MUTHAYAMMAL ENGINEERING COLLEGE RASIPURAM

TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

19BSS23

Prepared by : Department of Mathematics

SYLLABUS

- **UNIT – I FOURIER TRANSFORMS**

  Statement of Fourier integral theorem - Fourier transforms pair – Fourier sine and cosine transforms – Properties – Transforms of simple functions – Convolution theorem – Parseval's identity–Problems .

- **UNIT – II Z - TRANSFORMS AND DIFFERENCE EQUATIONS**

  Z- transforms - Elementary properties – Initial and final value theorem – Inverse Z - transforms – Partial fraction method – Residue method – Convolution theorem - Formation of difference equations – Solution of difference equations using Z – transforms

- **UNIT - III FOURIER SERIES**

  Dirichlet's conditions – General Fourier series – Odd and even functions – Half range sine series – Half range cosine series – Parseval's identity – Harmonic analysis

- **UNIT – IV BOUNDARY VALUE PROBLEMS**

  Classification of PDE - Solutions of one dimensional wave equation – One dimensional equation of heat conduction – Fourier series solution in Cartesian coordinates – Steady state solution of two dimensional equation of heat conduction (excluding insulated edges) on finite square plates (excluding circular plates).

- **UNIT – V PARTIAL DIFFERENTIAL EQUATIONS**

  Formation of partial differential equations – Singular integrals – Solutions of standard types of first order partial differential equations – Lagrange's linear equation - Linear partial differential equations of second and higher order with constant coefficients of homogeneous when the R.H.S is e^{ax+by}, $x^m y^n$ m,n>0, sin (ax+by), cos (ax+by)

Text Books:

- Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley and Sons, New Delhi-2014.
- Grewal. B.S, Higher Engineering Mathematics, 43rd Edition, Khanna Publications, Delhi-2014.

FOURIER TRANSFORM

Introduction

Jean-Baptiste Joseph Fourier (21st March 1768 – 16th May1830)was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law of conduction are also named in his honour. **Joseph Fourier** introduced the **transform**in his study of heat transfer, where Gaussian functions appear as solutions of the heat equation.

In the study of **Fourier** series, complicated but periodic functions are written as the sum of **simple** waves mathematically represented by sine and cosine functions. The **Fourier transform** is an extension of the **Fourier** series that results when the period of the represented function is lengthened and allowed to approach infinity. Fourier Transform maps a time series (eg. audio samples) into the series of frequencies (their amplitudes and phases) that composed the time series. Inverse Fourier Transform maps the series of frequencies (their amplitudes and phases) back into the corresponding time series. The two functions are inverses of each other. Shortly, The Fourier Transform is a mathematical technique that transforms a function of time, $f(t)$, to a function of frequency, $F(s)$.

Applications

- The Fourier transform has many applications, in fact any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier series and Fourier transforms. Here are some examples from physics, engineering, and signal processing.
    - Communication
    - Astronomy
    - Geology
    - Optics

- Fourier Transforms helps to analyze spectrum of the signals, helps in find the response of the LTI systems. (Continuous Time Fourier Transforms is for Analog signals and Discrete time Fourier Transforms is for discrete signals)

- Discrete Fourier Transforms are helpful in Digital signal processing for making convolution and many other signal manipulation.

Fourier Transforms

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty,\infty)$ $f : R \to C$ and be piece-wise continuous each finite partial interval then the complex Fourier transform of $f(x)$ is defined by

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Inverse Fourier Transform

Inverse complex Fourier transform of F(s) is given by

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

Properties of Fourier Transforms

1. Linearity property

If $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, then

$$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$$

2. Shifting property

If $F[f(x)] = F(s)$ then $F(f(x - a)) = e^{ias} F[f(x)] = e^{ias} F(s)$

3. Changeof scale property

$$\text{If } F[f(x)] = F(s) \text{ then } F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right) \text{ where } a \neq 0$$

4. Shifting in s

If $F[f(x)] = F(s)$ then $F(e^{iax} f(x)) = F(s+a)$

5. Modulation Property

$$\text{If } F(f(x)) = F(s) \text{ then } F[\cos ax f(x)] = \frac{1}{2}[F(s+a)+F(s-a)]$$

6. Fourier transform of Derivative

If $F[f(x)] = F(s)$ and derivative $f'(x)$ is continuous, absolutely integrable on $(-\infty, \infty)$, then $F[f'(x)] = -(is) F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

7. Derivative of transform

$$\text{If } F[f(x)] = F(s) \text{, then } F(x^n f(x)) = (-i)^n \frac{d^n F(s)}{ds^n}$$

Definition: Convolution of two functions.

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

PROBLEMS

Problem 1. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence evaluate $\int_0^{\infty} \frac{\sin s}{s} ds$.

Solution: Fourier transform of $f(x)$ is $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx \qquad (\because \sin sx \text{ is an odd } fn.)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a \cos sx \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin as}{s} \right]$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} (\cos sx - i \sin sx) \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx \, ds \qquad \left[\because \frac{\sin as}{s} \sin \, sx \; is \; odd \right]$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin as}{s}\right)\cos sx\, ds$$

Put $a = 1, x = 0$

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin s}{s}\, ds$$

$$\frac{\pi}{2}\times 1 = \int_0^\infty \frac{\sin s}{s}\, ds \;(\because f(x) = 1, -a \le x \le a)$$

$$\therefore \int_0^\infty \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

Definition: If the fourier transform of f(x) is equal to f(s) then the function f(x) is called **self-reciprocal**. i.e. F(f(x)) = f(s)

Problem 2: Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2)+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} dx \qquad \ldots (1)$$

Consider $a^2x^2 - isx = (ax)^2 - 2(ax)\frac{(is)}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad \ldots (2)$$

Substitute (2) in (1), we get

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2+\frac{s^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \qquad \text{Let } t = ax - \frac{is}{2a}, dt = a dx$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{\frac{s^2}{4a^2}} \qquad \ldots (3)$$

$$\text{Put } a = \frac{1}{\sqrt{2}} in\ (3)$$

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$\therefore e^{-s^2/2}$ is self-reciprocal with respect to Fourier Transform.

Problem 3: State and Prove convolution theorem on Fourier transform.

Solution:

Statement: If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$\text{i.e. } F[f(x)*g(x)] = F[f(x)]F[g(x)] = F(s)G(s)$$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}dx\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\,dx\right)dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F(g(x-t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, F(g(t)) dt \qquad [\therefore f(g(x-t) = e^{ist} F(g(t))]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt G(s) \qquad [\therefore F(g(t)) = G(s)]$$

$$F(f * g) = F(s).G(s). \qquad [\therefore F(f(t)) = F(s)].$$

Problem 4: Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ and hence evaluate

(i) $\int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right) dt$ **(ii)** $\int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right)^2 dt$

Solutions:

Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{isx} dx$$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-a}^{a} (a^2 - x^2) e^{isx} dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin x)\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2) \cos sx\, dx \left[\because (a^2 - x^2) \sin sx \text{ is an odd fn.} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(a^2 - x^2) \left(\frac{\sin sx}{s} \right) - (-2x) \left(\frac{-\cos sx}{s^2} \right) + (2) \left(\frac{\sin sx}{s^3} \right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{2a \cos as}{s^2} + \frac{2 \sin as}{s^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-2as \cos as + 2 \sin as}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \qquad \ldots (1)$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\,dx$$ (the second term is on odd function)

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\,dx$$

Put $a = 1$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos sx\,dx \qquad \left[f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases}\right]$$

Put $x = 0$

$$f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\,dx \qquad \begin{bmatrix} f(0) = 1 - 0 \\ = 1 \end{bmatrix}$$

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds$$

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4} \quad \text{[by changing s} \rightarrow \text{t]}$$

Using Parseval's identify

$$\int_{-\infty}^\infty |F(s)|^2\,ds = \int_{-\infty}^\infty |f(x)|^2\,dx$$

$$\int_{-\infty}^\infty \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-\infty}^\infty |a^2 - x^2|^2\,dx$$

$$\int_{-\infty}^\infty \frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^1 (1 - x^2)^2\,dx \text{ (put a = 1)}$$

$$2 \times \frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1 - x^2)^2\,dx$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{16}\times 2\left(\frac{8}{15}\right) = \frac{\pi}{15}$$

Put $s = \text{t}$, $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

Problem 5: Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ and hence find the value of **(i)** $\int_0^\infty \frac{\sin^2 t}{t^2} dt$. **(ii)** $\int_0^\infty \frac{\sin^4 t}{t^4} dt$.

Solution:

The Fourier transform of $F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\,(\cos sx + i\sin sx)dx$$

The Fourier transform of $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx \, [\therefore (1-|x|) \sin sx \text{ is an odd fn.}]$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ (1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ -\frac{\cos s}{s^2} + \frac{1}{s^2} \right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] \quad (1)$$

(i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] (\cos sx - i \sin sx) ds \text{ (by (1))}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx \, ds \quad \text{(Second term is odd)}$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx \, ds$$

Put $x = 0$

$$f(0) = 1 - |0| = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds$$

$$\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{2\sin^2(s/2)}{s^2} ds = \frac{\pi}{2}$$

Put $t = s/2 \qquad ds = 2dt$

$$\int_0^\infty \frac{2\sin^2 t}{(2t)^2} 2dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right]^2 ds = \int_{-1}^{1} (1-|x|)^2 \, dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2 \, dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1} (1-x)^2 \, dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^2 ds = \left[2\left(\frac{1-x}{-3}\right)^3\right]_0^1$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds = \frac{2}{3}; Let\ t = s/2, dt = \frac{ds}{2}$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 2dt = \frac{2}{3}$$

$$\frac{16}{16\pi}\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 dt = \frac{1}{3}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

Fourier Sine and Cosine Transform

Infinite Fourier Sine Transform of f(x) is denoted by F_s{f(x)] and is defined as

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

Inverse Fourier Sine Transform is

$$f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$$

The Fourier cosine Integral of f(x) in $(0,\infty)$ is

$$F_C[f(x)] = F_C(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

This is known as **Infinite Fourier Cosine Transform** of f(x).

The **Inverse Fourier Cosine Transform** is

$$f(x) = F^{-1}[F_C(s)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C(s) \cos sx \, ds$$

Properties of Fourier Sine and Cosine Transforms

1. Linearity Property

(i) $F_c[af(x) + bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

(ii) $F_s[af(x) + bg(x)] = aF_s[f(x)] + bF_s[g(x)]$ where a and b are constants.

2. Modulation property

If $F_c[f(x)] = F_c[s]$ and $F_s[f(x)] = F_s[s]$, then

(i) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iii) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(iv) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

3. Change of Scale Property

(i) $F_C[f(ax)] = \frac{1}{a}F_C\left(\frac{s}{a}\right) \quad if\ a > 0$ (ii) $F_S[f(ax)] = \frac{1}{a}F_S\left(\frac{s}{a}\right) \quad if\ a > 0$

4. Differentiation of sine and cosine transform

(i) $F_C[xf(x)] = \frac{d}{ds}[F_S(s)] = \frac{d}{ds}[F_S(f(x))]$

(ii) $F_S[xf(x)] = -\frac{d}{ds}[F_C(s)] = -\frac{d}{ds}[F_C(f(x))]$

5. Identities

If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively then

i) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds$$

ii) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)ds$$

iii) $$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds$$

Problem 1: Find the Fourier cosine and sine transformation of $f(x) = e^{-ax}$, $a > 0$. Hence deduce that $\int_0^\infty \frac{x\sin\alpha x}{1+x^2}dx = \frac{\pi}{2}e^{-\alpha}$. and $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt = \frac{\pi}{2a}e^{-a|x|}$

Problem 1: Find the Fourier cosine and sine transformation of $f(x) = e^{-ax}$, $a > 0$. Hence deduce that $\int_0^\infty \frac{x \sin \alpha x}{1+x^2} dx = \frac{\pi}{2} e^{-\alpha}$. and $\int_0^\infty \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$

Solution:

The Fourier cosine transform is $F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$

$$\Rightarrow F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a+0) \right] = \sqrt{\frac{2}{\pi}} . \frac{a}{a^2+s^2}$$

The Fourier sine transform is $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+s^2}(0-s)\right] = \sqrt{\frac{2}{\pi}}.\frac{s}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

By inverse Sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sx\, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}\, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}\, ds$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

Put $a = 1, x = \alpha$

$$\frac{\pi}{2} e^{-a} = \int_0^\infty \frac{s \sin sx}{s^2 + 1} ds$$

Replace 's' by 'x' and 'x' by 's'

$$\int_0^\infty \frac{x \sin sx}{1 + x^2} dx = \frac{\pi}{2} e^{-\alpha}.$$

Using Fourier inverse cosine transform,

$$\text{f(x)} = e^{-a|x|} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \cos sx\, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2 + s^2} ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace '}s\text{' by '}t\text{')}$$

$$\int_0^\infty \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

Problem 2: Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \left[\frac{\cos(s+1)x + \cos(s-1)x}{2} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right]$$, provided $s \neq 1, s \neq -1$.

Problem3: Using Parseval's Identity calculate

(a) $\int_0^\infty \frac{1}{(a^2+x^2)^2} dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$

Solution: (a) By Parseval's identity.

$$\int_0^\infty e^{-2ax} dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right]^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi} a^2 \int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{a^2+s^2}$$

i.e. $\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$ [Replace, s by x]

(b) By Parseval's identity.

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(f(x))|^2\, ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)^2 ds$$

$$\text{i.e}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\, ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\pi}{2}\times\frac{1}{2a}$$

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\, dx = \frac{\pi}{4a} \quad \text{[Replace, } s \text{ by } x\text{]}$$

Problem 4. Evaluate **(a)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$, using Fourier cosine and sine transform.

Solution: **(a)** Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

$$F_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{s^2+1} (-\cos x + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+1} \right] \qquad \ldots (1)$$

$$F_c(e^{-2x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2}{s^2+4} \right) \qquad \ldots (2)$$

$$\therefore \int_0^\infty f(x) g(x) dx = \int_0^\infty F_c(f(x)) F_c(g(x)) ds$$

$$\int_0^\infty e^{-x} e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \left(\frac{1}{s^2+1} . \frac{2}{s^2+4} \right) ds \ (from\ (1)\ \&\ (2))$$

$$\int_0^\infty e^{-3x} dx = \frac{4}{\pi} \int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} ds$$

$$\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{\pi}{4} \left[\frac{e^{-3x}}{-3} \right]_0^\infty = \frac{\pi}{4} \left(\frac{1}{3} \right)$$

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12} \quad \text{[Replace s to x]}$$

(b) To find $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx.$

Let

$$f(x) = e^{-ax}, g(x) = e^{-bx}$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2} \right) \quad \ldots (1)$$

$$F_s(g(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+b^2} \right) \quad \ldots (2)$$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s[f(x)].F_s[g(x)]ds \text{ From (1) and (2)}$$

$$\int_0^\infty e^{-ax}e^{-bx}dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds$$

i.e $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2(a+b)}$ [Replace s to x]

Problem 5. Find the Fourier Cosine Transform of e^{-x^2} and hence Show that $xe^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier sine transform.

Solution

The Fourier Cosine Transform of f(x) is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}.2\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\text{ R.P of } e^{-\frac{s^2}{4}}\int_{-\infty}^\infty \frac{e^{-x^2+isx}}{e^{\frac{-s^2}{4}}}dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-x^2+isx+\frac{s^2}{4}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2} dx$$

Put $x - \frac{is}{2} = y; \quad dx = dy$

When $x = -\infty, \quad y = -\infty$

$x = \infty, \quad y = \infty$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} 2\int_{0}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}} 2\frac{\sqrt{\pi}}{2} \qquad \left(\because \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$F_c\left(e^{-x^2}\right) = \frac{1}{\sqrt{2}} e^{\frac{-s^2}{4}}$$

Result : $F_s\left[x e^{\frac{-x^2}{2}}\right] = -\frac{d}{ds} F_c\left[e^{\frac{-x^2}{2}}\right]$

But $F_c\left[e^{\frac{-x^2}{2}}\right] = e^{\frac{-s^2}{2}}$

$$F_s\left[x e^{\frac{-x^2}{2}}\right] = -\frac{d}{ds}\left(e^{\frac{-s^2}{2}}\right)$$

$$= -e^{\frac{-s^2}{2}} \cdot \left(\frac{-2s}{2}\right)$$

$$= s e^{\frac{-s^2}{2}}$$

$\therefore \quad x e^{\frac{-s^2}{2}}$ is self reciprocal with respect to sine transform

Z-Transforms and Difference Equations

Definition:

Let $\{f(n)\}$be a sequence defined for $n = 0, \pm1, \pm2, \pm3, \ldots$then the Z-transform of $f(n)$is defined as $Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} = F(z)$ which is known as two sided or Bilateral Z-transform of $f(n)$.

If $f(n) = 0 \; for \; n < 0$, then the Z-transform reduces to one sided or Unilateral Z-transform and is defined as $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = F(z)$.

Z-transform for discrete values of t:

If the function $f(t)$ is defined at discrete values of t, where $t = nT,\; n = 0,1,2,3, \ldots \infty$,T being the sampling period, then $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n} = F(z)$.

Z-transform of standard functions:

1. $Z\{a^n\} = \frac{z}{z-a} \; if \; |z| > |a|$

   Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}$$

Note:

(i) When a = 1, $Z\{1\} = \frac{z}{z-1}$

(ii) When a = -1, $Z\{(-1)^n\} = \frac{z}{z+1}$

2. $Z(k) = k\frac{z}{z-1}$

Proof: By definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$= \sum_{n=0}^{\infty} k\, z^{-n} = k \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$Z\{k\} = k\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right] = k\left[1 - \frac{1}{z}\right]^{-1} = k\left[\frac{z-1}{z}\right]^{-1} = k\frac{z}{z-1}$$

3. $Z\{n\} = \frac{z}{(z-1)^2}$

Proof: By definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{n\} = \sum_{n=0}^{\infty} n\, z^{-n} = \frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right] = \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} = \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z^2}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\therefore Z\{n\} = \frac{z}{(z-1)^2}$$

4. $Z\left\{\frac{1}{n}\right\} = log\left(\frac{z}{z-1}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\left\{\frac{1}{n}\right\} = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n = \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 \ldots$$

$$= -log\left(1 - \frac{1}{z}\right) = -log\left(\frac{z-1}{z}\right) \left[\because \log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty\right]\right.$$

$$\therefore \; Z\left\{\frac{1}{n}\right\} = log\left(\frac{z}{z-1}\right)$$

5. $Z\left\{\frac{1}{n+1}\right\} = zlog\left(\frac{z}{z-1}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\left\{\frac{1}{n+1}\right\} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{z}\right)^{n} = 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^{2} + \frac{1}{4}\left(\frac{1}{z}\right)^{3} + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^{2} + \frac{1}{3}\left(\frac{1}{z}\right)^{3} + \cdots\right]$$

$$= -z\log\left(1 - \frac{1}{z}\right) = -z\log\left(\frac{z-1}{z}\right) \left[\because \log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty\right]\right.$$

$$\therefore \; Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right)$$

6. $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^{n} = 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^{2} + \frac{1}{3!}\left(\frac{1}{z}\right)^{3} + \cdots$$

$$\therefore \; Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$$

7. $Z\{na^n\} = \frac{az}{(z-a)^2}$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{na^n\} = \sum_{n=0}^{\infty} n\, a^n\, z^{-n} = \sum_{n=0}^{\infty} n\left(\frac{a}{z}\right)^n = \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= \frac{a}{z}\left(1 - \frac{a}{z}\right)^{-2} = \frac{a}{z}\left(\frac{z-a}{z}\right)^{-2} = \frac{az}{(z-a)^2}.$$

8. $Z\{cosn\theta\} = \frac{z(z-cos\theta)}{z^2-2zcos\theta+1}$ and $Z\{sinn\theta\} = \frac{zsin\theta}{z^2-2zcos\theta+1}$

Proof: Let $a = e^{i\theta}$

We know that $Z\{a^n\} = \frac{z}{z-a}$ $\Rightarrow Z\left\{\left(e^{i\theta}\right)^n\right\} = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(cos\theta+isin\theta)}$

$$Z\{(cos\theta + isin\theta)^n\} = \frac{z}{(z - cos\theta) - isin\theta}$$

$$Z\{cosn\theta + isinn\theta\} = \frac{z[(z - cos\theta) + isin\theta]}{\left((z - cos\theta)\right)^2 + sin^2\theta} = \frac{z(z - cos\theta) + izsin\theta}{z^2 - 2zcos\theta + cos^2\theta + sin^2\theta}$$

$$Z\{cosn\theta\} + iZ\{sinn\theta\} = \frac{z(z - cos\theta) + izsin\theta}{z^2 - 2zcos\theta + 1} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1} + i\frac{zsin\theta}{z^2 - 2zcos\theta + 1}$$

Equate the real and imaginary parts on both sides, we get

$$Z\{cosn\theta\} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}$$

$$Z\{sinn\theta\} = \frac{zsin\theta}{z^2 - 2zcos\theta + 1}$$

Note: When $\theta = \frac{\pi}{2}$

$$Z\left\{cosn\frac{\pi}{2}\right\} = \frac{z^2}{z^2+1} \text{ and } Z\left\{sinn\frac{\pi}{2}\right\} = \frac{z}{z^2+1}$$

9. $Z\{coshn\theta\} = \frac{z(z-cosh\theta)}{z^2-2zcosh\theta+1}$

Proof: $Z\{coshn\theta\} = Z\left\{\frac{e^{n\theta}+e^{-n\theta}}{2}\right\} = \frac{1}{2}Z\left\{(e^{\theta})^n + (e^{-\theta})^n\right\}$

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} + \frac{z}{z - e^{-\theta}}\right] = \frac{z}{2}\left[\frac{z - e^{-\theta} + z - e^{\theta}}{(z - e^{\theta})(z - e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (e^{\theta} + e^{-\theta})}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right] = \frac{z(z - cosh\theta)}{z^2 - 2zcosh\theta + 1}$$

10. $Z\{sinhn\theta\} = \frac{zsinh\theta}{z^2-2zcosh\theta+1}$

Proof: $Z\{sinhn\theta\} = Z\left\{\frac{e^{n\theta}-e^{-n\theta}}{2}\right\}$

$$= \frac{1}{2} Z\{(e^{\theta})^n - (e^{-\theta})^n\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}}\right] = \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right] = \frac{z sinh\theta}{z^2 - 2z cosh\theta + 1}$$

11. **Z-Transform of unit step function**:

Unit step function is denoted by $u(n)$ and is defined by $u(n) = \begin{cases} 1, for\, n \geq 0 \\ 0, for\, n < 0 \end{cases}$

By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore\ Z\{u(n)\} = \sum_{n=0}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} 1.\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots \infty = \left[1 - \frac{1}{z}\right]^{-1} = \left[\frac{z-1}{z}\right]^{-1} = \frac{z}{z-1}$$

$$\therefore\ Z\{u(n)\} = \frac{z}{z-1}$$

Note: $Z\{u(n-k)\} = z^{-k} Z\{u(n)\} = z^{-k}\frac{z}{z-1}$

Properties of Z-transform:

1. **Linearity property**

If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$ then $Z[af(n) \pm bg(n)] = aZ\{f(n)\} \pm bZ\{g(n)\}$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{[af(n) \pm bg(n)]\} = \sum_{n=0}^{\infty} [af(n) \pm bg(n)]\, z^{-n} = a\sum_{n=0}^{\infty} f(n) z^{-n} \pm b\sum_{n=0}^{\infty} g(n) z^{-n}$$

$$= aZ\{f(n)\} \pm bZ\{g(n)\} = aF(z) \pm bG(z)$$

2. **Damping Rule**

If $Z\{f(n)\} = F(z)$, then $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n)\, z^{-n} = \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$$

Note: $Z\{a^{-n} f(n)\} = F(az)$

3. Differentiation in Z-domain

If $Z\{f(n)\} = F(z)$,then $Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$

Proof: By the definition $\boldsymbol{F}(\boldsymbol{z}) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

Differentiate w.r.t. **z** on both sides we get,

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} f(n)(-n)\, z^{-n-1} = z^{-1}\sum_{n=0}^{\infty} -nf(n)\, z^{-n}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} nf(n)\, z^{-n}$$

$$-z\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} nf(n)\, z^{-n}$$

$$\therefore\ Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$$

4. Time shifting property

If $Z\{f(n)\} = F(z)$, then

(i) $Z\{f(n-k)\} = z^{-k}F(z)$

(ii) $Z\{f(n+k)\} = z^{k}\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$

Proof: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{f(n-k)\} = \sum_{n=0}^{\infty} f(n-k)\, z^{-n}$$

Put $n - k = m \Rightarrow n = m + k$

$$Z\{f(n-k)\} = \sum_{m=-k}^{\infty} f(m)\, z^{-(m+k)} = z^{-k} \sum_{m=-k}^{\infty} f(m)\, z^{-m} = z^{-k} \sum_{m=0}^{\infty} f(m)\, z^{-m}$$

$$\therefore\ Z\{f(n-k)\} = z^{-k} F(z)$$

Now

$$Z\{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)\, z^{-n}$$

Put $n + k = m \Rightarrow n = m - k$

$$Z\{f(n+k)\} = \sum_{m=k}^{\infty} f(m)\, z^{-(m-k)} = z^{k} \sum_{m=k}^{\infty} f(m)\, z^{-m}$$

$$= z^{k} \left[\sum_{m=k}^{\infty} f(m)\, z^{-m} + \sum_{m=0}^{k-1} f(m)\, z^{-m} - \sum_{m=0}^{k-1} f(m)\, z^{-m} \right]$$

$$= z^{k} \left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - \sum_{m=0}^{k-1} f(m)\, z^{-m} \right]$$

$$\therefore\ Z\{f(n+k)\} = z^k\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

Note:

$$Z\{f(t+kT)\} = Z\{f_{n+k}\} = z^k\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

Problems:

1. Find the Z-transform of $\frac{(n+1)(n+2)}{2}$

Solution: $Z\left\{\frac{(n+1)(n+2)}{2}\right\} = Z\left\{\frac{n^2+3n+2}{2}\right\} = \frac{1}{2}\left[Z\{n^2\} + 3Z\{n\} + 2Z\{1\}\right]$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}\right]$$

2. Find the Z-transform of $\frac{1}{n(n+1)}$

Solution: Let $f(n) = \frac{1}{n(n+1)}$

By partial fraction $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)}$

$\Rightarrow 1 = A(n+1) + Bn$

When n = -1 $\Rightarrow$ B = -1 and n = 0 $\Rightarrow$ A = 1

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1} \Rightarrow Z\left\{\frac{1}{n(n+1)}\right\} = Z\left\{\frac{1}{n}\right\} - Z\left\{\frac{1}{n+1}\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right) = (1-z)\log\left(\frac{z}{z-1}\right)$$

3. Find the Z-transform of $\frac{2n+3}{(n+1)(n+2)}$

Solution: Let $f(n) = \frac{2n+3}{(n+1)(n+2)}$

By partial fraction $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$

$\Rightarrow 2n+3 = A(n+2) + B(n+1)$

When n = -2 $\Rightarrow$ B = 1 and n = -1 $\Rightarrow$ A = 1

$$\therefore \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2} \Rightarrow Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = Z\left\{\frac{1}{n+1}\right\} + Z\left\{\frac{1}{n+2}\right\}$$

$$= z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z = (z^2+z)\log\left(\frac{z}{z-1}\right) - z$$

4. Find the Z-transform of $ab^n + 2n$.

Solution: $Z\{ab^n + 2n\} = aZ\{b^n\} + 2Z\{n\} = a\frac{z}{z-b} + 2\frac{z}{(z-1)^2}$.

5. Find the Z-transform of $f(n) = \begin{cases} 1, for\ n = k \\ 0, otherwise \end{cases}$

Solution: By the definition $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{f(n)\} = 1.z^{-k} = \frac{1}{z^k}$$

6. Find the Z- transform of $f(n-5)$

Solution: We know that $Z\{f(n-k)\} = z^{-k}F(z)$

$\therefore Z\{f(n-5)\} = z^{-5}Z\{f(n)\} = z^{-5}\frac{z}{z-1}$ (since f(n) is a unit step function)

7. Find the Z- transform of $2^n\delta(n-3)$

Solution: We know that $Z\{\delta(n-k)\} = z^{-k} = \frac{1}{z^k}$

$\therefore Z\{2^n\delta(n-3)\} = [z^{-3}Z\{\delta(n)\}]_{z\to\frac{z}{2}}$

$= \left[\frac{1}{z^3}\right]_{z\to\frac{z}{2}} = \frac{1}{\left(\frac{z}{2}\right)^3} = \frac{8}{z^3}$ (since $\delta(n)$ is a unit impulse function)

8. Find the Z-transform of $\frac{1}{(n+2)!}$

Solution: Let $f(n+2) = \frac{1}{(n+2)!} \Rightarrow f(n) = \frac{1}{n!}$

By shifting theorem

$$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right]$$

$$\therefore Z\{f(n+2)\} = z^2[F(z) - f(0) - f(1)z^{-1}]$$

$$F(z) = Z\{f(n)\} = Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$$

$$\therefore Z\left\{\frac{1}{(n+2)!}\right\} = z^2\left[e^{\frac{1}{z}} - 1 - z^{-1}\right]$$

9. Find the Z-transform of $r^n cosn\theta \; and \; r^n sinn\theta$

Solution: We know that

$$Z\{cosn\theta\} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}$$

By damping rule $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$

$$Z\{r^n cosn\theta\} = \left\{\frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}\right\}_{z \to \frac{z}{r}} = \frac{\frac{z}{r}\left(\frac{z}{r} - cos\theta\right)}{\left(\frac{z}{r}\right)^2 - 2\frac{z}{r}cos\theta + 1}$$

$$\therefore Z\{r^n cosn\theta\} = \frac{z(z - rcos\theta)}{z^2 - 2zrcos\theta + r^2}$$

Also $Z\{sinn\theta\} = \frac{zsin\theta}{z^2-2zcos\theta+1}$

$$Z\{r^n sinn\theta\} = \left\{\frac{zsin\theta}{z^2 - 2zcos\theta + 1}\right\}_{z\to\frac{z}{r}} = \frac{\frac{z}{r}sin\theta}{\left(\frac{z}{r}\right)^2 - 2\frac{z}{r}cos\theta + 1}$$

$$\therefore Z\{r^n sinn\theta\} = \frac{zrsin\theta}{z^2 - 2zrcos\theta + r^2}$$

10. Find the Z-transform of $n\ conn\theta$

Solution: We know that

$$Z\{cosn\theta\} = \frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}$$

By the property of Z-transform $Z\{nf(n)\} = -z\frac{d}{dz}[F(z)]$

$$Z\{n\ cosn\theta\} = -z\frac{d}{dz}\left\{\frac{z(z - cos\theta)}{z^2 - 2zcos\theta + 1}\right\}$$

$$= -z\left[\frac{(z^2 - 2zcos\theta + 1)(2z - cos\theta) - (z^2 - zcos\theta)(2z - 2cos\theta)}{(z^2 - 2zcos\theta + 1)^2}\right]$$

$$\therefore Z\{n\, cosn\theta\} = \frac{z(z^2 cos\theta - 2z + cos\theta)}{(z^2 - 2zcos\theta + 1)^2}$$

11. Find the Z-transform of $sin^2\left(\frac{n\pi}{4}\right)$

Solution:

$$sin^2\left(\frac{n\pi}{4}\right) = \frac{1 - cos2\left(\frac{n\pi}{4}\right)}{2} = \frac{1 - cos\left(\frac{n\pi}{2}\right)}{2}$$

$$Z\left\{sin^2\left(\frac{n\pi}{4}\right)\right\} = Z\left\{\frac{1 - cos\left(\frac{n\pi}{2}\right)}{2}\right\} = \frac{1}{2}\left[Z\{1\} - Z\left\{cos\left(\frac{n\pi}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

Theorems on Z-transform

1. **First Shifting Theorem**

If $Z\{f(t)\} = F(z)\ then\ Z\{e^{-at} f(t)\} = F(ze^{aT})$

Proof: By the definition of Z-transform $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n} = \sum_{n=0}^{\infty} f(nT)\, (ze^{aT})^{-n}$$

$$\therefore Z\{e^{-at} f(t)\} = F(ze^{aT})$$

Note: $Z\{e^{at} f(t)\} = F\left(\frac{z}{e^{aT}}\right)$

2. Second Shifting theorem

If $Z\{f(t)\} = F(z)$ *then* $Z\{f(t+T\} = z\{F(z) - f(0)\}$

Proof: By the definition of Z-transform $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{f(t+T)\} = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n} = \sum_{n=0}^{\infty} f((n+1)T)\, z^{-n}$$

Put n+1 = m $Z\{f(t+T)\} = \sum_{m=1}^{\infty} f(mT) z^{-(m-1)} = z \sum_{m=1}^{\infty} f(mT) z^{-m}$

$$= z\left[\sum_{m=1}^{\infty} f(mT) z^{-m} - f(0)\right] = z\{F(z) - f(0)\}$$

3. Initial Value theorem

If $Z\{f(n)\} = F(z)$, *then* $f(0) = \lim_{z\to\infty} F(z)$

Proof: By the definition $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots = f(0) + \frac{f(1)}{z} + \frac{f(1)}{z^2} + \cdots$$

Taking limit as $z \to \infty$ on both sides

$$\lim_{z\to\infty} F(z) = f(0) + 0 + \cdots$$

$$\Rightarrow f(0) = \lim_{z\to\infty} F(z)$$

4. Final Value Theorem

If $Z\{f(n)\} = F(z), then\ \lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)\, F(z)$

Proof: By the definition $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\{f(n+1) - f(n)\} = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

$$\Rightarrow z\{F(z) - f(0)\} - F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty} [f(n+1) - f(n)]\, z^{-n}$$

Taking limit $z \to 1$ on both sides

$$\lim_{z\to 1}[(z-1)F(z) - zf(0)] = \lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1) - f(n)]\,z^{-n}$$

$$\lim_{z\to 1}(z-1)\,F(z) - f(0) = f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \cdots .f(\infty)$$

$$\lim_{z\to 1}(z-1)\,F(z) = f(\infty) = \lim_{n\to\infty} f(n)$$

$$\therefore \lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)\,F(z)$$

Convolution of sequences

The convolution of two sequences $\{f(n)\}\ and\ \{g(n)\}$ is defined as
$f(n) * g(n) = \sum_{k=0}^{n} f(k)\,g(n-k)$

5. **Convolution Theorem**

If $Z\{f(n)\} = F(z)\ and\ Z\{g(n)\} = G(z), then\ Z\{f(n) * g(n)\} = F(z)G(z)$

Proof: By the definition $\boldsymbol{F(z)} = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\,z^{-n}$

$$Z\{f(n) * g(n)\} = \sum_{n=0}^{\infty}[f(n) * g(n)]\,z^{-n}$$

$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\,g(n-k)\,z^{-n}$ (By the definition of convolution)

By changing the order of summation

$$Z\{f(n) * g(n)\} = \sum_{k=0}^{n} f(k) \sum_{n=0}^{\infty} g(n-k) z^{-n}$$

$$= \sum_{k=0}^{\infty} f(k)\, Z\{g(n-k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k} G(z) = F(z)G(z)$$

Problems:

1. Find the Z-transform of

(i) $f(t) = e^{-at}$

(ii) $f(t) = e^{at}$

(iii) $f(t) = cosat$

(iv) $f(t) = sinat$

Solution:

(1) By first shifting property $Z\{e^{-at} f(t)\} = F(ze^{aT})$

$$\therefore Z\{e^{-at}(1)\} = Z\{1\}_{z \to ze^{aT}} = \left[\frac{z}{z-1}\right]_{z \to ze^{aT}}$$

$$= \frac{ze^{aT}}{ze^{aT} - 1}$$

(2) $Z\{e^{at}(1)\} = Z\{1\}_{z \to \frac{z}{e^{aT}}} = \left[\frac{z}{z-1}\right]_{z \to \frac{z}{e^{aT}}}$

$$= \frac{\frac{z}{e^{aT}}}{\frac{z}{e^{aT}} - 1} = \frac{z}{z - e^{aT}}$$

(3) By the definition $Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$$\therefore Z\{cosat\} = \sum_{n=0}^{\infty} cosanT\, z^{-n} = \sum_{n=0}^{\infty} cosn(aT)\, z^{-n}$$

$$= \frac{z(z - cosaT)}{z^2 - 2zcosaT + 1}$$

(4) $Z\{sinat\} = \sum_{n=0}^{\infty} sinanT\, z^{-n} = \sum_{n=0}^{\infty} sinn(aT)\, z^{-n}$

$$= \frac{zsinaT}{z^2 - 2zcosaT + 1}$$

2. Find the Z-transform of sin(t+T)

Solution: Let $f(t+T) = \sin(t+T)$ implies $f(t) = sint$

By second shifting theorem $Z\{f(t+T)\} = z[F(z) - f(0)]$

$$f(0) = sin0 = 0 \; and \; F(z) = Z\{f(t)\} = Z\{sint\} = \frac{zsinT}{z^2 - 2zcosT + 1}$$

$$\therefore Z\{\sin(t+T)\} = z\left[\frac{zsinT}{z^2 - 2zcosT + 1} - 0\right] = \frac{z^2 sinT}{z^2 - 2zcosT + 1}$$

3. Find the Z-transform of $(t+T)e^{-(t+T)}$

Solution:

Let $f(t+T) = \sin(t+T)$ implies $f(t) = te^{-t}$

By second shifting theorem $Z\{f(t+T)\} = z[F(z) - f(0)]$

$$f(0) = 0e^0 = 0 \; and \; F(z) = Z\{f(t)\} = Z\{te^{-t}\} = Z\{t\}_{z\to ze^T}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z\to ze^T} = \frac{Tze^T}{(ze^T - 1)^2}$$

$$\therefore Z\{(t+T)e^{-(t+T)}\} = z\left[\frac{Tze^T}{(ze^T-1)^2} - 0\right] = \frac{Tz^2e^T}{(ze^T-1)^2}$$

4. Find the initial value of $F(z) = \frac{z}{(z-1)(z-2)}$

Solution:

By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$$= \lim_{z\to\infty} \frac{z}{(z-1)(z-2)}$$

$$= \lim_{z\to\infty} \frac{z}{z^2\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)} = 0$$

5. If $F(z) = \frac{1+z^{-1}}{1-0.25z^{-2}}, find\ f(0) and\ f(\infty)$

   Solution: By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$$f(0) = \lim_{z\to\infty} \frac{1+z^{-1}}{1-0.25z^{-2}} = 1$$

   By final value theorem, $f(\infty) = \lim_{z\to 1}(z-1)\, F(z) = \lim_{z\to 1}(z-1)\frac{z(z+1)}{z^2-0.25} = 0$

6. If $F(z) = \frac{2z^2+3z+14}{(z-1)^4}, find\ f(2) and\ f(3)$

   Solution: By initial value theorem $f(0) = \lim_{z\to\infty} F(z)$

$$= \lim_{z\to\infty} \frac{2z^2+3z+14}{(z-1)^4} = \lim_{z\to\infty} \frac{z^2\left(2+\frac{3}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}$$

$\therefore f(0) = 0$

$$f(1) = \lim_{z\to\infty}[z(F(z) - f(0))]$$

$$= \lim_{z\to\infty} z\frac{2z^2 + 3z + 14}{(z-1)^4} = \lim_{z\to\infty} z^3 \frac{\left[2 + \frac{3}{z} + \frac{14}{z^2}\right]}{z^4\left(1 - \frac{1}{z}\right)^4}$$

$$\therefore f(1) = 0$$

$$f(2) = \lim_{z\to\infty}[z^2(F(z) - f(0) - f(1)z^{-1})]$$

$$= \lim_{z\to\infty} z^2\frac{2z^2 + 3z + 14}{(z-1)^4} = \lim_{z\to\infty} z^4 \frac{\left[2 + \frac{3}{z} + \frac{14}{z^2}\right]}{z^4\left(1 - \frac{1}{z}\right)^4}$$

$$\therefore f(2) = 2$$

$$f(3) = \lim_{z\to\infty}[z^3(F(z) - f(0) - f(1)z^{-1} - f(2)z^{-2})]$$

$$= \lim_{z\to\infty}\left[z^3\left(\frac{2z^2 + 3z + 14}{(z-1)^4} - \frac{2}{z^2}\right)\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^4 + 3z^3 + 14z^2 - 2z^4 + 8z^3 - 12z^2 + 8z - 2}{z^2(z-1)^4}\right]$$

$$= \lim_{z \to \infty} z^3 \left[\frac{2z^4 + 3z^3 + 14z^2 - 2z^4 + 8z^3 - 12z^2 + 8z - 2}{z^2(z-1)^4} \right]$$

$$= \lim_{z \to \infty} z^3 \left[\frac{11z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4} \right] = \lim_{z \to \infty} z^6 \left[\frac{11 + \frac{2}{z} + \frac{8}{z^2} - \frac{2}{z^3}}{z^6 \left(1 - \frac{1}{z}\right)^4} \right]$$

$$\therefore f(3) = 11$$

The Inverse Z-Transform

If $Z[f(n)] = F(Z)$ then $Z^{-1}F(Z) = f(n)$ is called inverse Z-transform of $F(Z)$

Example:

1. $Z[a^n] = \frac{z}{z-a} z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Methods of finding inverse Z-transforms:

1. Method of partial fraction
2. Method of residues
3. Long division method
4. Convolution method

Partial Fraction Method:

1. Find the inverse Z-transform of $\frac{10z}{z^2-3z+2}$

   Solution:

$$F(Z) = \frac{10z}{z^2 - 3z + 2} = \frac{10z}{(z-1)(z-2)}$$

$$\frac{F(z)}{10z} = \frac{10z}{z^2 - 3z + 2} = \frac{10z}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put z =1 A =1

Put z = 2 B = 1

$$\frac{F(z)}{10z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$F(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-10z}{z-1}\right] + Z^{-1}\left[\frac{10z}{z-2}\right]$$

$$= -10\, Z^{-1}\left[\frac{z}{z-1}\right] + 10 Z^{-1}\left[\frac{z}{z-2}\right]$$

$$Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1)^n + 10(2)^n, \quad n \geq 0$$

Method of Residues

To find inverse Z- transform using residue theorem

If $Z[f(n)] = F(Z)$, then f(n) which gives the inverse Z-transform of $F(Z)$ is obtained from the following result $f(n) = \frac{1}{2\pi i}\int_c z^{n-1}F(z)\, dz$

Where C is the closed contour which encloses all the poles of the integrand.

By Residue theorem,

$$\int_C z^{n-1}F(z)\, dz = 2\pi i\left[sum\ of\ residues\ of\ z^{n-1}F(z)\ at\ its\ poles\right]$$

Substituting (2) in (1)

1. Find the inverse Z – transform of $\frac{z}{(z-1)(z-2)}$

Solution: Let $F(z) = \frac{1}{(z-1)(z-2)}$ $f(n) = Z^{-1}F(Z)$

$$z^{n-1}F(z) = \frac{z^n}{(z-1)(z-2)}$$

The poles are z = 1, z = 2 (simple poles)

f(n)= sum of the residues $z^{n-1}F(z)$ at its

poles $\operatorname{Re}s\{z^{n-1}F(z)\}_{z=1} = \lim_{z\to 1}(z-1)\dfrac{z^n}{(z-1)(z-2)} = -(1)^n$

$$\operatorname{Re}s\{z^{n-1}F(z)\}_{z=2} = \lim_{z\to 2}(z-2)\frac{z^n}{(z-1)(z-2)} = 2^n$$

f(n)= sum of the residues $z^{n-1}F(z)$ at its poles

$= 2^n - (1)^n, \qquad n \geq 0$

Convolution Method:

1. Find the inverse Z-transform of $\frac{z^2}{(z-a)^2}$ using convolution theorem.

Solution:

By convolution theorem $Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\}Z^{-1}\{G(z)\}$

$$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = Z^{-1}\left\{\frac{z}{z-a}\cdot\frac{z}{z-a}\right\}$$

$$= Z^{-1}\left\{\frac{z}{z-a}\right\} Z^{-1}\left\{\frac{z}{z-a}\right\}$$

$$= a^n * a^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} a^k a^{n-k} = \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} 1 = (n+1)a^n$$

$$\therefore Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = (n+1)a^n$$

2. Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z+1)}$ using convolution theorem.

Solution:

By convolution theorem $Z^{-1}\{F(z) * G(z)\} = Z^{-1}\{F(z)\} Z^{-1}\{G(z)\}$

$$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z+1)}\right\} = Z^{-1}\left\{\frac{8z^2}{8\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right\} = Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)} \cdot \frac{z}{\left(z+\frac{1}{4}\right)}\right\}$$

$$= Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)}\right\} Z^{-1}\left\{\frac{z}{\left(z+\frac{1}{4}\right)}\right\}$$

$$= \left(\frac{1}{2}\right)^n * \left(-\frac{1}{4}\right)^n$$

$$= \sum_{k=0}^{n} f(k)\, g(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(-\frac{1}{4}\right)^{n-k} = \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[-\frac{\frac{1}{2}}{\frac{1}{4}}\right]^k = \left(-\frac{1}{4}\right)^n \sum_{k=0}^{n} [-2]^k$$

$$= \left(-\frac{1}{4}\right)^n [1 + (-2)^1 + (-2)^2 + \cdots + (-2)^n]$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right] \quad [Since\ a + ar + ar^2 + ar^3 + \cdots. + ar^n = \frac{1-r^{n+1}}{1-r}, when\ r < 1$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{3}\right] = \frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$$

$$\therefore Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z+1)}\right\} = \frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$$

Difference Equations:

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Example:

1. $y_{n+2} - y_{n+1} + y_n = 5$
2. $a_0 u_{n+1} + a_1 u_n = g(n)$

Formation of difference equation

1. Form a difference equation given $y_n = cosn\frac{\pi}{2}$

Solution:

Given $y_n = cosn\frac{\pi}{2}$.............(1)

$$y_{n+1} = \cos(n+1)\frac{\pi}{2} = cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -sin\frac{n\pi}{2}$$

$$y_{n+1} = -sin\frac{n\pi}{2} \text{..................(2)}$$

$$y_{n+2} = -sin\frac{(n+1)\pi}{2} = -sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -cos\frac{n\pi}{2}$$

$$y_{n+2} = -cos\frac{n\pi}{2} \text{..................(3)}$$

$y_{n+2} = -y_n$ (By (1)

$\therefore y_{n+2} + y_n = 0$

2. Form a difference equation given $u_n = \frac{1}{2}n(n+1)$

Solution:

Given $u_n = \frac{1}{2}n(n+1) = \frac{n^2}{2} + \frac{n}{2}$..........(1)

$u_{n+1} = \frac{(n+1)^2}{2} + \frac{(n+1)}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1$..........(2)

$u_{n+2} = \frac{(n+1)^2}{2} + \frac{3(n+1)}{2} + 1 = \frac{n^2}{2} + \frac{5n}{2} + 3$...............(3)

(3) - (2) gives $u_{n+2} - u_{n+1} = n + 2$.............(4)

(2) - (1) gives $u_{n+1} - u_n = n + 1$.............(5)

(4) – (5) gives $u_{n+2} - 2u_{n+1} + u_n = 1$ which is the required difference equation.

1. **Solve**

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ $given$ $y_0 = y_1 = 0$ using Z-transform.

Solution:

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Taking Z-transform on both sides,

$$Z\{y_{n+2}\} + 6Z\{y_{n+1}\} + 9Z\{y_n\} = Z\{2^n\}$$

$$z^2Y(z) - z^2y(0) - zy(1) + 6[zY(z) - zy(0)] + 9Y(z) = \frac{z}{z-2}$$

$$z^2Y(z) + 6zY(z) + 9Y(z) = \frac{z}{z-2}$$

$$Y(z)[z^2 + 6z + 9] = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

By partial fraction $\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$……………..(1)

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put z =-3 $\Rightarrow C = -\frac{1}{5}$ Put z =2 $\Rightarrow A = \frac{1}{25}$

Equate the co-efficient of z^2 on both sides, we get

A + B = 0 $\Rightarrow B = -A = -\frac{1}{25}$

Equation (1) becomes $\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{\frac{1}{25}}{z-2} - \frac{\frac{1}{25}}{z+3} - \frac{\frac{1}{5}}{(z+3)^2}$

$$Y(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$\Rightarrow y(n) = \frac{1}{25}Z^{-1}\left\{\frac{z}{z-2}\right\} - \frac{1}{25}Z^{-1}\left\{\frac{z}{z+3}\right\} - \frac{1}{5}Z^{-1}\left\{\frac{z}{(z+3)^2}\right\}$$

$$\therefore y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}n(-3)^n$$

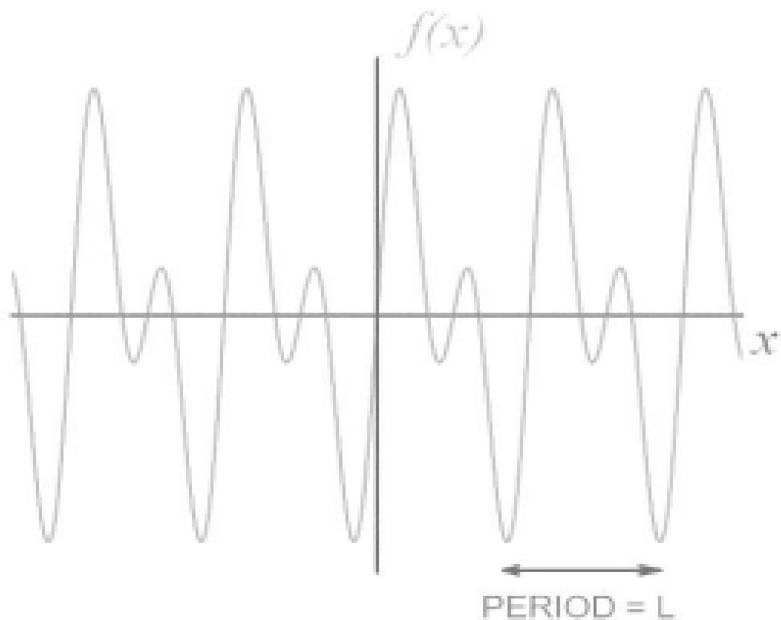
Fourier Series

Contents - Fourier series – Euler's formula – Dirichlet's conditions – Fourier series for a periodic function – Parseval's identity (without proof) – Half range cosine series and sine series – simple problems – Harmonic Analysis.

Periodic Functions

A function $f(x)$ is said to be periodic, if and only if $f(x+L) = f(x)$ is true for some value of L and for all values of x. The smallest value of L for which this equation is true for every value of x will be called the period of the function.

A graph of periodic function $f(x)$ that has period L exhibits the same pattern every L units along the $x-$ axis, so that $f(x+L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods). For example, $sinx$ and $cosx$ are periodic with period 2π and $tanx$ has period π.



Dirichlet's Conditions

(i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

(ii) $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c+2\pi)$

(iii) $f(x)$ has a finite number of maxima and minima in $(c, c+2\pi)$

Note 1: These conditions are not necessary but only sufficient for the existence of Fourier series.

Note 2: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(-\infty, \infty)$, then $f(x)$ has to be periodic of periodicity 2π for the existence of Fourier series of period 2π.

Note 3: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(c, c+2\pi)$, then $f(x)$ need not be periodic for the existence of Fourier series of period 2π.

Note 4: If $x = a$ is a point of continuity of $f(x)$, then the value of Fourier series at $x = a$ is $f(a)$. If $x = a$ is a point of discontinuity of $f(x)$, then the value of Fourier series at $x = a$ is $\frac{1}{2}[f(a+) + f(a-)]$. In other words, specifying a particular value of $x = a$ in a Fourier series, gives a series of constants that should equal $f(a)$. However, if $f(x)$ is discontinuous at this value of x, then the series converges to a value that is half-way between the two possible function values.

Fourier Series

Periodic functions occur frequently in engineering problems. Such periodic functions are often complicated. Therefore, it is desirable to represent these in terms of the simple periodic functions of sine and cosine. A development of a given periodic function into a series of sines and cosines was studied by the French physicist and mathematician Joseph Fourier (1768-1830). The series of sines and cosines was named after him.

If $f(x)$ is a periodic function with period 2π defined in $(c, c+2\pi)$ and the Dirichlet's conditions are satisfied, then $f(x)$ can be expanded as a **Fourier series** of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients a_0, a_n *and* b_n are calculate using Euler's formula.

Euler's Formula

(1) $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

(2) $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx dx$

(3) $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx dx$

Standard Integrals

1. $\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}[asinbx - bcosbx]$

2. $\int e^{ax} cosbx\, dx = \frac{e^{ax}}{a^2+b^2}[acosbx + bsinbx]$

3. Bernoulli's generalized formula of integration by parts

$\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Trigonometric results

1. $sinn\pi = 0,$ if n is an integer

2. $cosn\pi = (-1)^n,$ if n is an integer

Example 1

Obtain the Fourier series of the following function defined in $(0, 2\pi)$.

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \quad \text{and has period } 2\pi$$

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x \quad = \quad \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi \cdot \mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \Big(2\pi - \pi\Big) \\
&= \frac{\pi}{2} + \pi \\
\text{i.e. } a_0 &= \frac{3\pi}{2}.
\end{aligned}
$$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi \cdot \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{1}{n}\Big(\pi\sin n\pi - 0\cdot\sin n0\Big) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\
&\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi)
\end{aligned}
$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big)+\left(\frac{\cos n\pi}{n^2}-\frac{\cos 0}{n^2}\right)\right]+\frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1),$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx \, \mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx \, \mathrm{d}x$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right)+\left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right]-\frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n}+\left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right]-\frac{1}{n}(1-(-1)^n)$$

$$= -\frac{1}{n}(-1)^n + 0 - \frac{1}{n}(1-(-1)^n)$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{3\pi}{2}$, $a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $b_n = -\frac{1}{n}$

Even and Odd Functions

The function f(x) is said to be even, if f(-x) = f(x).

The function f(x) is said to be odd, if f(-x) = -f(x).

If $f(x)$ is an even function with period 2π defined in $(-\pi,\pi)$, then $f(x)$ can be expanded as a Fourier cosine series:

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx}$$

where the Fourier coefficients $\boldsymbol{a_0}$ and $\boldsymbol{a_n}$ are calculated by

(1) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

(2) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, cosnxdx$

If $f(x)$ is an odd function with period 2π defined in $(-\pi,\pi)$, then $f(x)$ can be expanded as a Fourier sine series:

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n\, sinnx}$$

where the Fourier coefficient $\boldsymbol{b_n}$ is calculated by $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, sinnxdx$

Example 2

Derive the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π and hence deduce $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi}\,[\frac{x^2}{2} + \frac{x^3}{3}]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{(-\pi)^2}{2} + \frac{(-\pi)^3}{3}\right)]$$

$$= \frac{1}{\pi}\,[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}]$$

$$a_0 = \frac{2\,\pi^2}{3}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)cosnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{sinnx}{n}\right) - (1 + 2x)\left(\frac{-cosnx}{n^2}\right) + (2)\left(\frac{-sinnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (1 - 2\pi)\left(\frac{cosn\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{(-1)^n}{n^2}\right) + 2\pi\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{4}{n^2}(-1)^n$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)sinnx\ dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{-cosnx}{n}\right) - (1 + 2x)\left(\frac{-sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) + 2\left(\frac{cosn\pi}{n^3}\right) - (-\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) - 2\left(\frac{cosn\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{-(-1)^n}{n}\right)\right]$$

$$b_n = \frac{2}{n}(-1)^{n+1}$$

Therefore, the Fourier series is of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\, cosnx + b_n\, sinnx)$$

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} (\frac{4}{n^2} (-1)^n\, cosnx + \frac{2}{n} (-1)^{n+1}\, sinnx) \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)$$

STEP FOUR

Deduction:

The end points of the range are $x = \pi$ and $x = -\pi$. Therefore, the value of Fourier series at $x = \pi$ is the average value of $f(x)$ at the points $x = \pi$ and $x = -\pi$. Hence put $x = \pi$ in (1),

$$\Rightarrow \frac{f(-\pi)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n\, cosn\pi$$

$$\Rightarrow \frac{(\pi + \pi^2) + (-\pi + \pi^2)}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{2n}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Half-Range Fourier Series

Example 3

Express $f(x) = x(\pi - x), 0 < x < \pi$ as a Fourier series of periodicity 2π containing (i) sine terms only and (ii) cosine terms only. Hence deduce, $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$ and $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Solution.

(i) **sine series:**

Let $f(x) = \sum_{1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[\{\pi x - x^2\}\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\{(-1)^n - 1\}\right]$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n \right]$$

$= 0$ if n is even

$$= \frac{8}{\pi n^3} \text{ if } n \text{ is odd}$$

$$\therefore \quad f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin (2n-1) x.$$

Setting $x = \pi/2$ which is a point of continuity we get first deduction .

(ii) cosine series:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{2}{\pi} \int_0^{\pi} x(\pi - x)\, dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n} \right) - (\pi - 2x)\left(- \frac{\cos nx}{n^2} \right) + (-2)\left(- \frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[- \frac{\pi}{n^2}(-1)^n - \frac{\pi}{n^2} \right] = - \frac{2}{n^2}[1 + (-1)^n]$$

$= 0$ for n odd

$$= -\frac{4}{n^2} \text{ for } n \text{ even}$$

$$x(\pi - x) = \frac{\pi^2}{6} - 4 \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n^2} \cos nx$$

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{1}^{\infty} \frac{1}{n^2} \cos 2nx.$$

Setting $x = \pi/2$ which is a point of continuity,

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - \sum \frac{1}{n^2}(-1)^n$$

$$\sum \frac{1}{n^2}(-1)^n = \frac{\pi^2}{6} - \frac{\pi^2}{4}$$

Harmonic Analysis

Example 4

Compute the first three harmonics of the Fourier series of f(x) given by the following table.

x	0	π/3	2π/3	π	4π/3	5π/3	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution.

We will form the table for the convenience of work.

We exclude the last point $x = 2\pi$.

x	f(x)	cos x	sin x	cos 2x	sin 2x	cos 3x	sin 3x
0	1.0	1	0	1	0	1	0
π/3	1.4	0.5	0.866	-0.5	0.866	-1	0
2π/3	1.9	-0.5	0.866	-0.5	-0.866	1	0
π	1.7	-1	0	1	0	-1	0
4π/3	1.5	-0.5	-0.866	-0.5	0.866	1	0
5π/3	1.2	0.5	-0.866	-0.5	-0.866	-1	0

$a_0 = 2/6 \sum f(x) = 1/3(1.0 + 1.4 + 1.9 + 1.7 + 1.5 + 1.2) = 2.9$

$a_1 = 2/6 \sum f(x) \cos x = 1/6(1 + 0.7 - 0.95 - 1.7 - 0.75 + 0.6) = -0.37$

$a_2 = 2/6 \sum f(x) \cos 2x = -0.1$

$a_3 = 2/6 \sum f(x) \cos 3x = 0.03$

$b_1 = 2/6 \sum f(x) \sin x = 0.17$

$b_2 = 2/6 \sum f(x) \sin 2x = -0.06$

$b_3 = 2/6 \sum f(x) \sin 3x = 0$

$f(x) = 1.45 - 0.33 \cos x - 0.1 \cos 2x + 0.03 \cos 3x + 0.17 \sin x - 0.06 \sin 2x$

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION

One-dimensional wave equation

Let us start with the wave equation. Imagine we have a tensioned guitar string of length L. Let us only consider vibrations in one direction. Let x denote the position along the string, let t denote time, and let y denote the displacement of the string from the rest position. See Fig. 1.

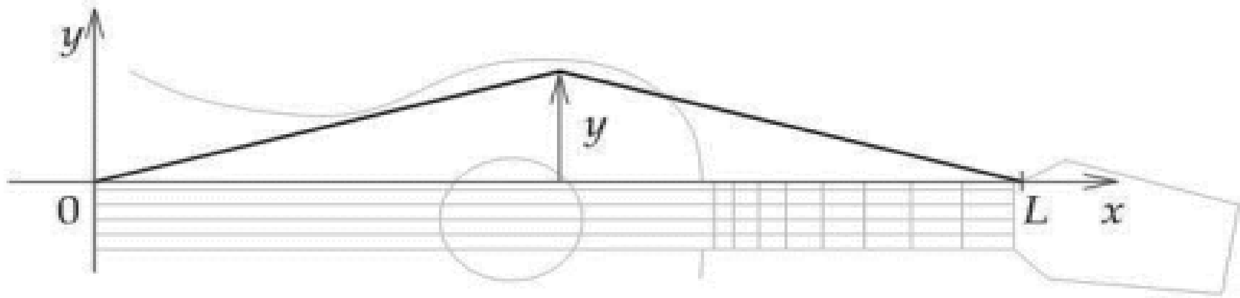


Figure 1: Vibrating string of length L, x is the position, y is displacement

Let $y(x, t)$ denote the displacement at point x at time t. The equation governing this setup is the so-called one-dimensional wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{....... (1)}$$

We generally use a more convenient notation for partial derivatives. We write y_t instead of $\frac{\partial y}{\partial t}$, and we write y_{xx} instead of $\frac{\partial^2 y}{\partial x^2}$.

With this notation the equation that governs this setup is the so-called one-dimensional wave equation, becomes $y_{tt} = a^2 y_{xx}$

for some constant $a > 0$. The intuition is similar to the heat equation, replacing velocity with acceleration: the acceleration at a specific point is proportional to the second derivative of the shape of the string. The wave equation is an example of a hyperbolic PDE.

Solution of the Wave Equation (by the method of separation of variables)

Let $y = X(x).T(t)$ be a solution of (1), where X(x) is a function of x only $T(t)$ is a function t only.

$$\frac{\partial^2 y}{\partial t^2} = XT'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X''T,$$

where $$X'' = \frac{d^2X}{dx^2} \text{ and } T'' = \frac{d^2T}{dt^2}.$$

Hence (1) becomes, $XT'' = a^2 X'' T$

i.e., $$\frac{X''}{X} = \frac{T''}{a^2T} \qquad \text{...(2)}$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

$\therefore \quad \frac{X''}{X} = \frac{T''}{a^2T} = k$ (say) where k is any constant.

Hence $X'' - kX = 0$ and $T'' - a^2 kT = 0$ ...(3)

Solutions of these equations depend upon the nature of the value of k.

Case 1. Let $k = \lambda^2$, a positive value .

Now the equation (3) are $X'' - \lambda^2 X = 0$ and $T'' - a^2\lambda^2 T = 0$.

Solving the ordinary differential equations we get,

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$$

and $$T = C_1 e^{\lambda at} + D_1 e^{-\lambda at}$$

Case 2. Let $k = -\lambda^2$, a negative number.

Then the equations (3) are $X'' + \lambda^2 X = 0$ and $T'' + a^2\lambda^2 T = 0$.

Solving, we get,

$$X = A_2 \cos \lambda x + B_2 \sin \lambda x$$

and

$$T = C_2 \cos \lambda at + D_2 \sin \lambda at.$$

Case 3. Let $k = 0$.

Now the equations (3) are $X'' = 0$ and $T'' = 0$.

Then integrating, $X = A_3 x + B_3$

and

$$T = C_3 t + D_3$$

Thus the various possible solutions of the wave equation are

$$y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda at} + D_1 e^{-\lambda at}) \quad \text{...(I)}$$

$$y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at) \quad \text{...(II)}$$

$$y = (A_3 x + B_3)(C_3 t + D_3) \quad \text{...(III)}$$

Heat on an insulated wire

Now let us consider with the heat equation. Consider a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and let t denote time. See Figure 2.

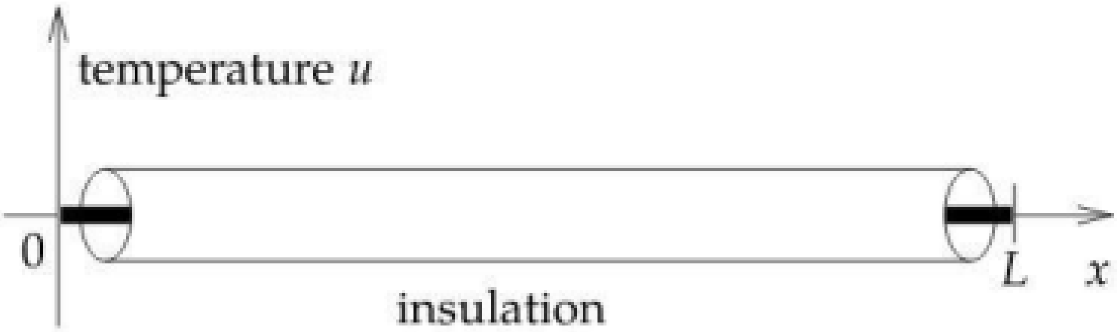


Figure 2: Insulated wire

Let $u(x, t)$ denote the temperature at point x at time t. The equation governing this setup is the so-called one-dimensional heat equation:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},}$$

where $k > 0$ is a constant (the thermal conductivity of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum and vice-versa.

Therefore, the heat equation is $u_t = ku_{xx}$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. If the ends of the wire are kept at temperature 0, then the conditions are:

(i) $u(0,t) = 0$ and $u(L,t) = 0$.

If, on the other hand, the ends are also insulated, the conditions are:

(ii) $u_x(0,t) = 0$ and $u_x(L,t) = 0$.

Let us see why that is so. If u_x is positive at some point x0, then at a particular time, u is smaller to the left of x0, and higher to the right of x0. Heat is flowing from high heat to low heat, that is to the left. On the other hand if ux is negative then heat is again flowing from high heat to low heat, that is to the right. So when ux is zero, that is a point through which heat is not flowing. In other words, ux(0,t)=0 means no heat is flowing in or out of the wire at the point x=0.

We have two conditions along the x-axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e., u or a derivative of u is set to zero). We also need an initial condition—the temperature distribution at time t=0. That is, u(x,0)=f(x), for some known function f(x).

Solution of heat equation by method of separation of variables

We have to solve the equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad \text{---------(1)}$$

where $k = \alpha^2$ is called the diffusivity of the substance.

Assume a solution of the form $u(x,t) = X(x).T(t)$ where X is a function of x and T is a function of t.

Then (1) becomes,

$$XT' = \alpha^2 X''T,$$

where $X'' = \frac{d^2X}{dx^2}$ and $T' = \frac{dT}{dt}$

i.e., $\frac{X''}{X} = \frac{T'}{\alpha^2 T}$ -----(2)

The LHS is a function of x alone and the RHS is the function of t alone when x and t are independent variables. Equation (2) can be true only if each expression is equal to a constant.

$\therefore$ Let $\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k$ (constant)

$\therefore\ X'' - kX = 0$, and $T' - \alpha^2 kT = 0$ ...(3)

The nature of solutions of (3) depends upon the values of k.

Case 1. Let $k = \lambda^2$, a positive number.

Then (3) becomes,

$$X'' - \lambda^2 X = 0, \text{ and } T' - \alpha^2\lambda^2 T = 0.$$

Solving, we get

$$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x} \text{ and } T = C_1 e^{\alpha^2\lambda^2 t}.$$

Case 2. Let $k = -\lambda^2$, a negative number. Then (3) becomes $X'' + \lambda^2 X = 0$, and $T' + \alpha^2\lambda^2 T = 0$.

Solving, we obtain

$$X = A_2 \cos\lambda x + B_2 \sin\lambda x, \text{ and } T = C_2 e^{-\alpha^2\lambda^2 t}.$$

Case 3. Let $k = 0$.

Then $X'' = 0$ and $T' = 0$.

Solving, we arrive at ,

$$X = A_3 x + B_3 \text{ and } T = C_3.$$

Hence the possible solutions of (1) are

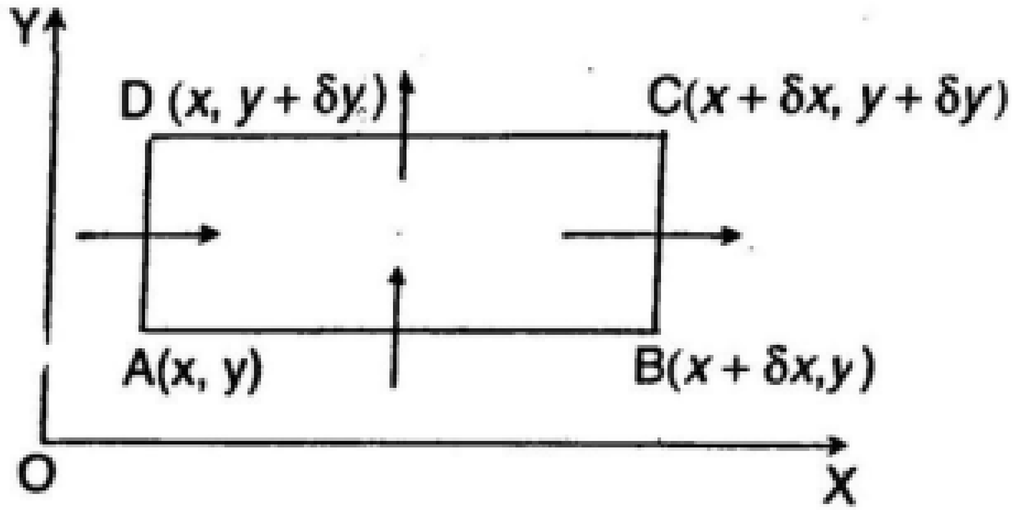
$$u(x, t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})\, C_1 e^{\alpha^2\lambda^2 t} \quad ...(I)$$

$$u(x, t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, C_2\, e^{-\alpha^2\lambda^2 t} \quad ...(II)$$

$$u(x, t) = (A_3 x + B_3) C_3$$

Two-Dimensional Heat Flow

When the heat flow is along curves instead of along straight lines, all the curves lying in parallel planes, then the flow is called two-dimensional. Let us consider now the flow of heat in a metal plate in the XOY plane. Let the plate be of uniform thickness h, density ρ, thermal conductivity k and the specific heat c. Since the flow is two dimensional, the temperature at any point of the plate is independent of the z-co-ordinate. The heat flow lies in the XOY plane and is zero along the direction normal to the XOY plane.



Now, consider a rectangular element *ABCD* of the plate with sides δx and δy, the edges being parallel to the coordinates axes, as shown in the figure. Then the quantity of heat *entering* the element *ABCD* per sec. through the surface *AB* is

Example 1 The vertices of a thin square plate are $(0,0), (l,0), (0,l), (l,l)$. The upper edge of the square is maintained at an arbitrary temperature given by $u(x,l) = f(x)$. The other three edges are kept at zero temperature. Find the steady state temperature at any point on the plate.

Solution.

Suppose that $u(x, y)$ is the temperature at any point (x, y) of the plate in steady-state.

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ....(1)

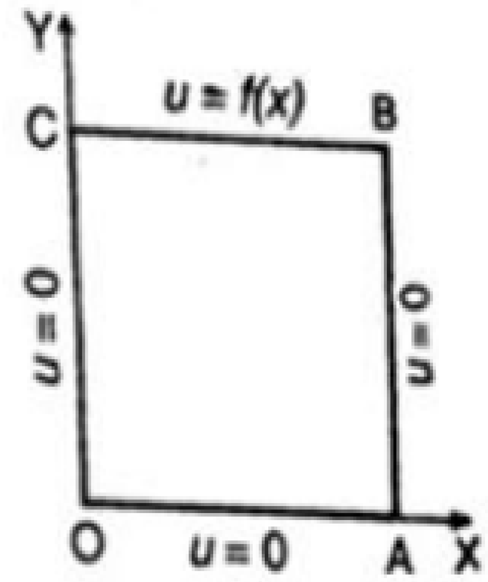
The boundary conditions are

$u(0, y) = 0, \quad \text{for } 0 \le y < l$ (i)

$u(l, y) = 0, \quad \text{for } 0 \le y < l$ (ii)

$u(x, 0) = 0, \quad \text{for } 0 \le x \le l$ (iii)

$u(x, l) = f(x), \quad \text{for } 0 < x < l$ (iv)



Solving (1), we get the three possible solutions,

$$u(x, y) = (Ae^{\lambda x} + Be^{-\lambda x})(C \cos \lambda y + D \sin \lambda y) \quad \text{....(I)}$$

$$u(x, y) = (A \cos \lambda x + B \sin \lambda x)(C e^{\lambda y} + D e^{-\lambda y}) \quad \text{....(II)}$$

$$u(x, y) = (Ax + B)(Cy + D) \quad \text{....(III)}$$

where A, B, C, D are different arbitary constants in each solution.

Now we shall select the solution II.

i.e., $u(x, y) = (A \cos \lambda x + B \sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$ ....(II)

Using the boundary condition (*i*) in (*II*),

$$A\,(Ce^{\lambda y} + De^{-\lambda y}) = 0, \text{ for } 0 \le y < l. \therefore A = 0$$

Using the condition (*ii*) in (*II*)

$$u(l, y) = B \sin \lambda l\ (Ce^{\lambda y} + De^{-\lambda y}) = 0. \text{ But } B \neq 0; \sin \lambda l = 0$$

i.e., $\lambda l = n\pi$

i.e., $\lambda = \dfrac{n\pi}{l}$ where *nn* is any integer.

Using (*iii*) in *II*,

$u(x, 0) = (C + D)(B \sin \lambda x) = 0$, for $0 \le x \le l$.

$B \neq 0$ Hence $C + D = 0$. $\therefore D = -C$.

Hence (*II*) reduces to,

$$u(x, y) = BC \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}} \right)$$

PARTIAL DIFFERENTIAL EQUATIONS

This unit covers topics that explain the formation of partial differential equations and the solutions of special types of first order partial differential equations (PDE).

1 Introduction

A partial differential equation (PDE) is one which involves one or more partial derivatives. The order of the highest derivative is called the order of the equation. A partial differential equation contains more than one independent variable. But, here we shall consider partial differential only equation two independent variables x and y so that $z = f(x, y)$. We shall denote

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

A partial differential equation is linear if it is of the first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous, otherwise it is non homogeneous. Partial differential equations are used to formulate and thus aid the solution of problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity.

2 Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of arbitrary constants or by the elimination of arbitrary functions.

(i) By the elimination of arbitrary constants

Let us consider the function f (x, y, z, a, b) = 0 ------------- (1)

where a & b are arbitrary constants

Differentiating equation (1) partially w.r.t x & y, we get

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Eliminating a and b from equations (1), (2) and (3), we get a partial differential equation of the first order of the form f (x, y, z, p, q) = 0.

(ii) By the elimination of arbitrary functions

Let u and v be any two functions which are arbitrary. This relation can be expressed as u = f(v) _______________ (1)

Differentiating (1) partially w.r.t x and y and eliminating the arbitrary functions from these relations, we get a PDE of the first order of the form f(x, y, z, p, q) = 0.

Example 1

Eliminate the arbitrary constants a and b from $z = ax + by + ab$ to construct a the PDE.

Solution. Consider $z = ax + by + ab$ ___________ (1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e,} \quad p = a \qquad \_\_\_\_\_\_\_\_\_ \quad (2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e,} \quad q = b \qquad \_\_\_\_\_\_\_ \quad (3)$$

Using (2) and (3) in (1), we get, $z = px + qy + pq$, which is the required PDE.

Example 2

Construct the partial differential equation by eliminating the arbitrary constants a and b from $z = (x^2 + a^2)(y^2 + b^2)$.

Solution. Given $z = (x^2 + a^2)(y^2 + b^2)$ _______(1)

Differentiating (1) partially w.r.t x and y, we get

$$p = 2x \ (y^2 + b^2)$$

$$q = 2y \ (x^2 + a^2)$$

Substituting the values of p and q in (1), we get, $4xyz = pq$, which is the PDE.

3 Solutions of a Partial Differential Equation

A solution or integral of a partial differential equation is a relation connecting the dependent and the independent variables which satisfies the given differential equation. A partial differential equation can result both from elimination of arbitrary constants and from elimination of arbitrary functions. But there is a basic difference in the two forms of solutions. A solution containing as many arbitrary constants as there are independent variables is called a complete integral. Here, the partial differential equations contain only two independent variables so that the complete integral will include two constants. The solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let $f(x,y,z,p,q) = 0$ ________ (1)

be the partial differential equation whose complete integral is

$f(x,y,z,a,b) = 0$ ________(2)

where a and b are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain

$$\frac{\partial \phi}{\partial a} = 0 \qquad \text{----------- (3)}$$

and

$$\frac{\partial \phi}{\partial b} = 0 \qquad \text{----------- (4)}$$

The eliminant of a and b from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put b = F(a), we get

f (x,y,z,a, F(a)) = 0 ---------- (5)

Differentiating (2), partially w.r.t. a, we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} F'(a) = 0 \qquad \text{-------- (6)}$$

The eliminant of a between (5) and (6), if it exists, is called the general integral of (1).

4 Lagrange's Linear Equation

Equations of the form Pp + Qq = R ________ (1), where P, Q and R are functions of x, y, z, are known as Lagrange equations. To solve this equation, let us consider the equations u = a and v = b, where a, b are constants and u, v are functions of x, y, z.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Comparing (2) and (3), we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (3)}$$

Similarly,

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (4)}$$

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$$

(or)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (5)}$$

Equation (5) represent a pair of simultaneous equations which are of the first order and of first degree. Therefore, the two solutions of (5) are $u = a$ and $v = b$. Thus, $f(u, v) = 0$ is the required solution of (1).

Note:

To solve the Lagrange's equation, we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping or by the method of multipliers.

Example

Find the general solution of $(mz - ny)p + (nx - lz)q = ly - mx$.

Solution.

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using the multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

$\therefore$ $xdx + ydy + zdz = 0$, which on integration gives

$$x^2/2 + y^2/2 + z^2/2 = \text{constant}$$

or $$x^2 + y^2 + z^2 = c_1 \quad \text{\_\_\_\_\_\_\_\_\_\_} (1)$$

Again using the multipliers l, m and n, we have

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

$\therefore$ $ldx + mdy + ndz = 0$, which on integration gives

$$lx + my + nz = c_2 \quad \text{\_\_\_\_\_\_\_\_\_\_} (2)$$

Hence, the required general solution is

$$\Phi(x^2 + y^2 + z^2, \ lx + my + nz) = 0$$

1 Some Special Types of Equations which can be Solved Easily by Methods other than the General Methods

The first order partial differential equation can be written as f(x, y, z, p, q) = 0, where $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$. In this section, we shall solve some standard forms of equations by special methods.

Type I: f (p, q) = 0. (Equations containing p and q only).
Suppose that z = ax + by +c is a solution of the equation f(p, q) = 0,
where f (a, b) = 0.

Solving this for b, we get b = F (a).
Hence the complete integral is z = ax + F(a) y + c __________(1)
To find the singular integral, differentiate (1) w.r.t. a, we get, 0 = x + yF'(a)_____(2)

Now, the singular integral is obtained by eliminating a and c from (1) and (2), we get 0 = 1.

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take c = F(a).

Then, z = ax + F(a) y + F(a) __________ (2)

Differentiating (2) partially w.r.t. a, we get

0 = x + F'(a). y + F '(a) ________ (3)

Eliminating a from (2) and (3), we get the general integral.

Type II: Equations of the form f (x,p,q) = 0, f (y,p,q) = 0 and f (z,p,q) = 0. (One of the variables x, y and z occurs explicitly)

(i) Let us consider the equation f (x,p,q) = 0.
Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $dz = pdx + qdy$

Assume that $q = a$.

Then the given equation takes the form $f\ (x, p, a\ ) = 0$.

Solving, we get p = F(x, a). Therefore, dz = F(x, a) dx + a dy.

(ii) Let us consider the equation f(y, p, q) = 0. Assume that p = a.

Then the equation becomes f (y, a, q) = 0 Solving, we get q = F (y, a).

Therefore, dz = adx + F(y,a) dy.

Integrating, z = ax + òF(y,a) dy + b, which is a complete Integral.

(iii) Let us consider the equation f(z, p, q) = 0.

Assume that q = ap.

Then the equation becomes f (z, p, ap) = 0
Solving, we get p = F(z,a). Hence dz = F(z,a) dx + a F(z, a) dy.

$$\text{ie,} \quad \frac{dz}{\Phi\ (z,a)} = dx + ady.$$

$$\text{Integrating,} \quad \int \frac{dz}{\Phi\ (z,a)} = x + ay + b, \text{ which is a complete Integral.}$$

Type III: f1(x, p) = f2 (y, q). ie, equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that f(x,p) = g(y,q) = a (say).

Solving for p and q, we get p = F(x, a) and q = G(y, a).

But $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Hence dz = pdx + qdy = F(x, a) dx + G(y, a) dy

Therefore, z = òF(x, a) dx + òG(y, a) dy + b , which is the complete integral of the given equation containing two constants a and b. The singular and general integrals are found in the usual way.

Example 7

Solve pq = xy

Solution.

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q} = a \text{ (say)}$$

Therefore, $\frac{p}{x} = a$ implies $p = ax$

and $\frac{y}{q} = a$ implies $q = \frac{y}{a}$

Since dz = pdx + qdy, we have

$$dz = axdx + \frac{y}{a} dy, \quad \text{which on integration gives.}$$

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Type IV (Clairaut's) form

Equation of the type $z = px + qy + f(p,q)$ ------(1) is known as Clairaut's form. Differentiating (1) partially w.r.t x and y, we get $p = a$ and $q = b$.

Therefore, the complete integral is given by
$z = ax + by + f(a,b)$.

2 EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non – linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By changing the variables suitably, we will reduce them into any one of the four standard forms.

Type (I): Equations of the form $F(x^m p, y^n q) = 0$ (or) $F(z, x^m p, y^n q) = 0$.

Case(i): If m '1 and n '1, then put $x^{1-m} = X$ and $y^{1-n} = Y$.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)\, x^{-m}$$

$$\text{Therefore, } x^m p = \frac{\partial z}{\partial X}(1-m) = (1-m)P, \text{ where } P = \frac{\partial z}{\partial X}$$

$$\text{Similarly, } y^n q = (1-n)Q, \text{ where } Q = \frac{\partial z}{\partial Y}$$

Hence, the given equation takes the form $F(P, Q) = 0$ (or) $F(z, P, Q) = 0$.

Case(ii) : If m = 1 and n = 1, then put log x = X and log y = Y.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \frac{1}{x}$$

$$\text{Therefore, } xp = \frac{\partial z}{\partial X} = P.$$

Similarly, yq = Q

Type (II) : Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y, z^k q)$.

Case (i) : If $k \neq -1$, put $Z = z^{k+1}$,

$$\text{Now } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \frac{\partial z}{\partial x} = (k+1)z^k \cdot \frac{\partial z}{\partial x} = (k+1)\, z^k p.$$

$$\text{Therefore, } z^k p = \frac{1}{k+1} \frac{\partial Z}{\partial x}$$

$$\text{Similarly, } z^k q = \frac{1}{k+1} \frac{\partial Z}{\partial y}$$

Case (ii) : If k = -1, put Z = log z.

$$\text{Now, } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \frac{\partial z}{\partial x} = \frac{1}{z} p$$

$$\text{Similarly, } \frac{\partial Z}{\partial y} = \frac{1}{z} q.$$

3 Charpit's Method

This is a general method to solve the most general non-linear PDE $f(x,y,z,p,q) = 0$____(1) of order one involving two independent variables. To solve (1), we solve the system of auxiliary equations called Charpit's equations.

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0} \quad \text{\_\_\_\_(2).}$$

Working rule of Charpit's Method:

Step 1: Transfer all the terms of the PDE to LHS and denote the entire expression by $f(x,y,z,p,q) = 0$.

Step 2: Write down Charpit's auxiliary equations.

Step 3: Find f_x, f_y, f_z, f_p and f_q. Put them in Step 2 and simplify.

Step 4: Choose two fractions such that the resulting integral is a simplest relation involving p or q or both.

Step 5: Use Step 4 to find p and q and put p and q in the equation $dz = pdx + qdy$, which on integration gives the complete integral.

Second and Higher Order Partial Differential Equations

This unit covers the following topics: Partial differential equations of second and higher order, Classification of linear partial differential equations of second order, Homogeneous and non-homogeneous equations with constant coefficients, Monge's methods.

1 Classification of linear partial differential equations of second order

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \_\_\_\_\_\_\_ (1)$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y, but do not depend on the unknown function u. The classification of second order linear PDEs is given by the following:

The second order linear PDE (1) is called

(i) Hyperbolic, if $B^2 - 4AC > 0$

(ii) Parabolic, if $B^2 - 4AC = 0$

(ii) Elliptic, if $B^2 - 4AC < 0$

2 Homogeneous Partial Linear Differential Equations with constant Coefficients.

A homogeneous linear partial differential equation of the n^{th} order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + c_n \frac{\partial^n z}{\partial y^n} = F(x,y) \qquad \text{---------} (1)$$

where $c_0, c_1, \text{---------}, c_n$ are constants and F is a function of 'x' and 'y'. It is

homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$(c_0 D^n + c_1 D^{n-1} D' + \ldots\ldots + c_n D'^n)\, z = F(x,y)$$

$$\text{or} \qquad f(D,D')\, z = F(x,y) \qquad \text{---------}(2),$$

$$\text{where, } \frac{\partial}{\partial x} \equiv D \text{ and } \frac{\partial}{\partial y} \equiv D'.$$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of f (D, D') z = 0-------(3), which must contain *n* arbitrary functions as the degree of the polynomial f(D, D'). The particular integral is the particular solution of equation (2).

Finding the complementary function

Let us now consider the equation f(D, D') z = F (x, y).

The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.

$$\text{i.e, } c_0 m^n + c_1 m^{n-1} + \ldots\ldots + c_n = 0 \qquad \text{---------(4)}$$

Solving equation (4) for m, we get n roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	**Nature of the roots**	**Complementary function(C.F)**
$m_1, m_2, m_3 \ldots\ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x)+\ldots\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = m,\ m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x)+f_3(y+m_3x)+\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = \ldots\ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx)+x^2f_3(y+mx)+\ldots\ldots+\ldots+x^{n-1}f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D,D')\,z = F(x,y)$.

Now, the P.I is given by $\dfrac{1}{f(D,D')} F(x,y)$

Case (i) : When $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

Replacing D by 'a' and D' by 'b', we have

$$P.I = \frac{1}{f(a,b)} e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

Case (ii) : When $F(x,y) = \sin(ax + by)$ (or) $\cos(ax + by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by), \text{ where } f(-a^2, -ab, -b^2) \neq 0.$$

Case (iii) : When $F(x,y) = x^m y^n$,

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv) : When F(x,y) is any function of x and y.

$$P.I = \frac{1}{f(D,D')} F(x,y).$$

Resolve $\frac{1}{f(D,D')}$

into partial fractions considering f (D,D') as a function of D alone.

Then operate each partial fraction on F(x,y) in such a way that

$$\frac{1}{D-mD'} F(x,y) = \int F(x,c-mx)\, dx ,$$

where c is replaced by y+mx after integration