or deform. The body is said to be strained.
- Direct Stress = $\frac{\text{Applied Force (F)}}{\text{Cross Sectional Area (A)}}$

Units: Usually N/m^2 (Pa), N/mm^2, MN/m^2, GN/m^2 or N/cm^2

Note: $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$

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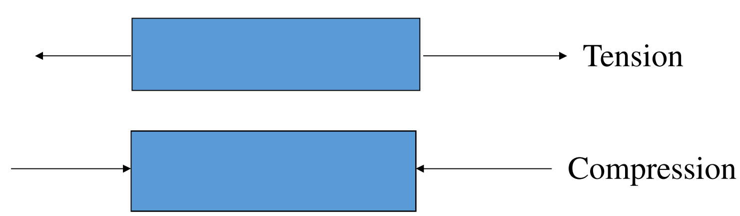
Direct Stress Contd.

- Direct stress may be tensile or compressive and result from forces acting perpendicular to the plane of the cross-section



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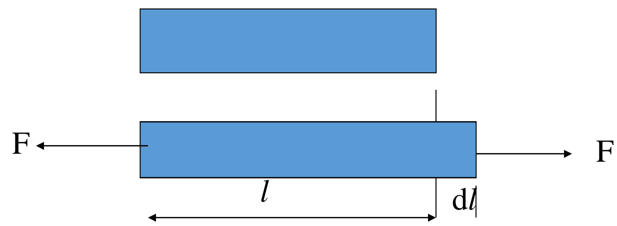
Direct or Normal Strain

- When loads are applied to a body, some deformation will occur resulting to a change in dimension.
- Consider a bar, subjected to axial tensile loading force, F. If the bar extension is d*l* and its original length (before loading) is *l*, then tensile strain is:



Direct Strain (ε) = $\frac{\text{Change in Length}}{\text{Original Length}}$

i.e. $\varepsilon = dl/l$

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Direct or Normal Strain Contd.

- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, dl: Compressive strain = - dl/L

Note: Strain is positive for an increase in dimension and negative for a reduction in dimension.

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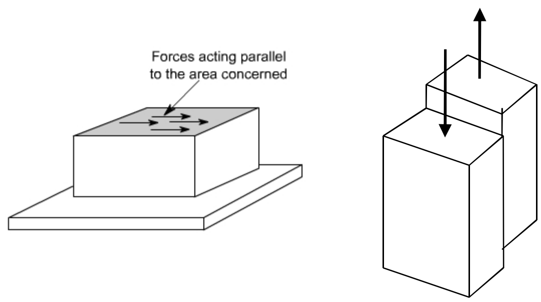
Shear Stress and Shear Strain

- Shear stresses are produced by equal and opposite parallel forces not in line.
- The forces tend to make one part of the material slide over the other part.
- Shear stress is tangential to the area over which it acts.



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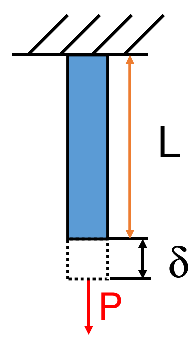
Strain

•It is defined as deformation per unit length

• it is the ratio of change in length to original length

•Tensile strain (+ Ve) (ε) = $\frac{\text{increase in length}}{\text{Original length}} = \frac{\delta}{L}$

Compressive strain (- Ve) (ε) = $\frac{\text{decrease in length}}{\text{Original length}} = \frac{\delta}{L}$



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Ultimate Strength

The strength of a material is a measure of the stress that it can take when in use. The ultimate strength is the measured stress at failure but this is not normally used for design because safety factors are required. The normal way to define a safety factor is :

$$safety\ factor = \frac{stress\ at\ failure}{stress\ when\ loaded} = \frac{Ultimate\ stress}{Permissible\ stress}$$

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Strain

We must also define **strain**. In engineering this is not a measure of force but is a measure of the deformation produced by the influence of stress. For tensile and compressive loads:

Strain is dimensionless, i.e. it is not measured in metres, kilograms etc.

$$strain \quad \varepsilon = \frac{increase\ in\ length\ x}{original\ length\ L}$$

For shear loads the strain is defined as the angle γ This is measured in radians

$$shear\ strain \quad \gamma \approx \frac{shear\ displacement\ x}{width\ L}$$

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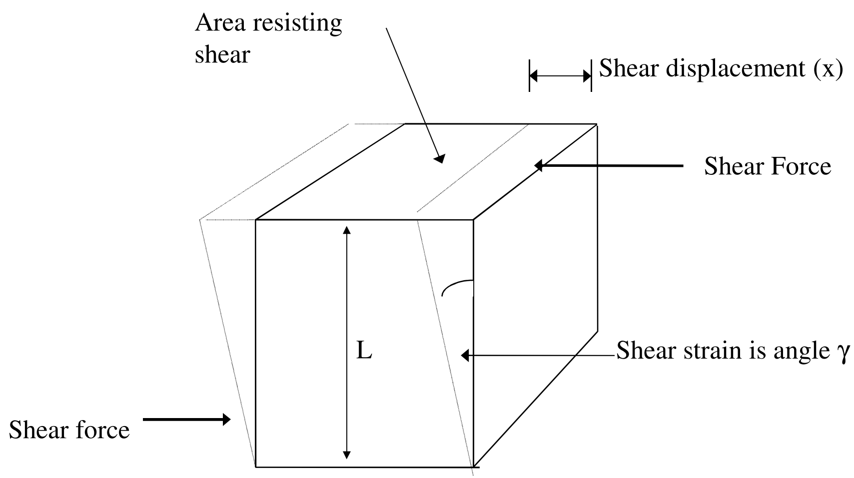
Shear stress and strain



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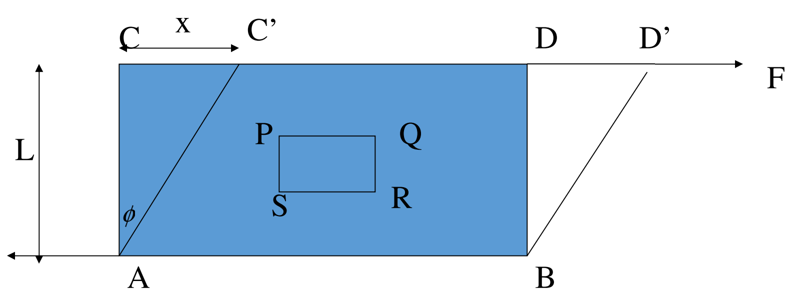
Shear Stress and Shear Strain Contd.



Shear strain is the distortion produced by shear stress on an element or rectangular block as above. The shear strain, γ (gamma) is given as:

$\gamma = x/L = \tan \phi$

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Shear Stress and Shear Strain Concluded

- For small ϕ $\gamma = \phi$
- Shear strain then becomes the change in the right angle.
- It is dimensionless and is measured in radians.

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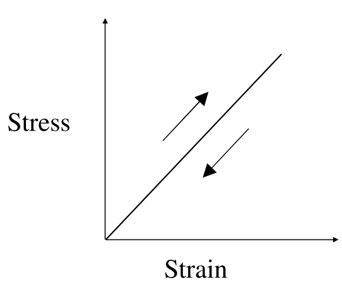
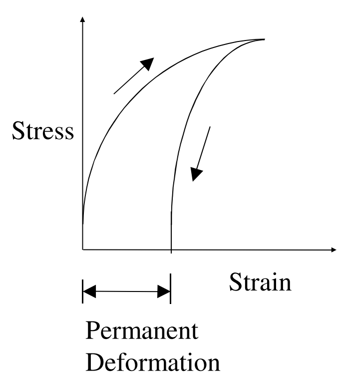
Elastic and Plastic deformation





Elastic deformation

Plastic deformation

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Modulus of Elasticity

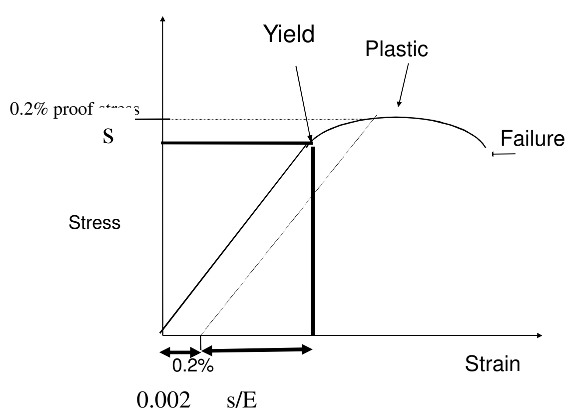
If the strain is "elastic" Hooke's law may be used to define

$$Youngs\ Modulus \quad E = \frac{Stress}{Strain} = \frac{W}{x} \times \frac{L}{A}$$

Young's modulus is also called the modulus of elasticity or stiffness and is a measure of how much strain occurs due to a given stress. Because strain is dimensionless Young's modulus has the units of stress or pressure

How to calculate deflection if the proof stress is applied and then partially removed.

If a sample is loaded up to the 0.2% proof stress and then unloaded to a stress s the strain x = 0.2% + s/E where E is the Young's modulus



Volumetric Strain

- Hydrostatic stress refers to tensile or compressive stress in all dimensions within or external to a body.
- Hydrostatic stress results in change in volume of the material.
- Consider a cube with sides x, y, z. Let dx, dy, and dz represent increase in length in all directions.
- i.e. new volume = (x + dx) (y + dy) (z + dz)

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Volumetric Strain Contd.

Neglecting products of small quantities:

New volume = x y z + z y dx + x z dy + x y dz

Original volume = x y z

= z y dx + x z dy + x y dz

Volumetric strain, $\Delta V = \dfrac{z\,y\,dx + x\,z\,dy + x\,y\,dz}{x\,y\,z}$ ε_v

ε_v = dx/x + dy/y + dz/z

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

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Elasticity and Hooke's Law

- All solid materials deform when they are stressed, and as stress is increased, deformation also increases.
- If a material returns to its original size and shape on removal of load causing deformation, it is said to be **elastic**.
- If the stress is steadily increased, a point is reached when, after the removal of load, not all the induced strain is removed.
- This is called the elastic limit.

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Hooke's Law

- States that providing the limit of proportionality of a material is not exceeded, the stress is directly proportional to the strain produced.
- If a graph of stress and strain is plotted as load is gradually applied, the first portion of the graph will be a straight line.
- The slope of this line is the constant of proportionality called modulus of Elasticity, E or Young's Modulus.
- It is a measure of the stiffness of a material.

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Hooke's Law

Modulus of Elasticity, E = $\frac{Direct\ stress}{Direct\ strain} = \frac{\sigma}{\varepsilon}$

Also: For Shear stress: Modulus of rigidity or shear modulus, G = $\frac{Shear\ stress}{Shear\ strain} = \frac{\tau}{\gamma}$

Also: Volumetric strain , is proportional to hydrostatic stress, within the elastic range i.e. :

$$\sigma \;/\; \varepsilon_v \;=\; K$$

'K' called **bulk modulus**.

Stress-Strain Relations of Mild Steel

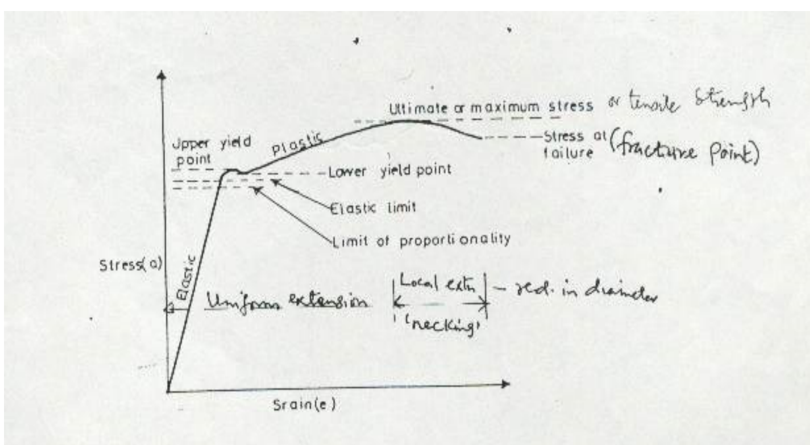

Fig: Behaviour of mild-steel rod under tension.

Equation For Extension

From the above equations:

$$E = \frac{\sigma}{\varepsilon} = \frac{F / A}{dl / L} = \frac{F\,L}{A\,dl}$$

$$dl = \frac{F\,L}{A\,E}$$

This equation for extension is very important

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Factor of Safety

- The load which any member of a machine carries is called working load, and stress produced by this load is the working stress.
- Obviously, the working stress must be less than the yield stress, tensile strength or the ultimate stress.
- This working stress is also called the permissible stress or the allowable stress or the design stress.

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Factor of Safety Contd.

- Some reasons for factor of safety include the inexactness or inaccuracies in the estimation of stresses and the non-uniformity of some materials.

$$\text{Factor of safety} = \frac{\textit{Ultimate or yield stress}}{\textit{Design or working stress}}$$

Note: Ultimate stress is used for materials e.g. concrete which do not have a well-defined yield point, or brittle materials which behave in a linear manner up to failure. Yield stress is used for other materials e.g. steel with well defined yield stress.

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Results From a Tensile Test

(a) Modulus of Elasticity, $E = \dfrac{\textit{Stress up to limit of proportionality}}{\textit{Strain}}$

(b) Yield Stress or Proof Stress (See below)

(c) Percentage elongation = $\dfrac{\textit{Increase in gauge length}}{\textit{Original gauge length}} \, x\, 100$

(d) Percentage reduction in area = $\dfrac{\textit{Original area} - \textit{area at fracture}}{\textit{Original area}} \, x\, 100$

(e) Tensile Strength = $\dfrac{\textit{Maximum load}}{\textit{Original cross sectional area}}$

The percentage of elongation and percentage reduction in area give an indication of the ductility of the material i.e. its ability to withstand strain without fracture occurring.

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Proof Stress

- High carbon steels, cast iron and most of the non-ferrous alloys do not exhibit a well defined yield as is the case with mild steel.
- For these materials, a limiting stress called proof stress is specified, corresponding to a non-proportional extension.
- The non-proportional extension is a specified percentage of the original length e.g. 0.05, 0.10, 0.20 or 0.50%.

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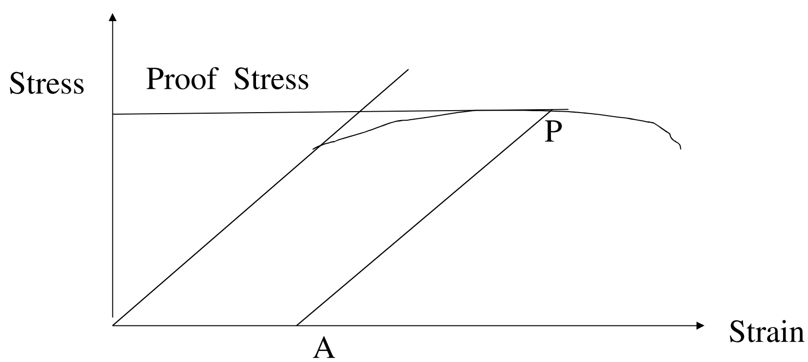
Determination of Proof Stress



The proof stress is obtained by drawing AP parallel to the initial slope of the stress/strain graph, the distance, OA being the strain corresponding to the required non-proportional extension e.g. for 0.05% proof stress, the strain is 0.0005.

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Thermal Strain

Most structural materials expand when heated,

in accordance to the law: $\varepsilon = \alpha\ T$

where ε is linear strain and

α is the coefficient of linear expansion;

T is the rise in temperature.

That is for a rod of Length, L;

if its temperature increased by t, the extension,

dl = α L T.

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Thermal Strain Contd.

As in the case of lateral strains, thermal strains

do not induce stresses unless they are constrained.

The total strain in a body experiencing thermal stress

may be divided into two components:

Strain due to stress, ε_σ and

That due to temperature, ε_T .

Thus: $\varepsilon = \varepsilon_\sigma + \varepsilon_T$

$$\varepsilon = \frac{\sigma}{E} + \alpha T$$

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Principle of Superposition

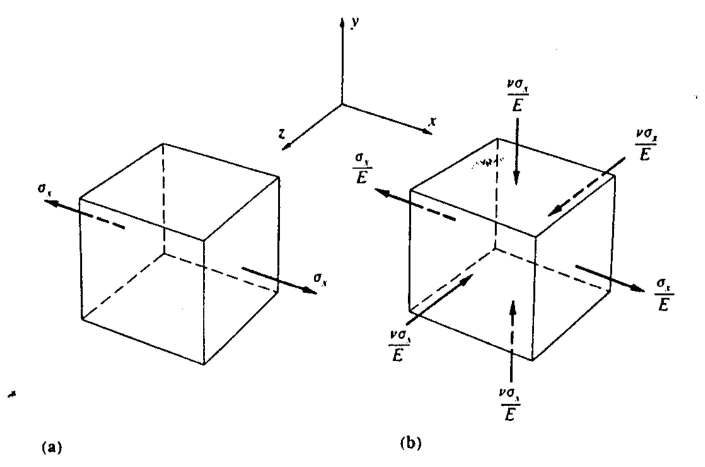
- It states that the effects of several actions taking place simultaneously can be reproduced exactly by adding the effect of each action separately.
- The principle is general and has wide applications and holds true if:

(i) The structure is elastic

(ii) The stress-strain relationship is linear

(iii) The deformations are small.

General Stress-Strain Relationships



Relationship between Elastic Modulus (E) and Bulk Modulus, K

It has been shown that : $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon(\sigma_y + \sigma_z)\right]$$

For hydrostatic stress, $\sigma_x = \sigma_y = \sigma_z = \sigma$

$$i.e.\quad \varepsilon_x = \frac{1}{E}[\sigma - 2\,\sigma\,\upsilon] = \frac{\sigma}{E}[1-2\,\upsilon]$$

Similarly, ε_y *and* ε_z *are each* $\frac{\sigma}{E}[1-2\,\upsilon]$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad = Volumetric\ strain$$

$$\varepsilon_v = \frac{3\,\sigma}{E}[1-2\,\upsilon]$$

$$E = \frac{3\,\sigma}{\varepsilon_v}[1-2\,\upsilon]$$

$$Bulk\ \ Modulus,\ K = \frac{Volumetric\ or\ hydrostatic\ stress}{Volumetric\ strain} = \frac{\sigma}{\varepsilon_v}$$

$$i.e.\quad E = 3\,K\left[1-2\,\upsilon\right] \quad and \quad K = \frac{E}{3\left[1-2\,\upsilon\right]}$$

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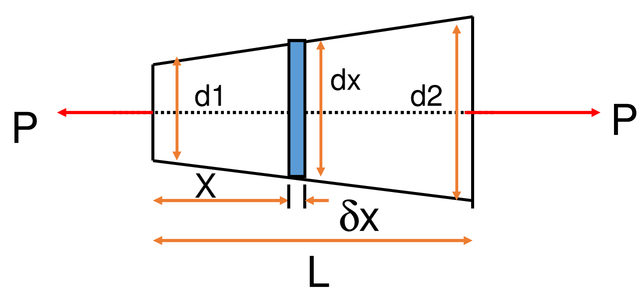
Extension of Bar of Tapering cross Section from diameter d1 to d2:-



Bar of Tapering Section:

dx = d1 + [(d2 - d1) / L] * X

$\delta\Delta = P\delta x / E[\pi /4\{d1 + [(d2 - d1) / L] * X\}^2]$

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$$\Delta = 4P\int_0^L dx / [E\,\pi\{d1+kx\}^2]$$

$$= - [4P/\pi E] \times 1/k\ [\{1/(d1+kx)\}]_0^L dx$$

$$=- [4PL/\pi E(d2-d1)]\ \{1/(d1+d2-d1) - 1/d1\}$$

$$\Delta = 4PL/(\pi E\ d1\ d2)$$

Check :-

When d = d1=d2

Δ =PL/ [(π /4)* d^2E] = PL /AE

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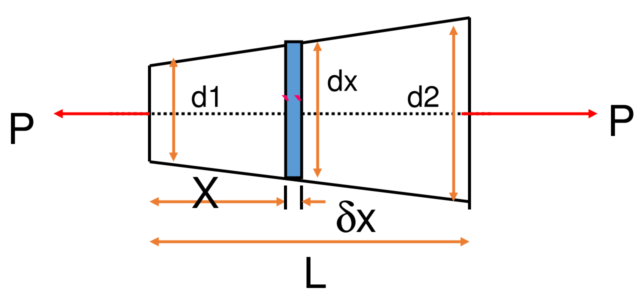
Q. Find extension of tapering circular bar under axial pull for the following data: d1 = 20mm, d2 = 40mm, L = 600mm, E = 200GPa. P = 40kN



ΔL = 4PL/(π E d1 d2)

= 4*40,000*600/(π* 200,000*20*40)

= 0.38mm. Ans.

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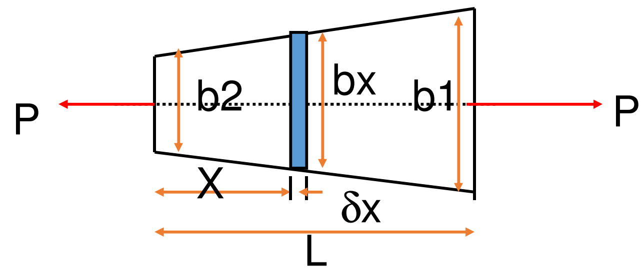
Extension of Tapering bar of uniform thickness t, width varies from b1 to b2:-



P/Et $\int \delta x / [(b1 + k^*X)]$,

Bar of Tapering Section:

$bx = b1 + [(b2 - b1) / L] * X = b1 + k^*x$,

$\delta\Delta = P\delta x / [Et(b1 + k^*X)]$, $k =$ **(b2 - b1) / L**

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$$\Delta L = \int_0^L \Delta L = \int_0^L P\delta x / [Et(b1 - k^*X)],$$

$$= P/Et \int \delta x / [(b1 - k^*X)],$$

$$= - P/E\,t\,k * \log_e [(b1 - k^*X)]_0^L,$$

$$= PL \log_e(b1/b2) / [Et(b1 - b2)]$$

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Example

- A steel tube having an external diameter of 36 mm and an internal diameter of 30 mm has a brass rod of 20 mm diameter inside it, the two materials being joined rigidly at their ends when the ambient temperature is 18 ^{0}C. Determine the stresses in the two materials: (a) when the temperature is raised to 68 ^{0}C (b) when a compressive load of 20 kN is applied at the increased temperature.

For brass: Modulus of elasticity = 80 GN/m^2; Coefficient of expansion = 17 x 10 -6 /^{0}C

For steel: Modulus of elasticity = 210 GN/m^2; Coefficient of expansion = 11 x 10 -6 /^{0}C

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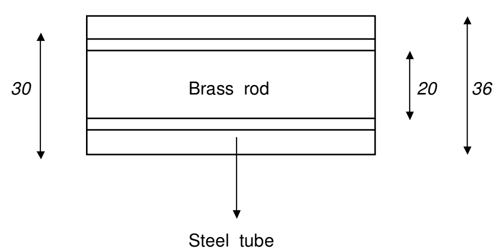
Solution



Area of brass rod (A_b) = $\frac{\pi\ x\ 20^2}{4} = 314.16\ mm^2$

Area of steel tube (A_s) = $\frac{\pi\ x\ (36^2 - 30^2)}{4} = 311.02\ mm^2$

$$A_s E_s = 311.02\ x\ 10^{-6} m^2\ x\ 210\ x\ 10^9\ N/m^2 = 0.653142\ x\ 10^8\ N$$

$$\frac{1}{A_s E_s} = 1.53106\ x\ 10^{-8}$$

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Solution Contd.

$$A_b E_b = 314.16 \, x \, 10^{-6} m^2 \; x \; 80 \, x \, 10^9 \, N / m^2 = 0.251327 \, x \, 10^8 \, N$$

$$\frac{1}{A_b E_b} = 3.9788736 \, x \, 10^{-8}$$

$$T(\alpha_b - \alpha_s) = 50(17 - 11) \, x \, 10^{-6} = 3 \, x \, 10^{-4}$$

With increase in temperature, brass will be in compression while steel will be in tension. This is because expands more than steel.

$$i.e. \quad F[\frac{1}{A_s E_s} + \frac{1}{A_b E_b}] = T(\alpha_b - \alpha_s)$$

i.e. F[1.53106 + 3.9788736] x 10^{-8} = 3 x 10^{-4}

F = 5444.71 N

Solution Concluded

Stress in steel tube = $\frac{5444.71N}{311.02 \, mm^2} = 17.51 N / mm^2 = 17.51 MN / m^2 (Tension)$

Stress in brass rod = $\frac{5444.71N}{314.16 \, mm^2} = 17.33 N / mm^2 = 17.33 MN / m^2 (Compression)$

(b) Stresses due to compression force, F' of 20 kN

$$\sigma_s = \frac{F' E_s}{E_s A_s + E_b A_b} = \frac{20 \, x \, 10^3 N \; x \; 210 \, x \, 10^9 \, N / m^2}{0.653142 + 0.251327 \, x \, 10^8} = 46.44 \, MN / m^2 (Compression)$$

$$\sigma_b = \frac{F' E_b}{E_s A_s + E_b A_b} = \frac{20 \, x \, 10^3 N \; x \; 80 \, x \, 10^9 \, N / m^2}{0.653142 + 0.251327 \, x \, 10^8} = 17.69 \, MN / m^2 (Compression)$$

Resultant stress in steel tube = - 46.44 + 17.51 = 28.93 MN/m^2 (Compression)

Resultant stress in brass rod = -17.69 - 17.33 = 35.02 MN/m^2 (Compression)

Example

A composite bar, 0.6 m long comprises a steel bar 0.2 m long and 40 mm diameter which is fixed at one end to a copper bar having a length of 0.4 m.

i. Determine the necessary diameter of the copper bar in order that the extension of each material shall be the same when the composite bar is subjected to an axial load.

ii. What will be the stresses in the steel and copper when the bar is subjected to an axial tensile loading of 30 kN? (For steel, E = 210 GN/m^2; for copper, E = 110 GN/m^2)

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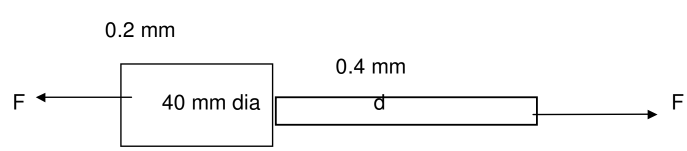
Solution



Let the diameter of the copper bar be d mm

Specified condition: Extensions in the two bars are equal

$$dl_c = dl_s$$

$$dl = \varepsilon\, L = \frac{\sigma}{E}\, L = \frac{FL}{AE}$$

Thus: $\frac{F_c L_c}{A_c E_c} = \frac{F_s L_s}{A_s E_s}$

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Solution

Also: Total force, F is transmitted by both copper and steel

i.e. F_c = Fs = F

$$i.e.\quad \frac{L_c}{A_c E_c} = \frac{L_s}{A_s E_s}$$

Substitute values given in problem:

$$\frac{0.4\,m}{\pi\, d^2/4\, m^2\ 110\, x\, 10^9\, N/m^2} = \frac{0.2\,m}{\pi/4\, x\, 0.040^2\ x\, 210\, x\, 10^9\ N/m^2}$$

$$d^2 = \frac{2\, x\, 210\, x\, 0.040^2}{110} m^2;\quad d = 0.07816\, m = 78.16\, mm.$$

Thus for a loading of 30 kN

Stress in steel, $\sigma_s = \dfrac{30\, x\, 10^3\, N}{\pi/4\, x\, 0.040^2\ x\, 10^{-6}} = 23.87\ MN/m^2$

Stress in copper, $\sigma_c = \dfrac{30\, x\, 10^3\, N}{\pi/4\, x\, 0.07816^2\ x\, 10^{-6}} = 9\ MN/m^2$

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Elastic Strain Energy

- If a material is strained by a gradually applied load, then work is done on the material by the applied load.
- The work is stored in the material in the form of strain energy.
- If the strain is within the elastic range of the material, this energy is not retained by the material upon the removal of load.

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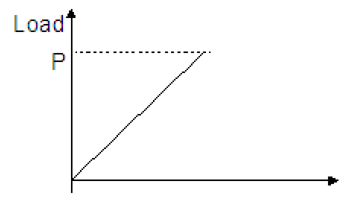
Elastic Strain Energy Contd.

Figure below shows the load-extension graph of a uniform bar. The extension dl is associated with a gradually applied load, P which is within the elastic range. The shaded area represents the work done in increasing the load from zero to its value



Work done = strain energy of bar = shaded area

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Elastic Strain Energy Concluded

W = U = 1/2 P dl (1)

Stress, σ = P/A i.e P = σ A

Strain = Stress/E

i.e dl/L = σ/E , dl = (σL)/E L= original length

Substituting for P and dl in Eqn (1) gives:

W = U = 1/2 σ A . (σ L)/E = σ^2/2E x A L

A L is the volume of the bar.

i.e U = σ^2/2E x Volume

The units of strain energy are same as those of work i.e. Joules. Strain energy per unit volume, σ^2/2E is known as resilience. The greatest amount of energy that can stored in a material without permanent set occurring will be when σ is equal to the elastic limit stress.

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UNIT 2

Shear Force and Bending Moment

SHEAR FORCE AND BENDING MOMENT

Definition of beam – Types of beams – Concept of shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and overhanging beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – Point of contraflexure – Relation between S.F., B.M and rate of loading at a section of a beam.

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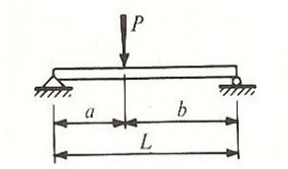
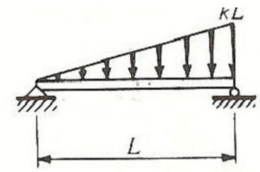
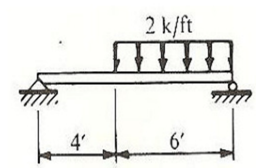
4-Classification of Beams:

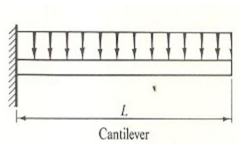
1) Simple Beam

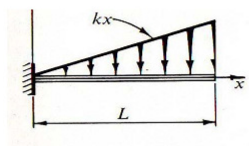


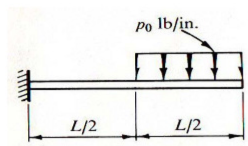




Cantilever Beam







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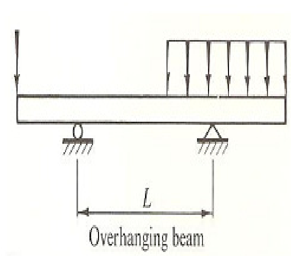
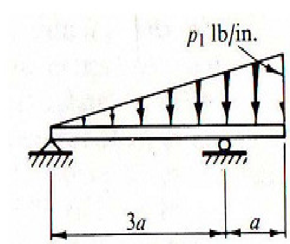
3) Simple Beam with Overhanging OR "Overhanging Beam"





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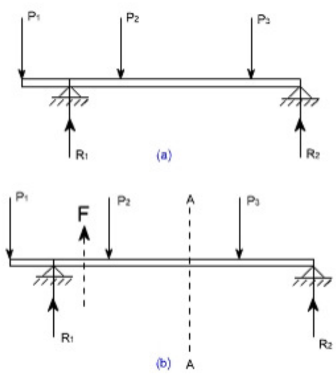
Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms



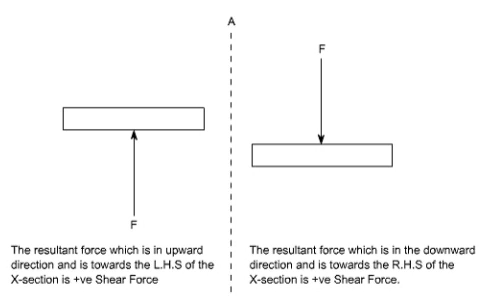
Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:

At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.



Positive Shear Force

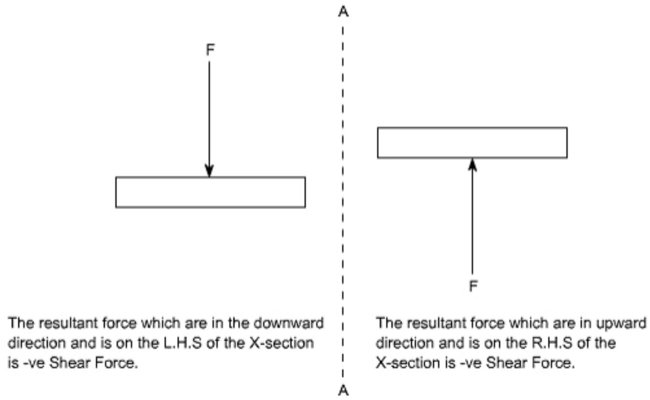
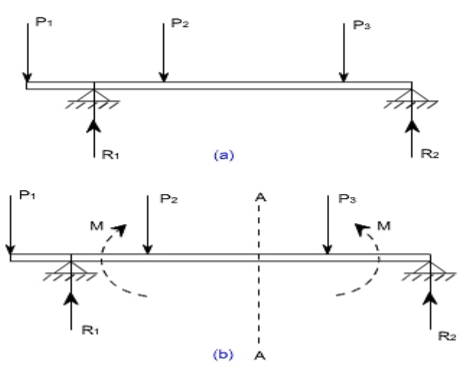


Fig 3: Negative Shear Force

BENDING MOMENT



Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as <u>the algebraic sum of the moments about an x-section of all the forces acting on either side of the section</u>

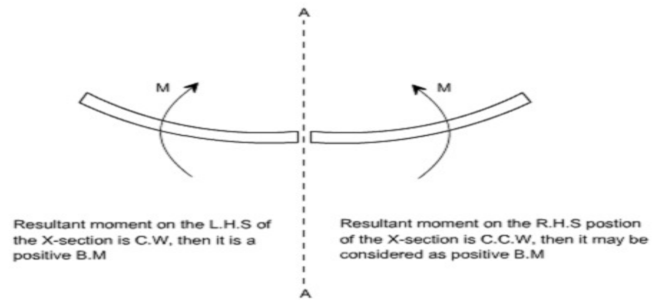
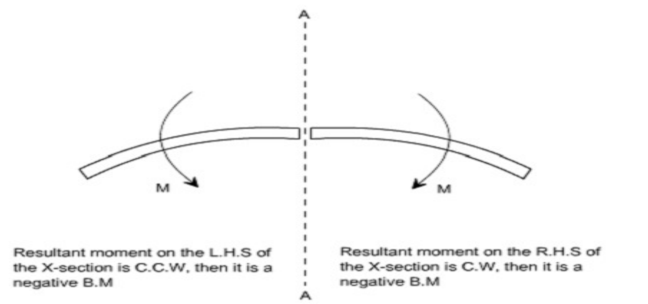
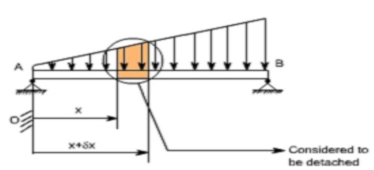


Fig 5: Positive Bending Moment

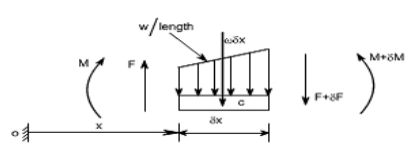


Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established. Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ dF at the section x and x + dx respectively.
- The bending moment at the sections x and x + dx be M and M + dM respectively.
- Force due to external loading, if 'w' is the mean rate of loading per unit length then the total loading on this slice of length dx is w. dx, which is approximately acting through the centre 'c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre 'c'. This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point 'c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots (2)$$

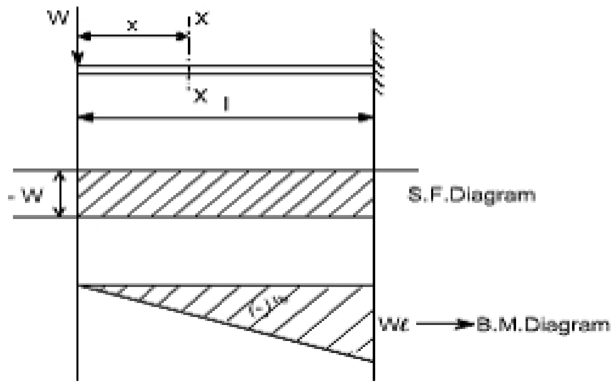
A cantilever of length carries a concentrated load 'W' at its free end. Draw shear force and bending moment.

Solution:

At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative.

Taking moments about the section gives (obviously to the left of the section) M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e.

M = -Wl From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



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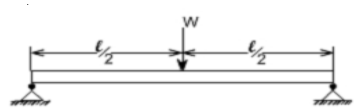
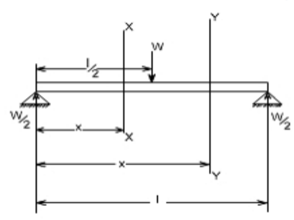
Simply supported beam subjected to a central load (i.e. load acting at the mid-



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.



.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$S.F_{Y-Y} = \frac{W}{2} - W = \frac{-W}{2}$ for all values greater = l/2

Hence S.F diagram can be plotted as,

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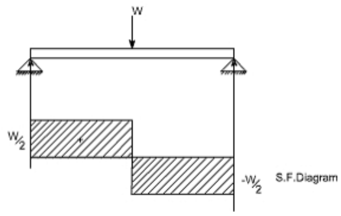


.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{x\text{-}x} = \frac{W}{2}x \text{ for x lies between 0 and l/2}$$

$$B.M_{at\ x=\frac{l}{2}} = \frac{W}{2}\ \frac{l}{2} \text{ i.e B.M at x = 0}$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}x - W\left(x - \frac{l}{2}\right)$$

Again

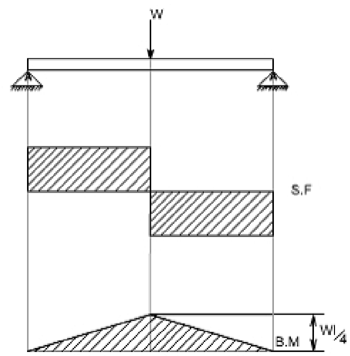
$$= \frac{W}{2}x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}x + \frac{Wl}{2}$$

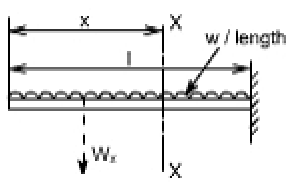
$$B.M_{at\ x=l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

$$= 0$$

Which when plotted will give a straight relation i.e.



A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx}$ = -Wx for all values of 'x'. ---------- **(1)**

$S.F_{xx}$ = 0

$S.F_{xx\ at\ x=l}$ = -Wl

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

$$B.M_{X\text{-}X} = -\,W\,x\,\frac{x}{2}$$

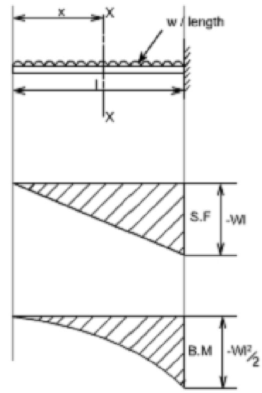
$$= -\,W\frac{x^2}{2}$$

The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

The extreme values of this would be at x = 0 and x = l

$$B.M_{at\,x=l} = -\frac{Wl^2}{2}$$

$$= \frac{Wl}{2} - Wx$$



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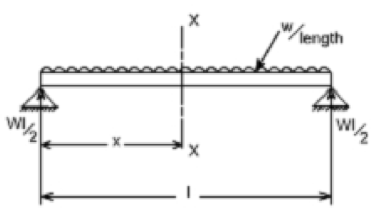
Simply supported beam subjected to a uniformly distributed load U.D.L

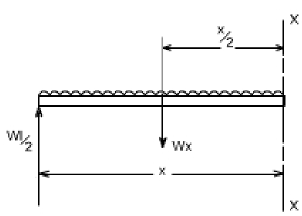


S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



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$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

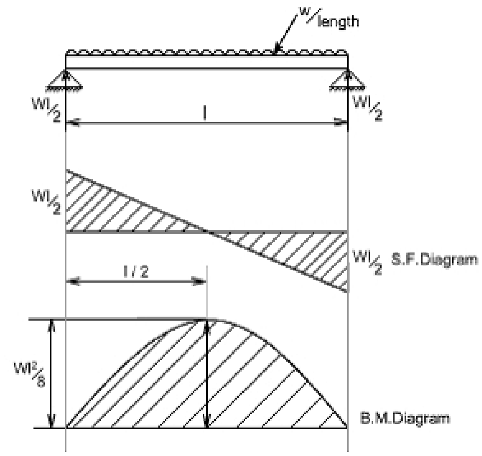
so the

$$= W.\frac{x}{2}(l - 2) \quad ....(2)$$

$B.M_{at\ x = 0} = 0$

$B.M_{at\ x = l} = 0$

$$B.M\Big|_{at\ x = l} = -\frac{Wl^2}{8}$$



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An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span.

Determine the

(i). The second moment of area of the cross-section of the girder

(ii). The maximum stress set up.

Solution:

The second moment of area of the cross-section can be determined as follows :

For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (b.d3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

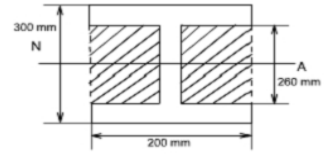
$$= 1.86 \times 10^{-4}\ m^4$$

The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\, y_{max^m}$$



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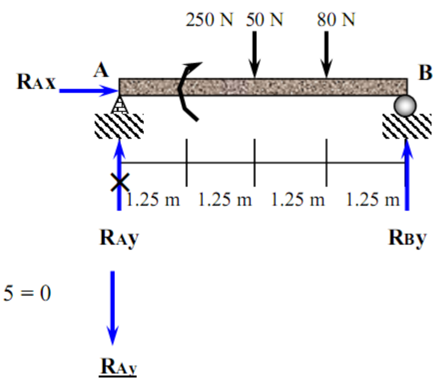
Calculations of Beam Reactions

Ex3:



$\rightarrow \sum F_x = 0$ --- (1)

$R_{AX} = 0$

$\circlearrowleft + \sum M_{@A} = 0$ --- (2)

$250 + 80 \times 2.5 + 80 \times 3.75 - R_B \times 5 = 0$

$\therefore R_{By} = +135 \text{ N} \uparrow$

$\uparrow \sum F_y = 0$ --- (3)

$R_{Ay} = -5 \text{ N} \uparrow \Rightarrow R_{Ay} = 5 \text{ N} \downarrow$

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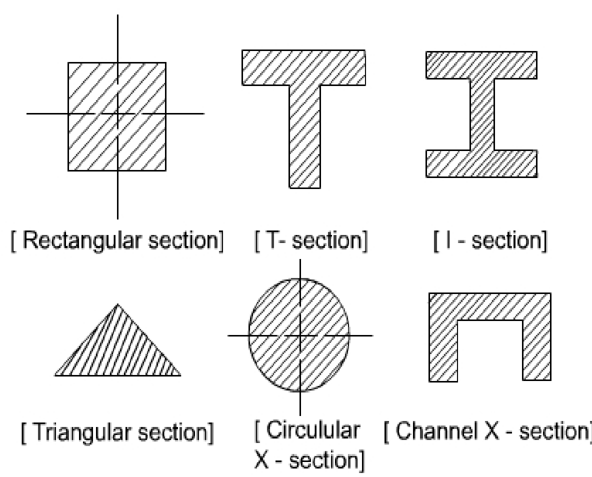
UNIT -3

Flexural and shear stresses in beams

- **Members Subjected to Flexural Loads**
- **Introduction:**
- In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are termed as beam.
- There are various ways to define the beams such as
- **Definition I:** A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.
- **Definition II:** A beam is nothing simply a bar which is subjected to forces or couples that lie in a plane containing the longitudinal axis of the bar. The forces are understood to act perpendicular to the longitudinal axis of the bar.
- **Definition III:** A bar working under bending is generally termed as a beam.
- **Materials for Beam:**
- The beams may be made from several usable engineering materials such commonly among them are as follows:
- Metal
- Wood
- Concrete
- Plastic

Geometric forms of Beams:

- The Area of X-section of the beam may take several forms some of them have been shown below:



Loading restrictions:

Concept of pure bending:

- As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since   or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

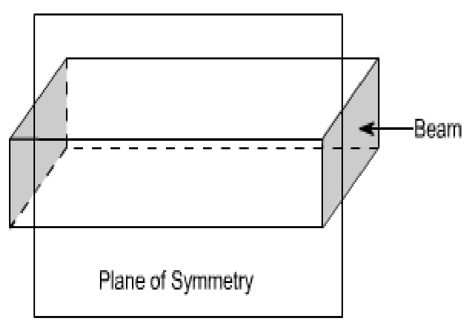


Fig (1)

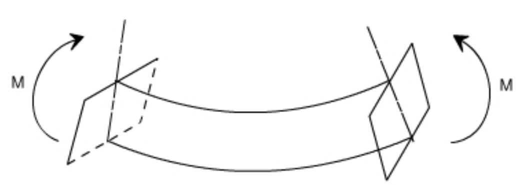


Fig (2)

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

- In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle
- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'
- Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

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$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'
then the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R} y\, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A$$

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- Now the term is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.
- Therefore $M = \frac{E}{R} I$ ....(2)

  combining equation 1 and 2 we get

  $$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

- **This equation is known as the Bending Theory Equation.** The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

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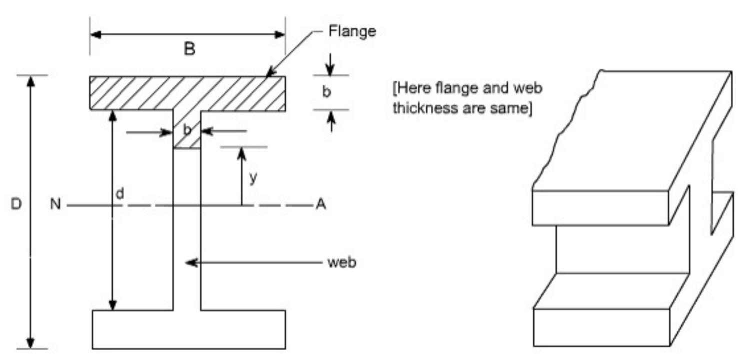
Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area

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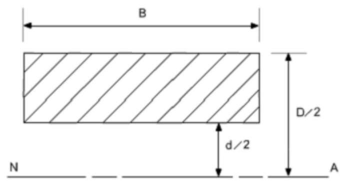


Flange area

Area of the flange $= B\left(\frac{D-d}{2}\right)$

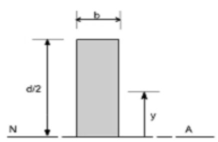
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

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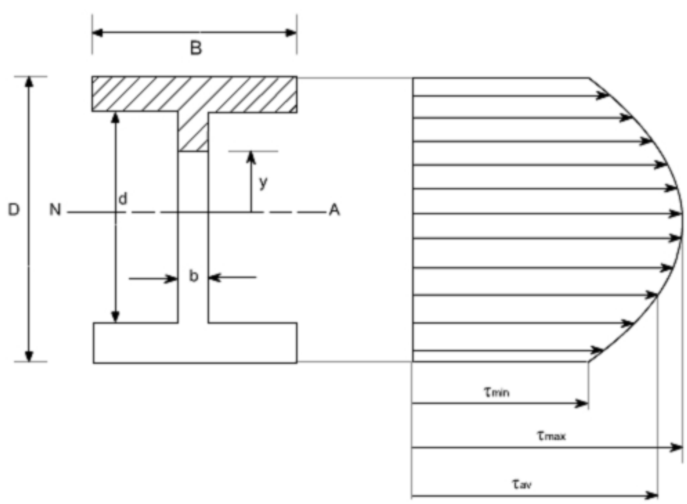

$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

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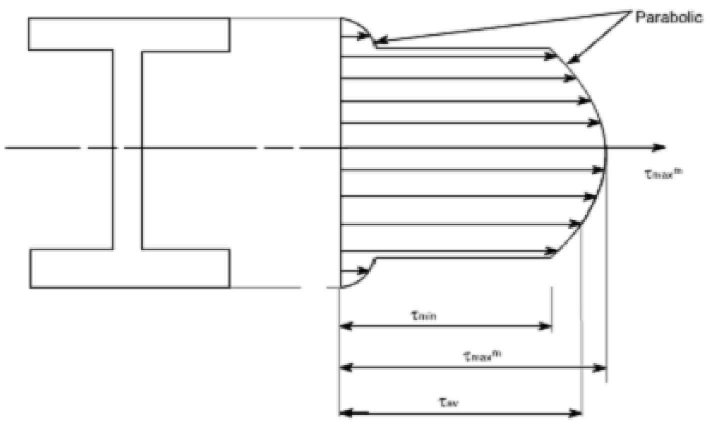

This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

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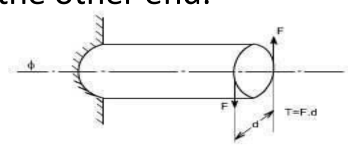
UNIT -4
TORSION OF CIRCULAR SHAFTS

Torsion of circular shafts

- **Definition of Torsion:** Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.
- **Effects of Torsion:** The effects of a torsional load applied to a bar are section with respect to the other end.nt of one end cross 1 section with respect to the other end.



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Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a 1 b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to a b'. The angle θ measured between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.

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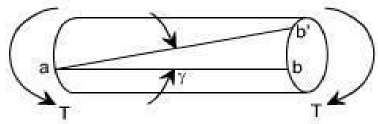

Modulus of Elasticity in shear: T he ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol

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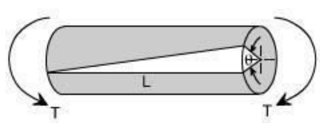
Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its l ngth, than the angle θ

through which one end of t he bar will twist relative to the other is known is the angle of twist.



Despite the difference s in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.

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Relationship in Torsion:

1st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area.

2nd Term: This refers to stress, and the stress increases as the distance from the axis increases.

3rd Term: it refers to the deformation and contains in which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross sectional area of the shaft.

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Assumption:

The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

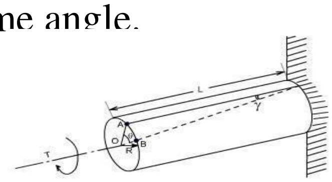
The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

The stress does not exceed the elastic limit.

The circular section remains circular

Cross section remain plane.

Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



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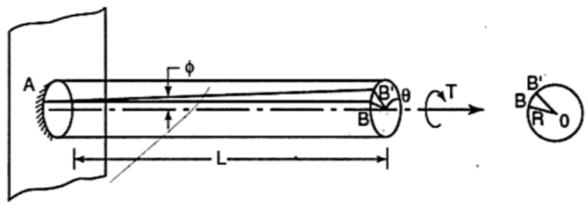
DERIVATION OF TORSIONAL EQUATIONS

Consider a shaft of length L, radius *R* fixed at one end and subjected to a torque T at the other end as shown in Fig.

Let O be the centre of circular section and B a point on surface. AB be the line on the shaft parallel to the axis of shaft. Due to torque T applied, let *B* move to *B'*. If γ is shear strain *(angle BOB')* and θ is the angle of twist in length *L,* then



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$$R\theta = BB' = L\gamma$$

If τ_s is the shear stress and *G* is modulus of rigidity then,

$$\gamma = \frac{\tau}{G}$$

$$R\theta = L\frac{\tau_s}{G}$$

$$\frac{\tau_s}{R} = \frac{G\theta}{L}$$

Similarly if the point *B* considered is at any distance *r* from centre instead of on the surface, it can be shown that

$$\frac{\tau}{r} = \frac{G\theta}{L} \qquad \ldots \text{(i)}$$

$$\frac{\tau_s}{R} = \frac{\tau}{r}$$

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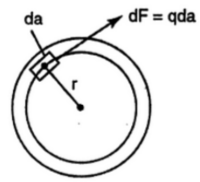
Thus shear stress increases linearly from zero at axis to the maximum value τ_s at surface.

Now consider the torsional resistance developed by an elemental area 'δa' at distance r from centre.

If τ is the shear stress developed in the element the resisting force is

$$dF = \tau da$$



Resisting torsional moment, dT= dF x r

= τrda

$$\tau = \tau_s \frac{r}{R}$$

Therefore,

$$dT = \tau_s \frac{r^2}{R} da$$

Total resisting torsional moment,

$$T = \sum \tau_s \frac{r^2}{R} da$$

$$T = \frac{\tau_s}{R} \sum r^2 da$$

But $\sum r^2 da$ is nothing but polar moment of inertia of the section. Representing it by notation J we get,

$$T = \frac{\tau_s}{R} J$$

$$\frac{T}{J} = \frac{\tau_s}{R}$$

$$\frac{\tau_s}{R} = \frac{\tau}{r}$$

$$\frac{T}{J} = \frac{\tau}{r}$$

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

Where,

T - torsional moment , N-mm

J - polar moment of inertia, mm^4

τ - shear stress in the element, N/mm^2

r- distance of element from centre of shaft, mm

G - modulus of rigidity, N/mm^2

θ- angle of twist, rad

L- length of shaft, mm

Power Transmitted by a shaft : If T is the applied to the shaft, then the power transmitted by the shaft is

Distribution of shear stresses in circular Shafts subjected to torsion :

This states that the shearing stress varies directly as the distance 1r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.

101

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transverse ly for instance a wooden shaft, with the fibers parallel t o axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.

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Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

Important types of springs are:

There are various types of springs such as

Helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.

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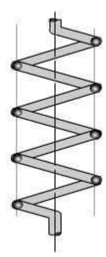
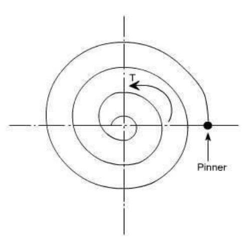


Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compression due to bending.



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Uses of springs :

To apply forces and to control motions as in brakes and clutches.
To measure forces as in spring balance.
To store energy as in clock springs.
To reduce the effect of shock or impact loading as in carriage springs.
To change the vibrating characteristics of a member as inflexible mounting of motors.

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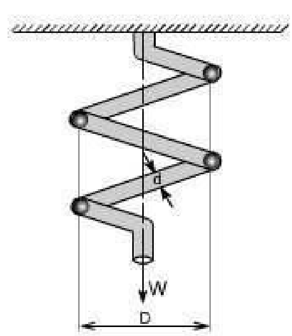
Derivation of the Formula :

In order to derive a necessary formula which governs the behavior of springs, consider a closed coiled spring subjected to an axial load W.



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Let
W = axial load
D = mean coil diameter d = diameter of spring wire
n = number of active coils
C = spring index = D / d For circular wires

l = length of spring wire G = modulus of rigidity
x = deflection of spring q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

$x = D / 2 . \theta$

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UNIT-5

COLUMNS & STRUTS

108

Introduction

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

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Struts

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

The strut may not be perfectly straight initially.

The load may not be applied exactly along the axis of the Strut.

One part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

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Euler's Theory

The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A

Strut with pinned ends

Consider an axially loaded strut, shown below, and is subjected to an axial load 1P' this load 1P' produces a deflection 1y' at a distance 1x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment at either end.

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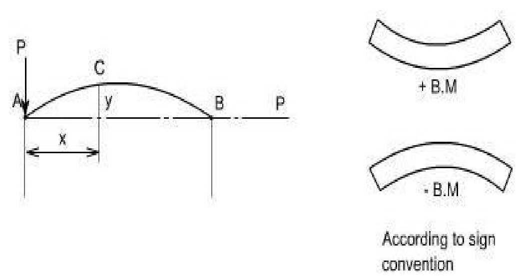
Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.



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Let us define a operator

D = d/dx

(D2 +n 2) y =0 where n2 = P/EI

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus y = A cos (nx) + B sin (nx) Where A and B are some constants.

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COLUMN BOTH ENDS PINNED (OR HINGED)

Consider a column AB of length l and uniform cross-sectional area, hinged at both of its ends A and B. Let P be the crippling load at which the column has just buckled. Due to the crippling load, the column will deflect into a curved form ACB as shown in fig. 8.5. Consider any section at a distance x from the end A.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section = - P . y

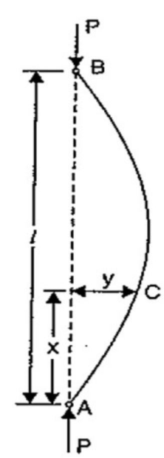

Fig. 8.5

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But moment $= EI \dfrac{d^2y}{dx^2}$.

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{(i)}$$

*The equation $\dfrac{d^2y}{dx^2} + \dfrac{P}{EI} \times y = 0$ can be written as $\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \dfrac{P}{EI}$ or $\alpha = \sqrt{\dfrac{P}{EI}}$

The solution of the equation is $y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) \text{ as } \alpha = \sqrt{\frac{P}{EI}}$$

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Where C_1 and C_2 are the constants of integration. The values of C_1 and C_2 are obtained as given below:

(i) At A, $x = 0$ and $y = 0$

Substituting these values in equation (i), we get

$$0 = C_1 . \cos 0° + C_2 \sin 0$$

$$= C_1 \text{ x } 1 + C_2 \text{ x } 0$$

Therefore, $C_1=0$ (ii)

(ii) *At B, $x=l$ and $y = 0$*

Substituting these values in equation (i), we

$$0 = C_1 \cdot \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right)$$

$$= 0 + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \qquad [\because C_1 = 0 \text{ from equation } (ii)]$$

$$= C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right) \qquad ...(iii)$$

From equation (iii), it is clear that either $C_2 = 0$

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As $C_1 = 0$, then if C_2 is also equal to zero, then from equation (i) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. This is not true.

$$\therefore \quad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or}$$

$$l\sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \ldots$$

Taking the least practical value,

$$l\sqrt{\frac{P}{EI}} = \pi$$

$$P = \frac{\pi^2 EI}{l^2}.$$

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COLUMN ONE END FIXED AND OTHER END FREE

Consider a column AB, of length l and uniform cross-sectional area, fixed at the end A and free at the end B. The free end will sway sideways when load is applied at free end and curvature in the length l will be similar to that of upper half of the column whose both ends are hinged. Let P is the crippling load at which the column has just buckled. Due to the crippling load P, the column will deflect as shown in Fig. 8.6 in which AB is the original position of the column and AB', is the deflected position due to crippling load P.

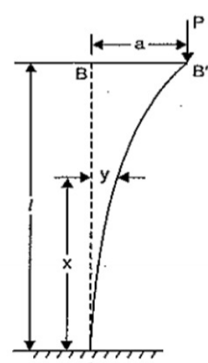

Fig.8.6

Consider any section at a distance x from the fixed end A.

Let y = Deflection (or lateral displacement) at the section

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Let y = Deflection (or lateral displacement) at the section

a = Deflection at the free end B. Then moment at the section due to the crippling load = P (a - y)

But moment, $M = EI\frac{d^2y}{dx^2}$

Equating the two moments, we get

$$EI\frac{d^2y}{dx^2} = P(a - y) = P.a - P.y$$

$$EI\frac{d^2y}{dx^2} + P.y = P.a$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P}{EI}.a.$$

The solution* of the differential equation is

$$y = C_1 . \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 . \sin\left(x\sqrt{\frac{P}{EI}}\right) + a \quad \text{(i)}$$

Where C_1 and C_2 are constant of integration; the values of C_1 and C_2 are obtained from boundary conditions. The boundary conditions are:

(i) For a fixed end, the deflection as well as slope is zero.

Hence at end A (which is fixed), the deflection $y = 0$ and also slope $\frac{dy}{dx} = 0$

Hence at A, $x = 0$ and $y = 0$

Substituting these values in equation *(i),* we get

$$0 = C_1 \cos 0 + C_2 \sin 0 + a$$

$$= C_1 \times 1 + C_2 \times 0 + a$$

$$= C_1 + a \quad \text{(ii)}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$

Differentiating the equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = C_1 . (-1) \sin\left(x\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 . \sqrt{\frac{P}{EI}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 . \sqrt{\frac{P}{EI}} \cos\left(x\sqrt{\frac{P}{EI}}\right)$$

But at $x = 0$ and $\frac{dy}{dx} = 0$

The above equation becomes as

$$0 = -C_1 . \sqrt{\frac{P}{EI}} \sin 0 + C_2 \sqrt{\frac{P}{EI}} \cos 0$$

$$= -C_1 \sqrt{\frac{P}{EI}} \times 0 + C_2 . \sqrt{\frac{P}{EI}} \times 1 = C_2 \sqrt{\frac{P}{EI}}$$

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From the above equation it is clear that either $C_2 = 0$,

or $\sqrt{\frac{P}{EI}} = 0$

But for the crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

Therefore, $C_2 = 0$.

Substituting the values of $C_1 = -a$ and $C_2 = 0$ in equation (i) we get

$$y = -a \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + a \qquad \text{(iii)}$$

But at the free end of the column, $x = l$ and $y = a$. Substituting these values in equation *(iii)* we get

$$a = -a . \cos\left(l . \sqrt{\frac{P}{EI}}\right) + a$$

$$0 = -a . \cos\left(l . \sqrt{\frac{P}{EI}}\right) \text{ or } a \cos\left(l . \sqrt{\frac{P}{EI}}\right) = 0$$

But 'a' cannot be equal to zero

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$$\therefore \quad \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = 0 = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2} \text{ or } \cos\frac{5\pi}{2} \text{ or } \ldots\ldots$$

$$\therefore \quad l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \ldots\ldots$$

Taking the least practical value,

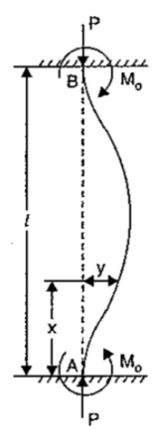
$$l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{2l}$$

$$P = \frac{\pi^2 EI}{4l^2}$$

123

COLUMN BOTH THE ENDS OF THE COLUMN ARE FIXED

Consider a column AB, of length l and uniform cross-sectional area, both ends A and B fixed. Let P is the crippling load at which the column has just buckled. Due to the crippling load P, the column will deflect as shown in Figure. Due to fixed ends, there will be fixed end moments (say Mo) at the ends A and B. The fixed end moments will be acting in such direction so that slope at the fixed ends becomes zero.



Consider any section at a distance x from the fixed end A.

Let y = Deflection (or lateral displacement) at the section
As the both the ends of the column are fixed and column carries a crippling load, there will be some fixed end moments at A and B.

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Let M_0 = Fixed end moments at A and B.
Then moment at the section = $M_0 - P.y$

But moment at the section is also = $EI\ \dfrac{d^2y}{dx^2}$

∴ Equating the two moments, we get

$$EI\ \frac{d^2y}{dx^2} = M_0 - P.y$$

or $$EI\ \frac{d^2y}{dx^2} + P\ .\ y = M_0$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI}\ .\ y = \frac{M_0}{EI} \qquad \text{...(A)}$$

$$= \frac{M_0}{EI} \times \frac{P}{P} = \frac{P}{EI}\ .\ \frac{M_0}{P}$$

The solution* of the above differential equation is

$$y = C_1\ .\ \cos\left(x\ .\sqrt{\frac{P}{EI}}\right) + C_2\ .\ \sin\left(x\ .\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad \text{...(i)}$$

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where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. Boundary conditions are :

(*i*) At A, $x = 0$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as A is a fixed end.

(*ii*) At B, $x = l$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as B is also a fixed end.

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{M_0}{P} \qquad (\because \quad \cos 0 = 1)$$

$$= C_1 + \frac{M_0}{P}$$

∴ $$C_1 = -\frac{M_0}{P} \qquad \text{...(ii)}$$

Differentiating the equation (*i*), with respect to x, we get

$$\frac{dy}{dx} = C_1\ .\ (-1)\sin\left(x\ .\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2\cos\left(x\ .\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1\sin\left(x\ .\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2\cos\left(x\ .\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}}$$

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Substituting the value $x = 0$ and $\frac{dy}{dx} = 0$, the above equation becomes

$$0 = -C_1 \times 0 + C_2 \times 1 \times \sqrt{\frac{P}{EI}} \qquad (\because \sin 0 = 0 \text{ and } \cos 0 = 1)$$

$$= C_2 \sqrt{\frac{P}{EI}}.$$

From the above equation, it is clear that either $C_2 = 0$ or $\sqrt{\frac{P}{EI}} = 0$. But for a given crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$$\therefore \qquad C_2 = 0.$$

Now substituting the values of $C_1 = -\frac{M_0}{P}$ and $C_2 = 0$ in equation (i), we get

$$y = -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + 0 + \frac{M_0}{P}$$

$$= -\frac{M_0}{P} \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(iii)$$

At the end B of the column, $x = l$ and $y = 0$.
Substituting these values in equation (iii), we get

$$0 = -\frac{M_0}{P} \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

or $$\frac{M_0}{P} \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}$$

or $$\cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P} \times \frac{P}{M_0} = 1 = \cos 0, \cos 2\pi, \cos 4\pi, \cos 6\pi, ...$$

$$\therefore \qquad l \cdot \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, ...$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P = \frac{\pi^2 EI}{l^2}.$$

COLUMN ONE END FIXED AND OTHER END HINGED

Consider a column AB of length l and uniform cross-sectional area fixed at the end A and hinged at the end B as shown in Fig. 19.7. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.7.

There will be fixed end moment (M_0) at the fixed end A. This will try to bring back the slope of deflected column zero at A. Hence it will be acting anticlock wise at A. The fixed end moment M_0 at A is to be balanced. This will be balanced by a horizontal reaction (H) at the top end B as shown in Fig. 19.7.

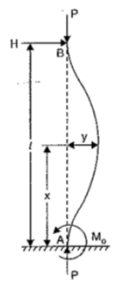

Fig. 19.7

Consider a section at a distance x from the end A.

Let y = Deflection of the column at the section,

M_0 = Fixed end moment at A, and

H = Horizontal reaction at B.

The moment at the section = Moment due to crippling load at B + Moment due to horizontal reaction at B

$$= -P \cdot y + H \cdot (l - x)$$

But the moment at the section is also

$$= EI \frac{d^2y}{dx^2}$$

Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = -P \cdot y + H(l - x)$$

or $EI \frac{d^2y}{dx^2} + P \,.\, y = H(l - x)$

or $\frac{d^2y}{dx^2} + \frac{P}{EI} \,.\, y = \frac{H}{EI}(l - x)$ (Dividing by EI) ...(A)

$$= \frac{H}{EI}(l - x) \times \frac{P}{P} = \frac{P}{EI} \,.\, \frac{H(l - x)}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x) \quad ...(i)$$

where C_1 and C_2 are constants of integration and their values are obtained from boundary conditions. Boundary conditions are :

(*i*) At the fixed end A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$

(*ii*) At the hinged end B, $x = l$ and $y = 0$.

Substituting the value $x = 0$ and $y = 0$ in equation (*i*), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{H}{P}(l - 0) = C_1 + \frac{H \,.\, l}{P}$$

$$\therefore \quad C_1 = -\frac{H}{P} \,.\, l \quad ...(ii)$$

Differentiating the equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1(-1)\sin\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$= -C_1 \sin\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} + C_2 \cos\left(x \,.\, \sqrt{\frac{P}{EI}}\right) \,.\, \sqrt{\frac{P}{EI}} - \frac{H}{P}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$.

$$\therefore \quad 0 = -C_1 \times 0 + C_2 \,.\, 1 \,.\, \sqrt{\frac{P}{EI}} - \frac{H}{P} \quad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2 \sqrt{\frac{P}{EI}} - \frac{H}{P} \quad \text{or} \quad C_2 = \frac{H}{P} \sqrt{\frac{EI}{P}} .$$

Substituting the values of $C_1 = -\frac{H}{P}$, l and $C_2 = \frac{H}{P} \sqrt{\frac{EI}{P}}$ in equation (i), we get

$$y = -\frac{H}{P} \cdot l \cos\left(x \sqrt{\frac{P}{EI}}\right) + \frac{H}{P} \sqrt{\frac{EI}{P}} \sin\left(x \sqrt{\frac{P}{EI}}\right) + \frac{H}{P} (l - x)$$

At the end B, $x = l$ and $y = 0$.

Hence the above equation becomes as

$$0 = -\frac{H}{P}\, l \cos\left(l \sqrt{\frac{P}{EI}}\right) + \frac{H}{P} \sqrt{\frac{EI}{P}} \sin\left(l \sqrt{\frac{P}{EI}}\right) + \frac{H}{P} (l - l)$$

$$= -\frac{H}{P}\, l \cos\left(l \sqrt{\frac{P}{EI}}\right) + \frac{H}{P} \sqrt{\frac{EI}{P}} \sin\left(l \sqrt{\frac{P}{EI}}\right) + 0$$

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or $$\frac{H}{P} \sqrt{\frac{EI}{P}} \sin\left(l \sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\, l \cos\left(l \sqrt{\frac{P}{EI}}\right)$$

or $$\sin\left(l \sqrt{\frac{P}{EI}}\right) = \frac{H}{P} \cdot l \times \frac{P}{H} \times \sqrt{\frac{P}{EI}} \cdot \cos\left(l \sqrt{\frac{P}{EI}}\right)$$

$$= l \cdot \sqrt{\frac{P}{EI}} \cdot \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right)$$

or $$\tan\left(l \sqrt{\frac{P}{EI}}\right) = l \cdot \sqrt{\frac{P}{EI}} .$$

The solution to the above equation is, $l \cdot \sqrt{\frac{P}{EI}} = 4.5$ radians

Squaring both sides, we get

$$l^2 \cdot \frac{P}{EI} = 4.5^2 = 20.25$$

$\therefore$ $$P = 20.25 \frac{EI}{l^2}$$

But approximately $20.25 = 2\pi^2$

$\therefore$ $$P = \frac{2\pi^2 EI}{l^2}$$

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EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column,

l = Actual length of the column, and

P = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^2}. \qquad \text{...(19.5)}$$

The crippling load (P) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table

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S.No.	End conditions of column	Crippling load in terms of		Relation between effective length and actual length
		Actual length	Effective length	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia *i.e.*, I_{xx} and I_{yy}.

The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

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Crippling Stress in Terms of Effective Length and Radius of Gyration. The moment of inertia (I) can be expressed in terms of radius of gyration (k) as

$$I = Ak^2 \text{ where } A = \text{Area of cross-section.}$$

As I is the least value of moment of inertia, then

k = Least radius of gyration of the column section.

Now crippling load P in terms of effective length is given by

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 E \times Ak^2}{L_e^2} \quad (\because \; I = Ak^2)$$

$$= \frac{\pi^2 E \times A}{\frac{L_e^2}{k^2}} = \frac{\pi^2 E \times A}{\left(\frac{L_e}{k}\right)^2} \quad ...(19.6)$$

And the stress corresponding to crippling load is given by

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}} = \frac{P}{A}$$

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$$= \frac{\pi^2 E \times A}{A\left(\frac{L_e}{k}\right)^2} \quad \text{(Substituting the value of } P\text{)}$$

$$= \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} \quad ...(19.7)$$

Slenderness Ratio. The ratio of the actual length of a column to the least radius of gyration of the column, is known as slenderness ratio.

Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}} = \frac{l}{k} \quad ...(19.8)$$

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