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MUTHAYAMMAL ENGINEERING COLLEGE
Rasipuram - 637 408

COURSE CODE & TITLE – 19ECC02 / Signals and Systems

Presentation by
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AP-ECE

Signals and systems

Text Books:

1. A.V. Oppenheim, A.S. Willsky and S.H. Nawab, “Signals and Systems”, 2nd Edition, PHI, 2009.

2. Simon Haykin and Van Veen, “Signals & Systems”, 2nd Edition, Wiley, 2005.

References:

1. BP Lathi, “Principles of Linear Systems and Signals”, 2nd Edition, Oxford University Press, 015.

2. Matthew Sadiku and Warsame H. Ali, “Signals and Systems A primer with MATLAB”, CRC Press, 2016.

3. Hwei Hsu, “Schaum's Outline of Signals and Systems”, 4th Edition, TMH, 2019.

Signals and systems overview

What is Signal?

It is representation of physical quantity (Sound, temperature, intensity, Pressure, etc..,) which varies with respect to time or space or independent or dependent variable.

or

It is single valued function which carries information by means of Amplitude, Frequency and Phase.

Example: voice signal, video signal, signals on telephone wires etc.

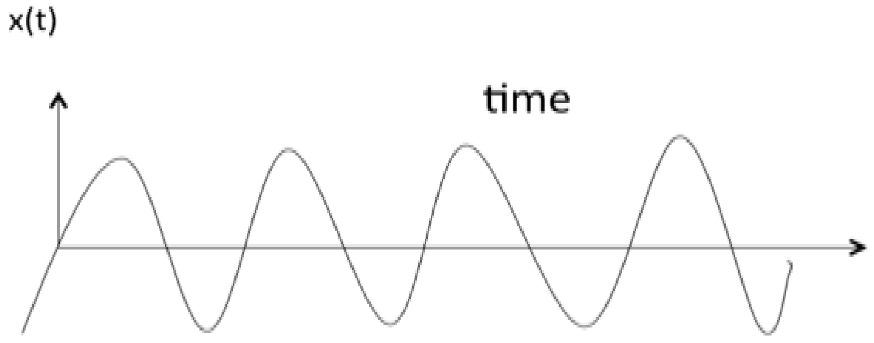
$$X(t) = A\,Sin(\omega t + \phi) = A\,Sin(2\pi f t + \phi)$$

Where A = Peak Amplitude

f = Frequency = $\dfrac{1}{T(Time\ Period)}$

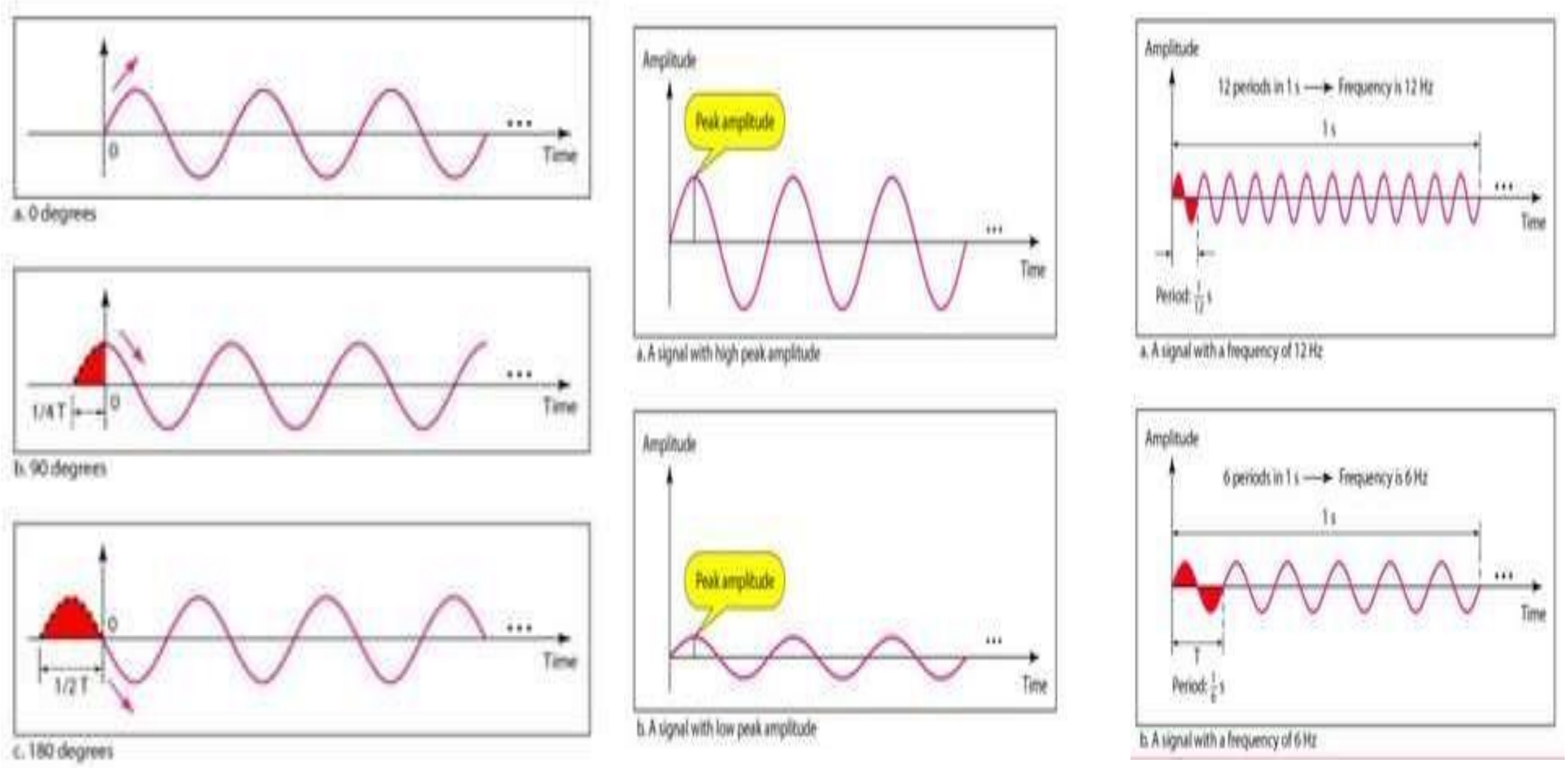
ϕ = Phase angle

Angular Frequency $\omega = 2\pi f$ (Linear Frequency)



Signals and systems overview

Signal with different Phases, Amplitudes and Frequencies

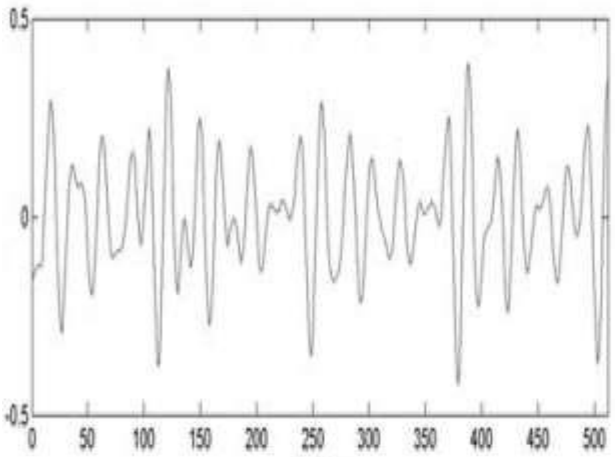


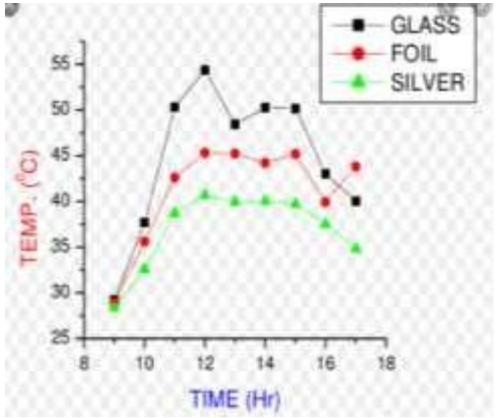
Classification of Signals

Types of Signals with respect to no. of variables or dimensions

- <u>One Dimensional or 1-D Signal:</u> If the signal is function of only one variable or If Signal value varies with respect to only one variable then it is called "One Dimensional or 1-D Signal"

  Examples: Audio Signal, Biomedical Signals, temperature Signal etc.., in which signal is function "time"





Classification of Signals

- <u>Two Dimensional or 2-D Signal:</u> If the signal is function of two variable or If Signal value varies with respect to two variable then it is called "Two Dimensional or 2-D Signal"

  Examples: Image Signal in which intensity is function of two spatial co-ordinates "X" & "Y" i,.e I(X,Y)

- <u>Three Dimensional or 3-D Signal:</u> If the signal is function of three variable or If Signal value varies with respect to three variable then it is called "Three Dimensional or 3-D Signal"

  Examples: Video Signal in which intensity is function of two spatial co-ordinates "X" & "Y" and also time "t" i.e v(x,y,t)

Classification of Signals

Types of Signal with respect to nature of the signal

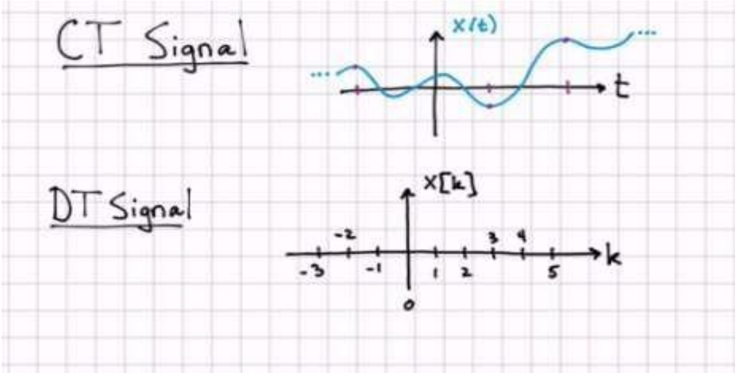
Continuous Time Signal (CTS) or Analog Signal :

If the signal values continuously varies with respect to time then it is called "Continuous Time Signal (CTS) or Analog Signal ". It contains infinite set of values and it is represented as shown below.

Digital Signal: If the signal contains only two values then it is called "Digital Signal".

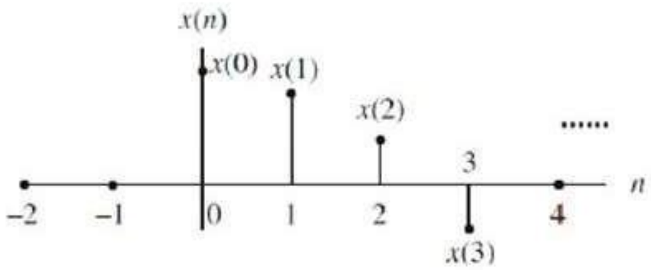
Discrete Time Signal (DTS):

If signal contain discrete set of values with respect to time then it is called "Discrete Time Signal (DTS)". It contains finite set of values. Sampling process converts Continuous time signal in to Discrete time signal.



Representation of Discrete Time Signal (DTS)

A discrete-time signal $x(n)$ is a function of an integer variable n. In the DS processor, the signal is represented by the discrete encoded values with a finite precision.



Graphical representation of a discrete-time signal $x(n)$

$$x(n) = \begin{cases} 1, & for\ n = 1,3 \\ 4, & for\ n = 2 \\ 0, & elsewhere \end{cases}$$

Functional representation

n	…	−2	−1	0	1	2	3	4	5	…
$x(n)$	…	0	0	0	1	4	1	0	0	…

Tabular representation

$x(n) = \{\dots, 0, \mathbf{0}, 1, 4, 1, 0, 0, \dots\}$ $\quad$ $infinite - duration\ signal$

$x(n) = \{\mathbf{0}, -2, 1, 4, -1,\}$ $\quad$ $finite - duration\ signal$

Sequence representation (bold or arrow for origin n=0)

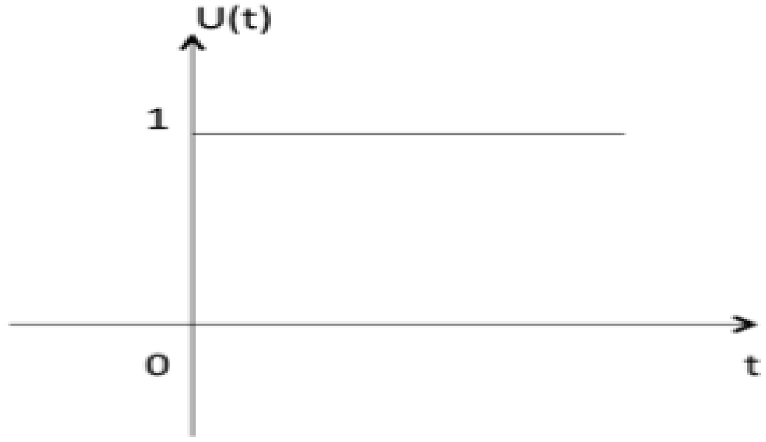
Mathematically a discrete-time signal $x(n)$ can be determined by

$$x(n) = x(t)|_{t=nT} = x(nT)$$

Basic Types of Signals

- **Unit Step Function**

Unit step function is denoted by u(t). It is defined as u(t) = 1 when $t \geq 0$
and 0 when $t < 0$

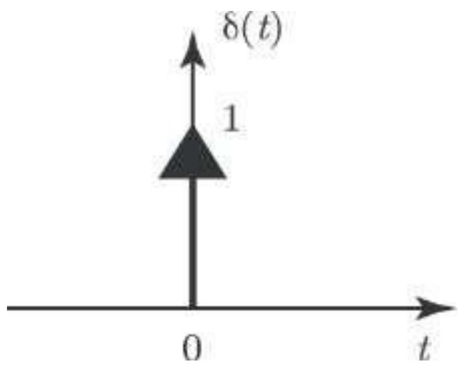


- It is used as best test signal.
- Area under unit step function is unity.

Basic Types of Signals

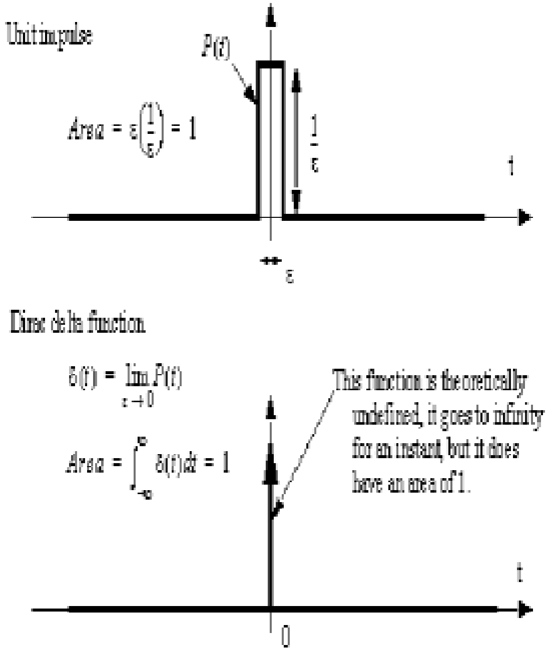
➢ Unit Impulse Function

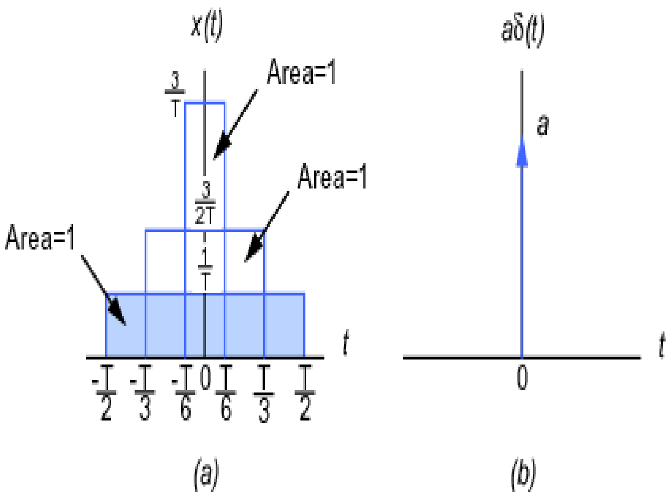
Impulse function is denoted by δ(t). and it is defined as $\delta(t) = \begin{cases} 0; & t \neq 0 \\ \infty; & t = 0 \end{cases}$



$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

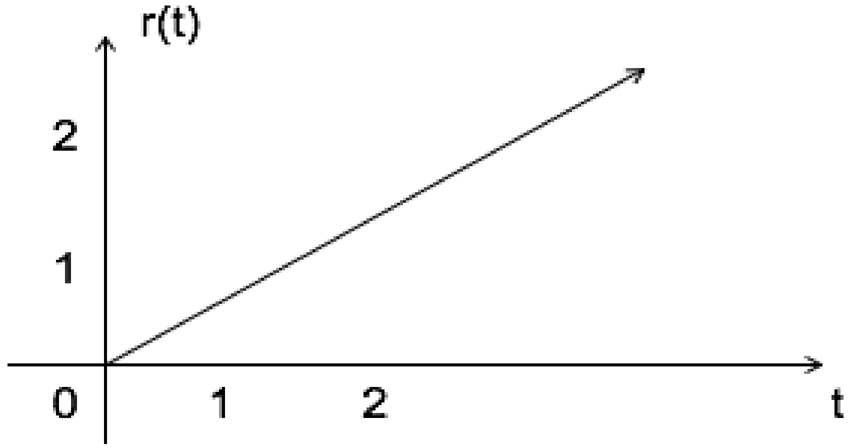




Basic Types of Signals

- **Ramp Signal**

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



$$\int u(t) = \int 1 = t = r(t)$$

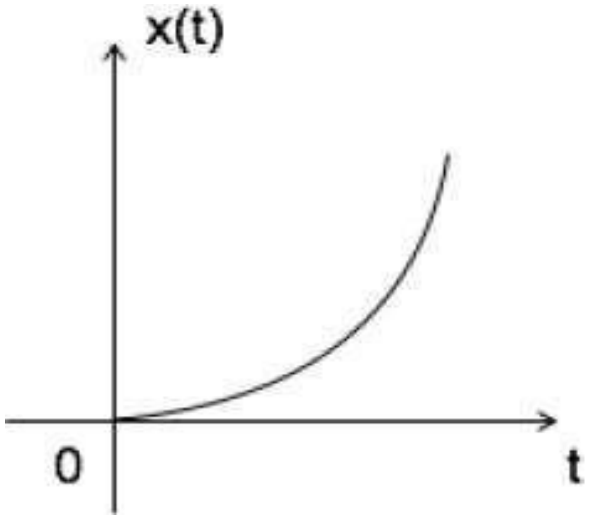
$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

Basic Types of Signals

- **Parabolic Signal**

Parabolic signal can be defined as x(t) = $\begin{cases} t^2/2 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



$$\iint u(t)dt = \int r(t)dt = \int tdt = \frac{t^2}{2} = parabolicsignal$$

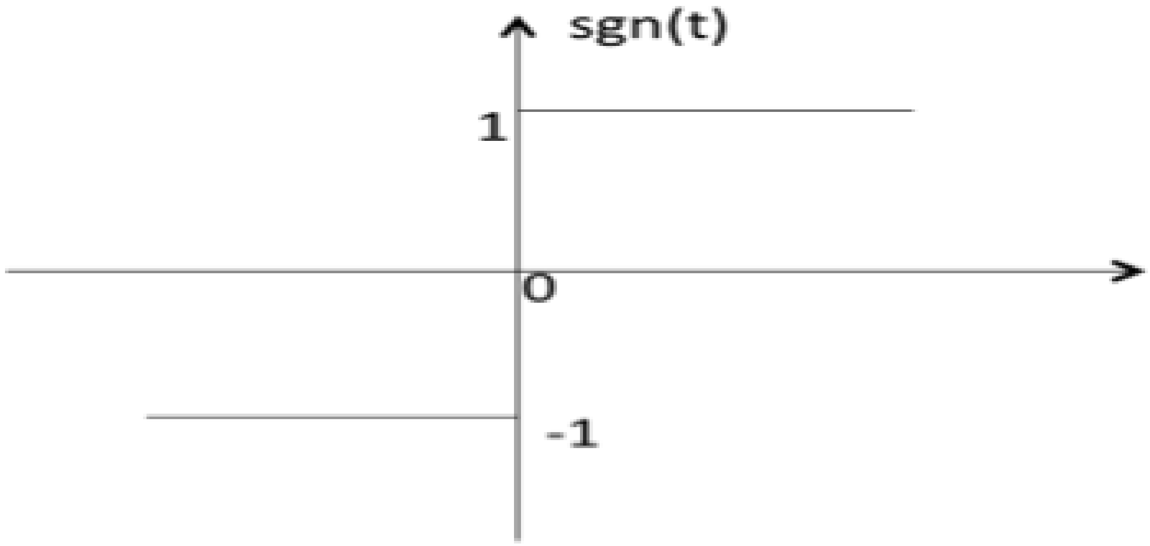
$$\Rightarrow u(t) = \frac{d^2x(t)}{dt^2}$$

$$\Rightarrow r(t) = \frac{dx(t)}{dt}$$

Basic Types of Signals

- **Signum Function**

Signum function is denoted as sgn(t). It is defined as sgn(t) = $\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



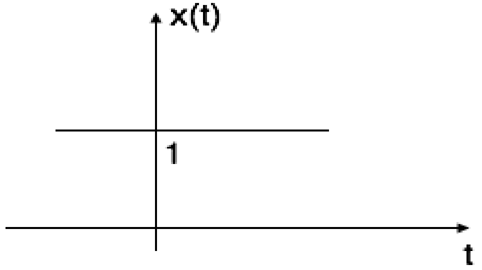
Basic Types of Signals

- **Exponential Signal**

Exponential signal is in the form of $x(t) = e^{at}$

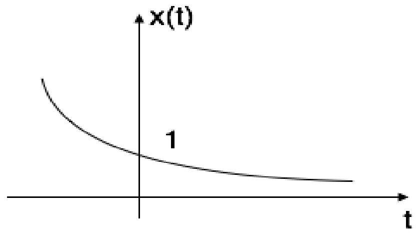
The shape of exponential can be defined by a.

Case i: if $a = 0 \rightarrow x(t) = e^{0} = 1$



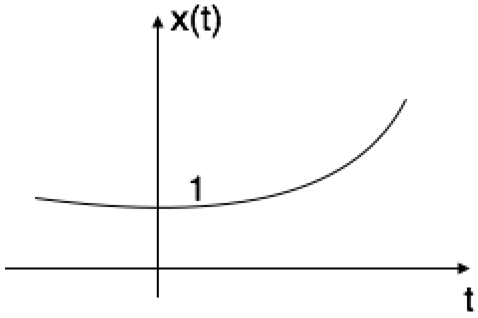
Case ii: if $a < 0$ i.e. -ve then $x(t) = e^{-at}$,

The shape is called decaying exponential.



Case iii: if $a > 0$ i.e. +ve then $x(t) = e^{at}$,

The shape is called raising exponential.



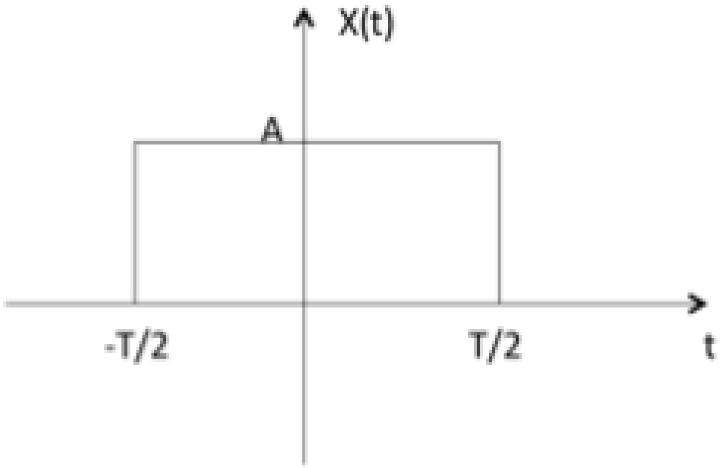
Basic Types of Signals

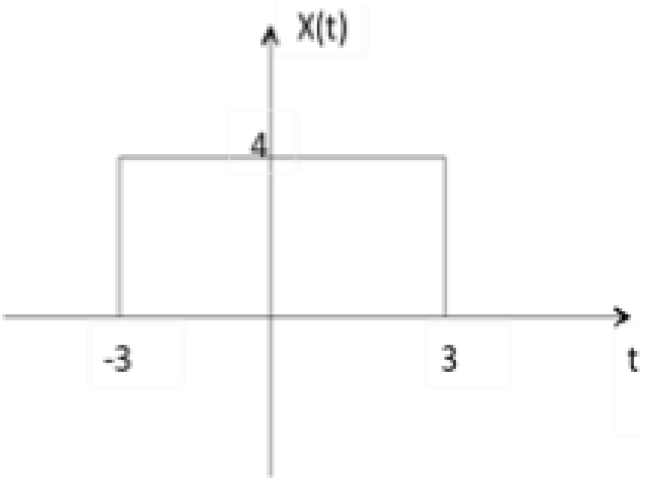
Rectangular Signal

Let it be denoted as x(t) and it is defined as

$$x(t) = A\,rect\left[\frac{r}{T}\right]$$

$$\text{ex: } 4\,rect\left[\frac{r}{6}\right]$$



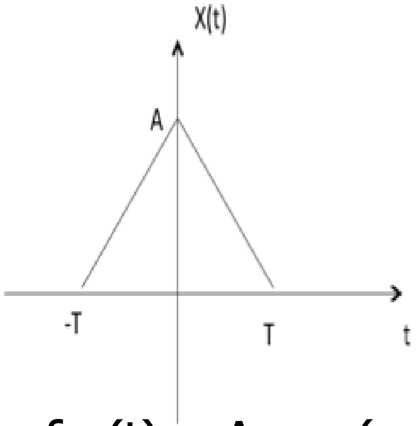


Basic Types of Signals

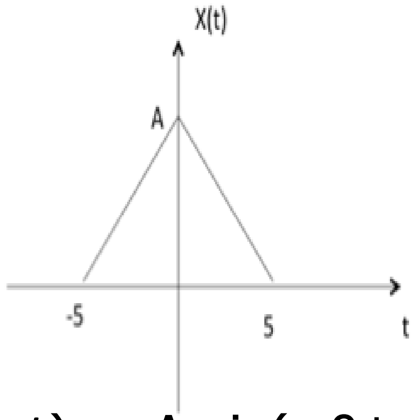
Triangular Signal

Let it be denoted as x(t),

$$x(t) = A\left[1 - \frac{|t|}{T}\right]$$

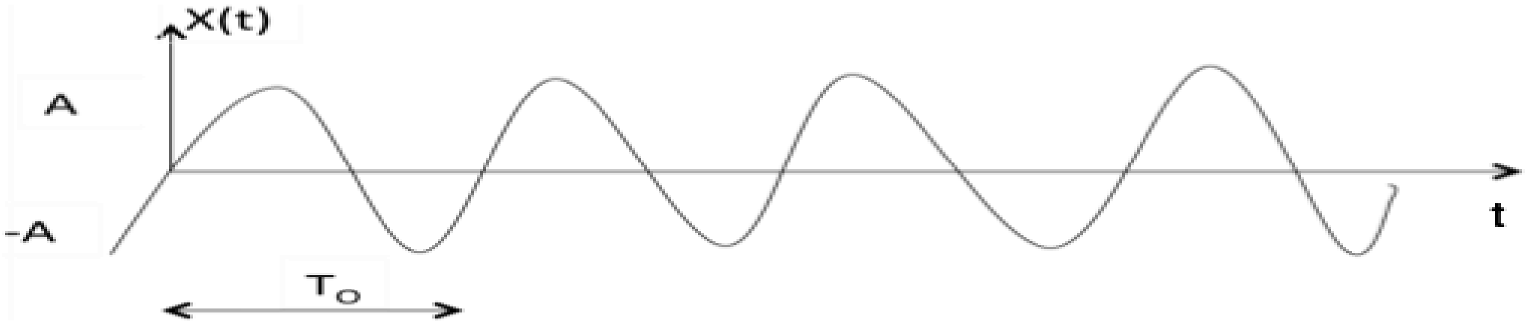


$$\text{ex:}\ x(t) = A\left[1 - \frac{|t|}{5}\right]$$



Sinusoidal Signal

Sinusoidal signal is in the form of x(t) = A cos($w0 \pm \phi$) or A sin($w0 \pm \phi$)



Where T0 = $2\pi / w0$

Classification of Signals

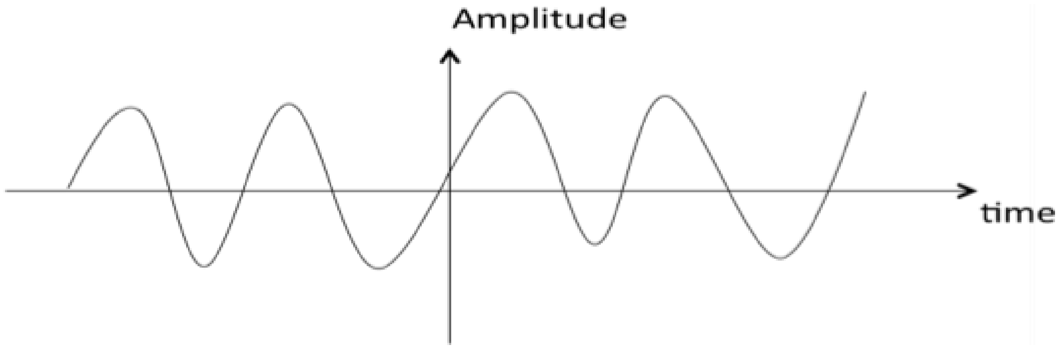
Signals are classified into the following categories:

- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

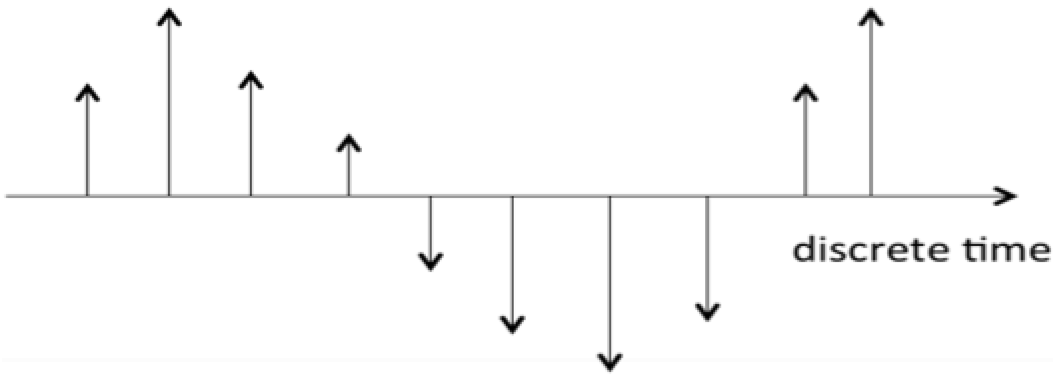
Classification of Signals

- **Continuous Time and Discrete Time Signals**

A signal is said to be continuous when it is defined for all instants of time.



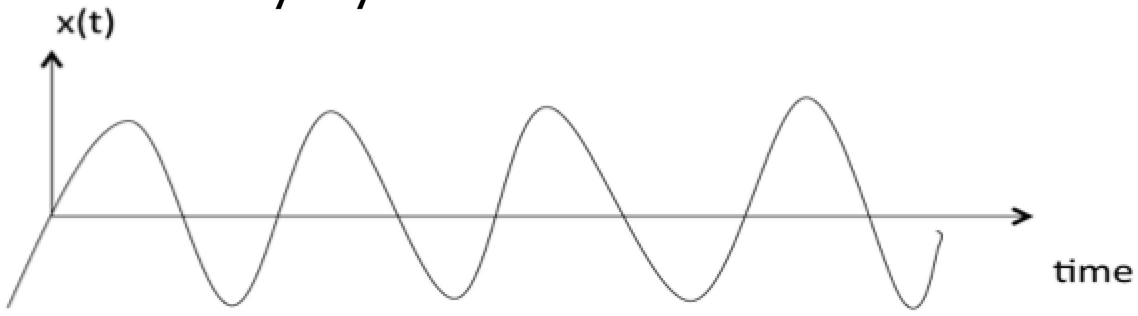
A signal is said to be discrete when it is defined at only discrete instants of time.



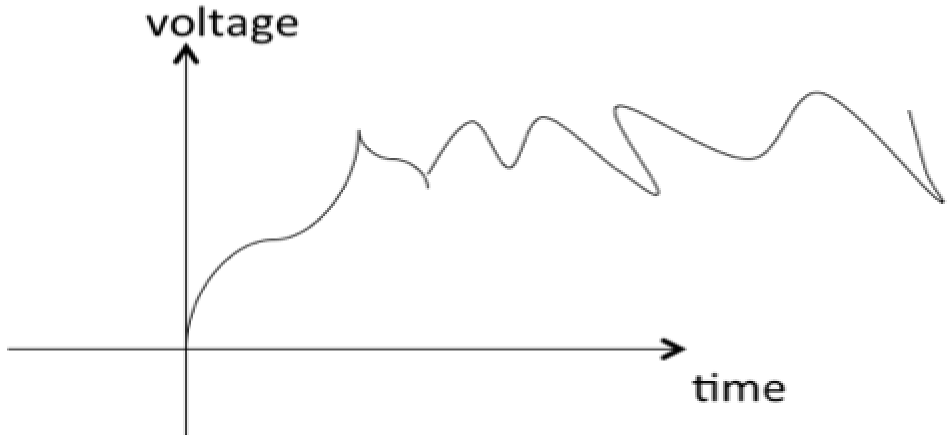
Classification of Signals

Deterministic and Non-deterministic Signals

A signal is said to be **deterministic** if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.



A signal is said to be **non-deterministic** if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called **random signals**. Random signals cannot be described by a mathematical equation.



They are modelled in probabilistic terms.

Classification of Signals

Even and Odd signals

A signal is said to be **even** when it satisfies the condition $x(t) = x(-t)$

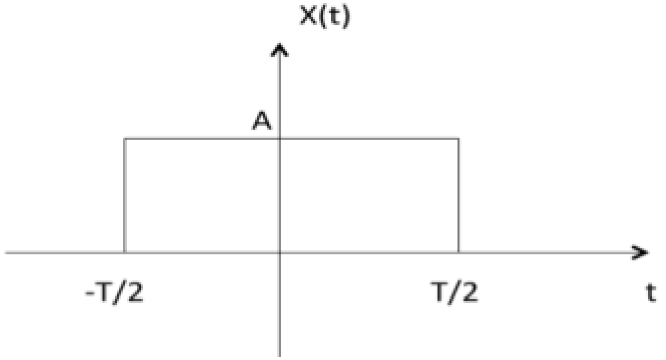
Example 1: t^2, t^4… $\cos t$ etc.

Let $x(t) = t^2$

$x(-t) = (-t)^2 = t^2 = x(t)$

$\therefore t^2$ is even function

Example 2: As shown in the following diagram, rectangle function $x(t) = x(-t)$ so it is also even function.



A signal is said to be **odd** when it satisfies the condition $x(t) = -x(-t)$

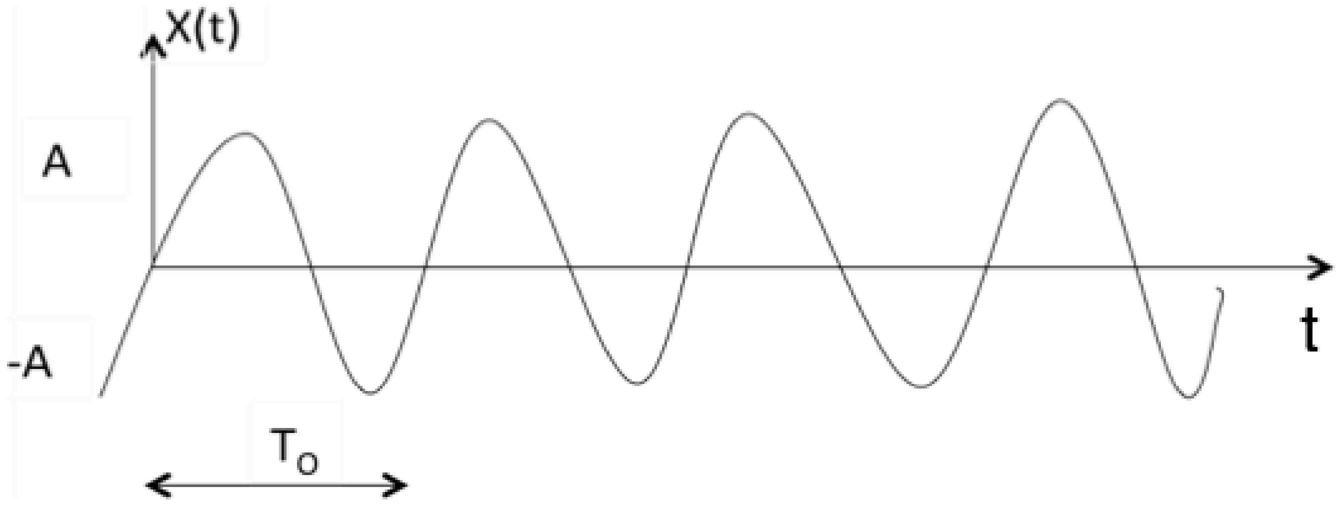
Classification of Signals

Periodic and Aperiodic Signals

A signal is said to be **periodic** if it satisfies the condition $x(t) = x(t + T)$ or $x(n) = x(n + N)$.

Where, T = fundamental time period,

$1/T = f =$ fundamental frequency.



The above signal will repeat for every time interval T0 hence it is periodic with period T0.

Classification of Signals

Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2\,(t)dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

NOTE: A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0 and Energy of power signal = ∞

Classification of Signals

Real and Imaginary Signals

A signal is said to be **real** when it satisfies the condition $x(t) = x^*(t)$

A signal is said to be **odd** when it satisfies the condition $x(t) = -x^*(t)$

Example:

If $x(t)= 3$ then $x^*(t)=3^*=3$, here $x(t)$ is a real signal.

If $x(t)= 3j$ then $x^*(t)=3j^* = -3j = -x(t)$, hence $x(t)$ is a odd signal.

Note: For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero.

Basic Operations on Signals

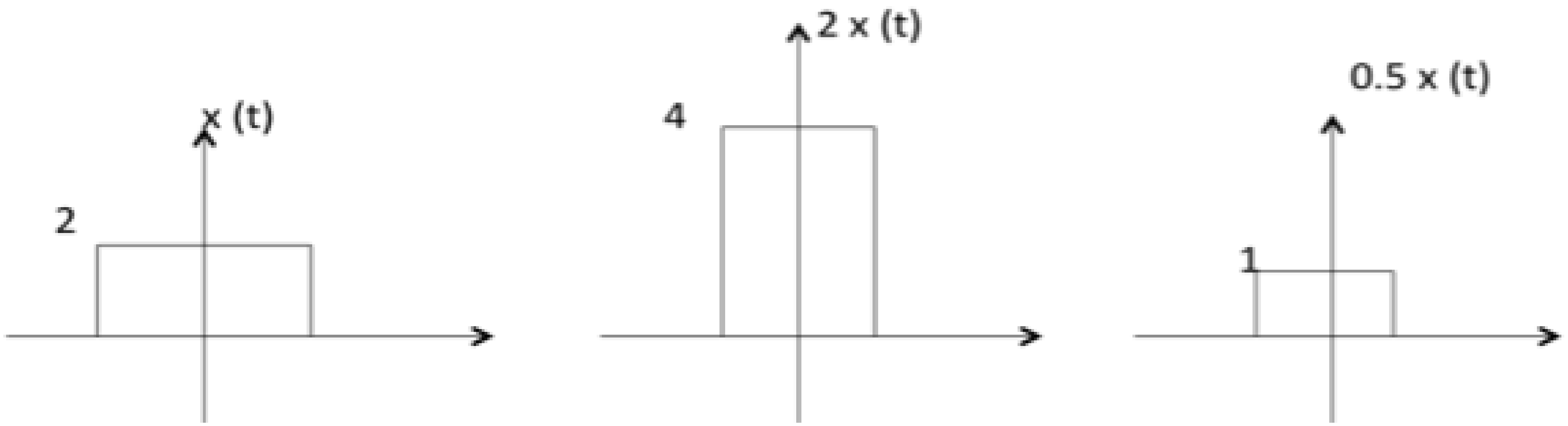
There are two variable parameters in general:

- Amplitude
- Time

The following operation can be performed with amplitude:

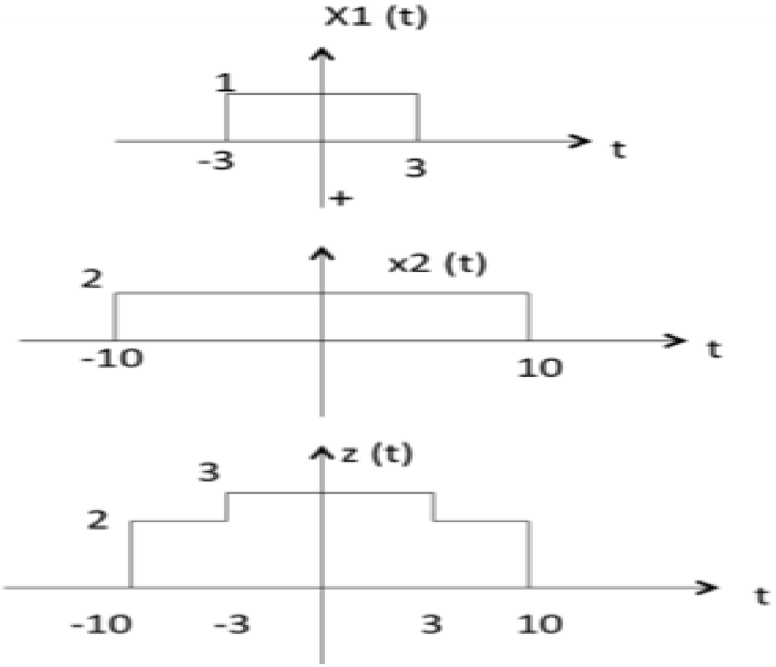
Amplitude Scaling

C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



As seen from the previous diagram,

$-10 < t < -3$ amplitude of $z(t) = x1(t) + x2(t) = 0 + 2 = 2$

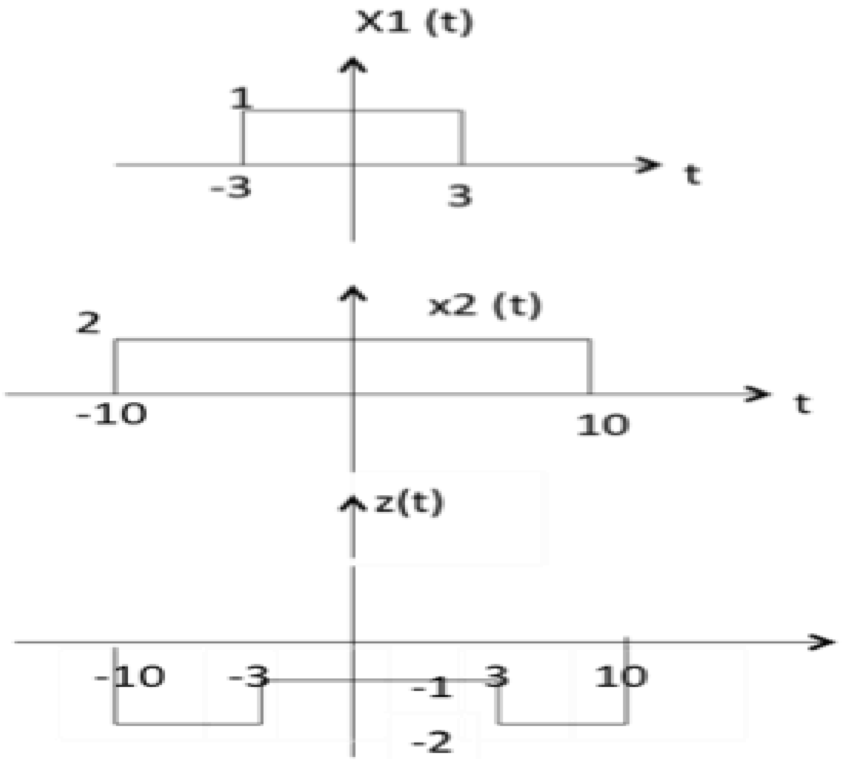
$-3 < t < 3$ amplitude of $z(t) = x1(t) + x2(t) = 1 + 2 = 3$

$3 < t < 10$ amplitude of $z(t) = x1(t) + x2(t) = 0 + 2 = 2$

Basic Operations on Signals

Subtraction

subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

$-10 < t < -3$ amplitude of z (t) = x1(t) - x2(t) = 0 - 2 = -2

$-3 < t < 3$ amplitude of z (t) = x1(t) - x2(t) = 1 - 2 = -1

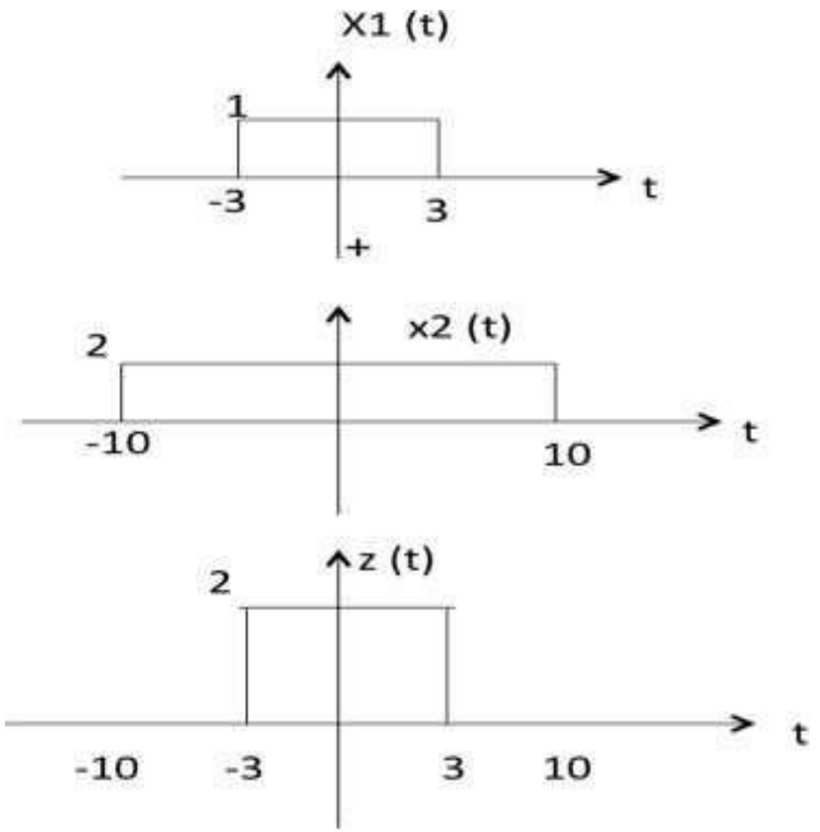
$3 < t < 10$ amplitude of z (t) = x1(t) - x2(t) = 0 - 2 = -2

Basic Operations on Signals

Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes.

This can be best explained by the following example:



As seen from the diagram above,

$-10 < t < -3$ amplitude of $z(t) = x1(t) \times x2(t) = 0 \times 2 = 0$

$-3 < t < 3$ amplitude of $z(t) = x1(t) - x2(t) = 1 \times 2 = 2$

$3 < t < 10$ amplitude of $z(t) = x1(t) - x2(t) = 0 \times 2 = 0$

Basic Operations on Signals

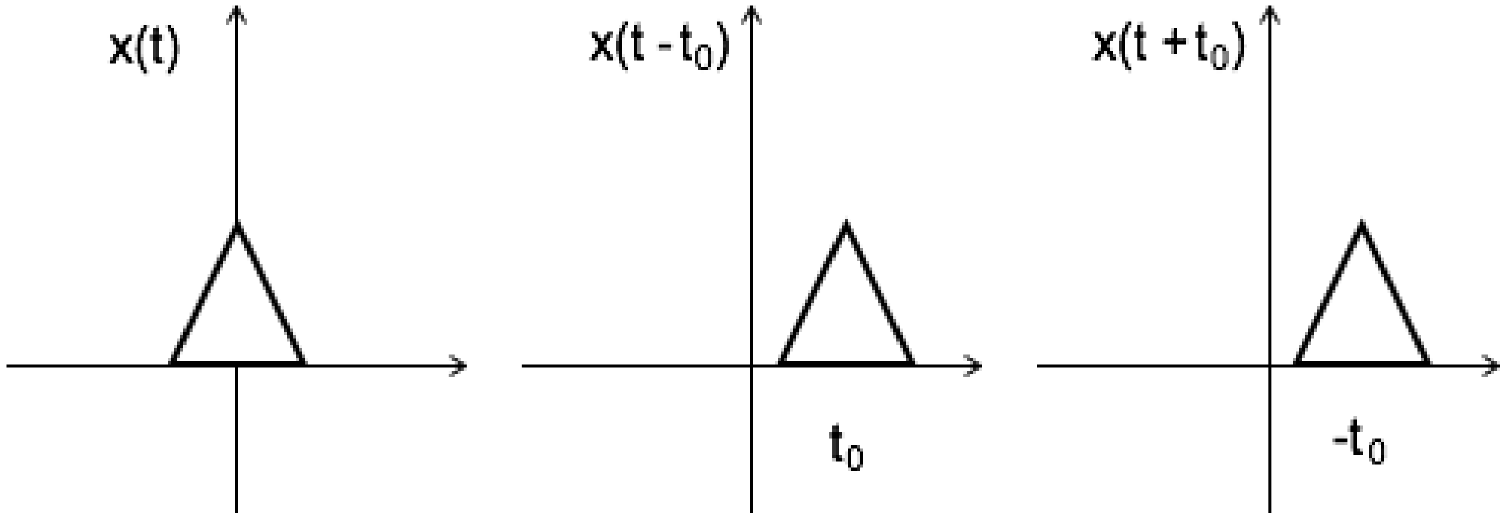
The following operations can be performed with time:

Time Shifting

$x(t \pm t0)$ is time shifted version of the signal $x(t)$.

$x(t + t0) \rightarrow$ negative shift

$x(t - t0) \rightarrow$ positive shift



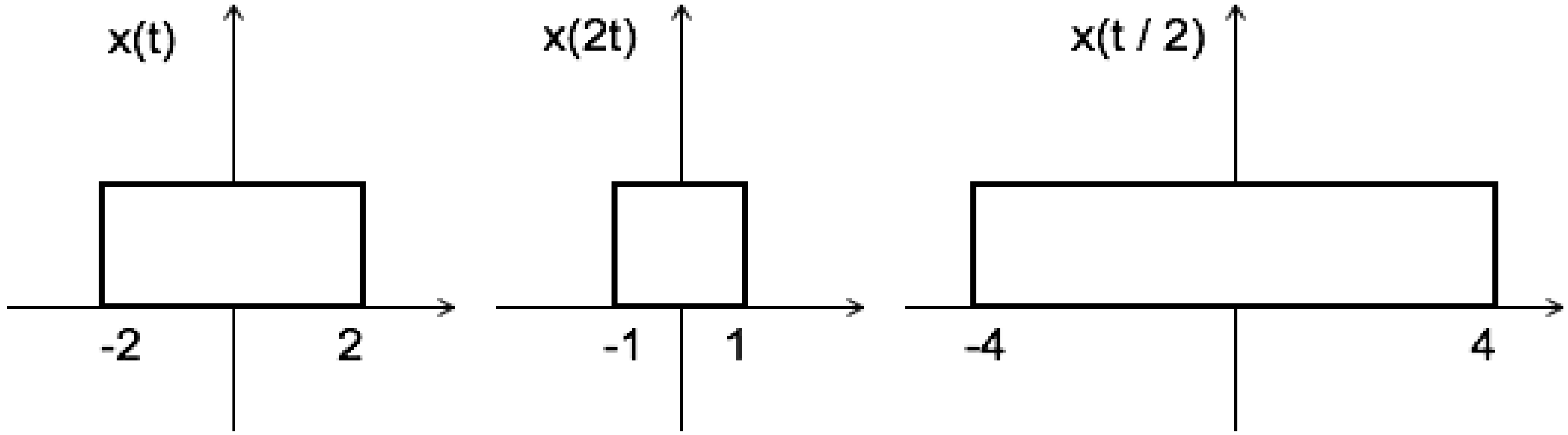
Basic Operations on Signals

Time Scaling

x(At) is time scaled version of the signal x(t). where A is always positive.

$|A| > 1 \rightarrow$ Compression of the signal

$|A| < 1 \rightarrow$ Expansion of the signal

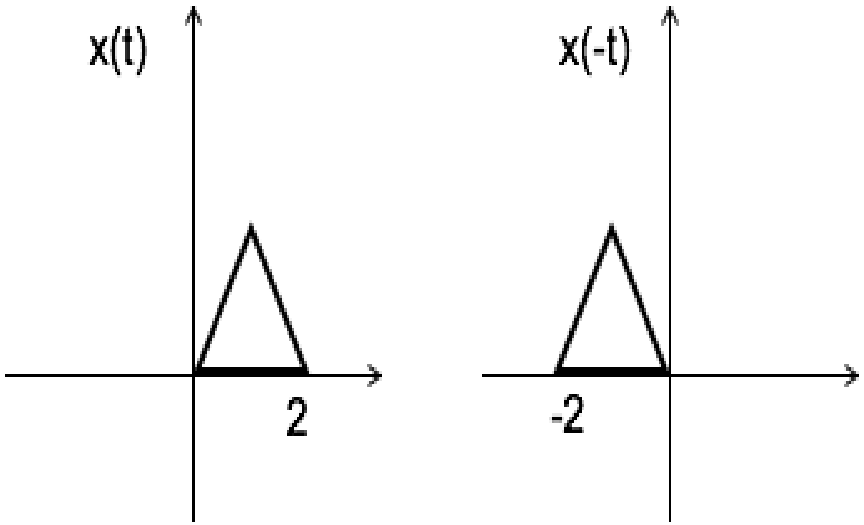


Note: u(at) = u(t) time scaling is not applicable for unit step function.

Basic Operations on Signals

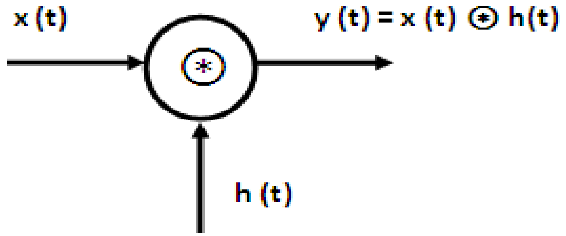
Time Reversal

x(-t) is the time reversal of the signal x(t).



Basic Operations on Signals

Convolution: Convolution between two continuous time signals can be written as



$$y(t) = x(t) \circledast h(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)h(t-\tau)d\tau = \int_{\tau=-\infty}^{\tau=+\infty} h(\tau)x(t-\tau)d\tau$$

The following operations are required to compute convolution

1. Time reversal
2. Time Shifting (Delay & Advance)
3. Signal Multiplication
4. Integration

Note: If two signals are finite duration then Graphical Method is used and Else Function Method is employed to compute Convolution

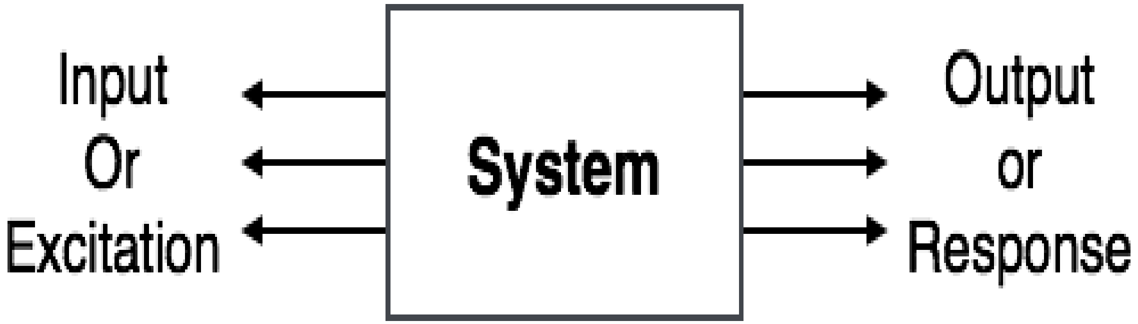
System Definition

What is System?

System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response.

For one or more inputs, the system can have one or more outputs.

Example: Communication System



Classification of Systems

Systems are classified into the following categories:

- linear and Non-linear Systems
- Time Variant and Time Invariant Systems
- linear Time variant and linear Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

Classification of Systems

Linear and Non-linear Systems

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as x1(t), x2(t), and outputs as y1(t), y2(t) respectively. Then, according to the superposition and homogenate principles,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$\therefore T[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example: $y(t) = x^2(t)$

Solution:

$$y_1(t) = T[x_1(t)] = x_1^2(t)$$

$$y_2(t) = T[x_2(t)] = x_2^2(t)$$

$$T[a_1 x_1(t) + a_2 x_2(t)] = [a_1 x_1(t) + a_2 x_2(t)]^2$$

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be non linear.

Classification of Systems

Time Variant and Time Invariant Systems

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant. The condition for time invariant system is:

$$y(n, t) = y(n-t)$$

The condition for time variant system is:

$$y(n, t) \neq y(n-t)$$

Where $y(n, t) = T[x(n-t)]$ = input change

$y(n-t)$ = output change

Example:

$y(n) = x(-n)$

$y(n, t) = T[x(n-t)] = x(-n-t)$

$y(n-t) = x(-(n-t)) = x(-n + t)$

$\therefore y(n, t) \neq y(n-t)$. Hence, the system is time variant.

Classification of Systems

Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

Example 1: $y(t) = 2\,x(t)$

For present value t=0, the system output is $y(0) = 2x(0)$. Here, the output is only dependent upon present input. Hence the system is memory less or static.

Example 2: $y(t) = 2\,x(t) + 3\,x(t-3)$

For present value t=0, the system output is $y(0) = 2x(0) + 3x(-3)$.

Here $x(-3)$ is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

Classification of Systems

Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non causal system, the output depends upon future inputs also.

Example 1: $y(n) = 2\,x(t) + 3\,x(t-3)$

For present value t=1, the system output is $y(1) = 2x(1) + 3x(-2)$.

Here, the system output only depends upon present and past inputs. Hence, the system is causal.

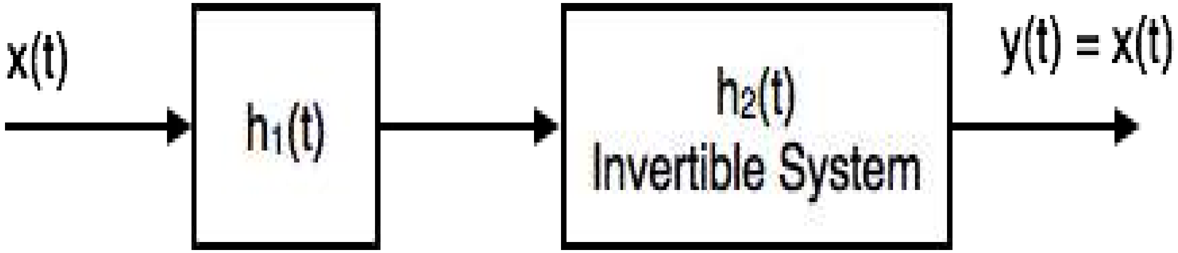
Example 2: $y(n) = 2\,x(t) + 3\,x(t-3) + 6x(t + 3)$

For present value t=1, the system output is $y(1) = 2x(1) + 3x(-2) + 6x(4)$ Here, the system output depends upon future input. Hence the system is non-causal system.

Classification of Systems

Invertible and Non-Invertible systems

A system is said to invertible if the input of the system appears at the output.



$$Y(S) = X(S)\ H1(S)\ H2(S)$$

$$= X(S)\ H1(S) \cdot 1(H1(S))$$

Since $H2(S) = 1/(H1(S))$

$\therefore Y(S) = X(S)$

$\rightarrow y(t) = x(t)$

Hence, the system is invertible.

If $y(t) \neq x(t)$, then the system is said to be non-invertible.

Classification of Systems

Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

Note: For a bounded signal, amplitude is finite.

Example 1: $y(t) = x^2(t)$

Let the input is u(t) (unit step bounded input) then the output y(t) = u2(t) = u(t) = bounded output.

Hence, the system is stable.

Example 2: $y(t) = \int x(t)dt$

Let the input is u (t) (unit step bounded input) then the output $y(t) = \int u(t)dt$ = ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when t → infinite).

Hence, the system is unstable.

Convolution and correlation of signals

Convolution

Convolution is a mathematical operation used to express the relation between input and output of an LTI system. It relates input, output and impulse response of an LTI system as

$$y(t) = x(t) * h(t)$$

Where y (t) = output of LTI

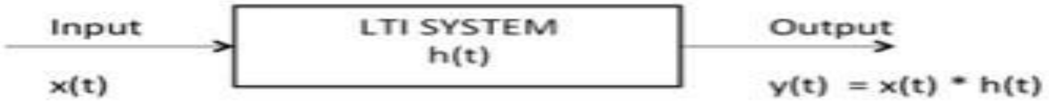
x (t) = input of LTI

h (t) = impulse response of LTI

There are two types of convolutions:

- Continuous convolution
- Discrete convolution

Continuous Convolution



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

(or)

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

Convolution and correlation of signals

Discrete Convolution

$$y(n) = x(n) * h(n)$$

$$= \Sigma_{k=-\infty}^{\infty} x(k)h(n-k)$$

(or)

$$= \Sigma_{k=-\infty}^{\infty} x(n-k)h(k)$$

By using convolution we can find zero state response of the system.

Deconvolution

Deconvolution is reverse process to convolution widely used in signal and image processing.

Properties of Convolution

Commutative Property

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive Property

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

Convolution and correlation of signals

Associative Property

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shifting Property

$$x_1(t) * x_2(t) = y(t)$$

$$x_1(t) * x_2(t - t_0) = y(t - t_0)$$

$$x_1(t - t_0) * x_2(t) = y(t - t_0)$$

$$x_1(t - t_0) * x_2(t - t_1) = y(t - t_0 - t_1)$$

Convolution with Impulse

$$x_1(t) * \delta(t) = x(t)$$

$$x_1(t) * \delta(t - t_0) = x(t - t_0)$$

Convolution and correlation of signals

Convolution of Unit Steps

$$u(t) * u(t) = r(t)$$

$$u(t - T_1) * u(t - T_2) = r(t - T_1 - T_2)$$

$$u(n) * u(n) = [n + 1]u(n)$$

Scaling Property

If $x(t) * h(t) = y(t)$

then $x(at) * h(at) = \frac{1}{|a|} y(at)$

Differentiation of Output

if $y(t) = x(t) * h(t)$

then $\frac{dy(t)}{dt} = \frac{dx(t)}{dt} * h(t)$

or

$$\frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt}$$

Convolution and correlation of signals

Note:

- Convolution of two causal sequences is causal.
- Convolution of two anti causal sequences is anti causal.
- Convolution of two unequal length rectangles results a trapezium.
- Convolution of two equal length rectangles results a triangle.
- A function convoluted itself is equal to integration of that function.

Example: You know that $u(t) * u(t) = r(t)$

According to above note, $u(t) * u(t) = \int u(t)dt = \int 1dt = t = r(t)$

Here, you get the result just by integrating $u(t)$.

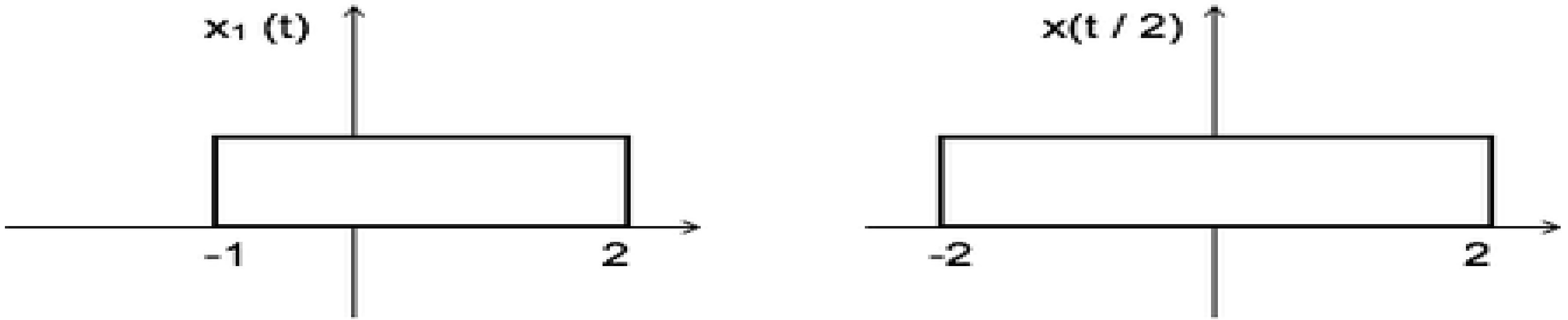
Convolution and correlation of signals

Limits of Convoluted Signal

If two signals are convoluted then the resulting convoluted signal has following range:

Sum of lower limits < t < sum of upper limits

Ex: find the range of convolution of signals given below



Here, we have two rectangles of unequal length to convolute, which results a trapezium.

The range of convoluted signal is:

Sum of lower limits < t < sum of upper limits

$$-1 + -2 < t < 2 + 2$$

$$-3 < t < 4$$

Hence the result is trapezium with period 7.

Convolution and correlation of signals

Area of Convoluted Signal

The area under convoluted signal is given by $A_y = A_x A_h$

Where A_x = area under input signal

A_h = area under impulse response

A_y = area under output signal

Proof: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Take integration on both sides

$$\int y(t)dt = \int \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt$$

$$= \int x(\tau)d\tau \int_{-\infty}^{\infty} h(t-\tau)dt$$

We know that area of any signal is the integration of that signal itself.

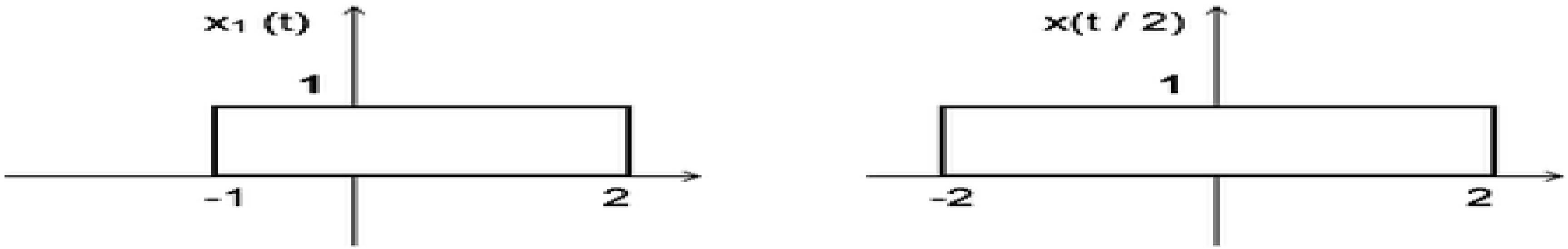
$\therefore A_y = A_x \, A_h$

DC Component

DC component of any signal is given by

$$\text{DC component} = \frac{\text{area of the signal}}{\text{period of the signal}}$$

Ex: what is the dc component of the resultant convoluted signal given below?



Here area of $x_1(t)$ = length × breadth = 1 × 3 = 3

area of $x_2(t)$ = length × breadth = 1 × 4 = 4

area of convoluted signal = area of $x_1(t)$ × area of $x_2(t)$

= 3 × 4 = 12

Duration of the convoluted signal = sum of lower limits < t < sum of upper limits

= -1 + -2 < t < 2+2

= -3 < t < 4

Period=7

$\therefore$ Dc component of the convoluted signal = $\dfrac{\text{area of the signal}}{\text{period of the signal}}$

Convolution and correlation of signals

i. To calculate discrete linear convolution:

Convolute two sequences x[n] = {a,b,c} & h[n] = [e,f,g]

	a	b	c
e	ea	eb	ec
f	fa	fb	fc
g	ga	gb	gc

Convoluted output = [ea, eb+fa, ec+fb+ga, fc+gb, gc]

Note: if any two sequences have m, n number of samples respectively, then the resulting convoluted sequence will have [m+n-1] samples.

Example: Convolute two sequences x[n] = {1,2,3} & h[n] = {-1,2,2}

×	1	2	3
-1	-1	-2	-3
2	2	4	6
2	2	4	6

Convoluted output y[n] = [-1, -2+2, -3+4+2, 6+4, 6]

= [-1, 0, 3, 10, 6]

Here x[n] contains 3 samples and h[n] is also having 3 samples so the resulting sequence having 3+3-1 = 5 samples.

Convolution and correlation of signals

ii. To calculate periodic or circular convolution:

Periodic convolution is valid for discrete Fourier transform. To calculate periodic convolution all the samples must be real. Periodic or circular convolution is also called as fast convolution.

If two sequences of length m, n respectively are convoluted using circular convolution then resulting sequence having max [m,n] samples.

Ex: convolute two sequences x[n] = {1,2,3} & h[n] = {-1,2,2} using circular convolution

×	1	2	3
-1	-1	-2	-3
2	2	4	6
2	2	4	6

Normal Convoluted output y[n] = [-1, -2+2, -3+4+2, 6+4, 6].

= [-1, 0, 3, 10, 6]

Here x[n] contains 3 samples and h[n] also has 3 samples. Hence the resulting sequence obtained by circular convolution must have max[3,3]= 3 samples.

Now to get periodic convolution result, 1st 3 samples [as the period is 3] of normal convolution is same next two samples are added to 1st samples as shown below:

$$
\begin{array}{cccc}
 & -1 & 0 & 3 \\
+ & 10 & 6 & \\
\hline
 & 9 & 6 & 3
\end{array}
$$

$\therefore$ Circular convolution result $y[n] = [9 \quad 6 \quad 3]$

Convolution and correlation of signals

Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

There are two types of correlation:

- Auto correlation
- Cros correlation

Auto Correlation Function

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with R(τ).

Consider a signals x(t). The auto correlation function of x(t) with its time delayed version is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad [\text{-ve shift}]$$

Where τ = searching or scanning or delay parameter.

Convolution and correlation of signals

Properties of Auto-correlation Function of Energy Signal

- Auto correlation exhibits conjugate symmetry i.e. R (τ) = R*(- τ)
- Auto correlation function of energy signal at origin i.e. at τ =0 is equal to total energy of that signal, which is given as:

  R (0) = E = $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$

- Auto correlation function $\infty \frac{1}{\tau}$,
- Auto correlation function is maximum at τ =0 i.e |R (τ) | ≤ R (0) ∀ τ
- Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

  $$F.T\,[R(\tau)] = \Psi(\omega)$$

  $$\Psi(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau} d\tau$$

- $R(\tau) = x(\tau) * x(-\tau)$

Convolution and correlation of signals

Auto Correlation Function of Power Signals

The auto correlation function of periodic power signal with period T is given by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x*(t-\tau)dt$$

Properties

- Auto correlation of power signal exhibits conjugate symmetry i.e. $R(\tau) = R*(-\tau)$
- Auto correlation function of power signal at $\tau = 0$ (at origin)is equal to total power of that signal. i.e.

  $R(0) = \rho$
- Auto correlation function of power signal $\infty \frac{1}{\tau}$,
- Auto correlation function of power signal is maximum at τ = 0 i.e.,

  $|R(\tau)| \leq R(0) \,\forall\, \tau$
- Auto correlation function and power spectral densities are Fourier transform pairs. i.e.,

  $F.T[R(\tau)] = s(\omega)$

- $$R(\tau) = x(\tau) * x(-\tau)$$

Density Spectrum

Let us see density spectrums:

Energy Density Spectrum

Energy density spectrum can be calculated using the formula:

$$E = \int_{-\infty}^{\infty} |\, x(f) \,|^2 df$$

Power Density Spectrum

Power density spectrum can be calculated by using the formula:

$$P = \Sigma_{n=-\infty}^{\infty} \,|\, C_n \,|^2$$

Convolution and correlation of signals

Cross Correlation Function

Cross correlation is the measure of similarity between two different signals.

Consider two signals $x_1(t)$ and $x_2(t)$. The cross correlation of these two signals $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2(t)\,dt \qquad [\text{-ve shift}]$$

If signals are complex then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2^*(t)\,dt \qquad [\text{-ve shift}]$$

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)x_1^*(t-\tau)\,dt \qquad [\text{+ve shift}]$$

$$= \int_{-\infty}^{\infty} x_2(t+\tau)x_1^*(t)\,dt \qquad [\text{-ve shift}]$$

Convolution and correlation of signals

Properties of Cross Correlation Function of Energy and Power Signals

- Auto correlation exhibits conjugate symmetry i.e. $R_{12}(\tau) = R^*_{21}(-\tau)$.
- Cross correlation is not commutative like convolution i.e.

$$R_{12}(\tau) \neq R_{21}(-\tau)$$

- If $R_{12}(0) = 0$ means, if $\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = 0$, then the two signals are said to be orthogonal.

  For power signal if $\lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x^*(t)\,dt$ then two signals are said to be orthogonal.

- Cross correlation function corresponds to the multiplication of spectrums of one signal to the complex conjugate of spectrum of another signal. i.e.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega)X_2^*(\omega)$$

This also called as correlation theorem.

Parseval's Theorem

Parseval's theorem for energy signals states that the total energy in a signal can be obtained by the spectrum of the signal as

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Note: If a signal has energy E then time scaled version of that signal x(at) has energy E/a.

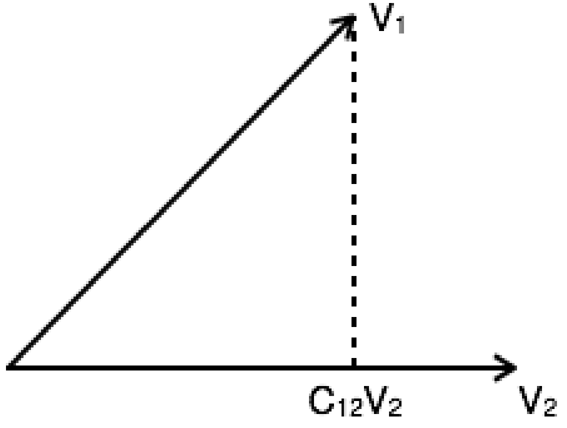
Analogy between vectors and signals

There is a perfect analogy between vectors and signals.

Vector

A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

Example: V is a vector with magnitude V. Consider two vectors V1 and V2 as shown in the following diagram. Let the component of V1 along with V2 is given by C12V2. The component of a vector V1 along with the vector V2 can obtained by taking a perpendicular from the end of V1 to the vector V2 as shown in diagram:



The vector V1 can be expressed in terms of vector V2

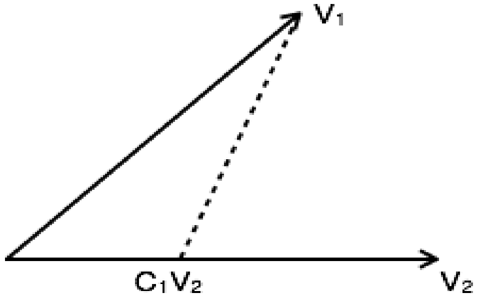
V1=C12V2 +Ve

- Where Ve is the error vector.

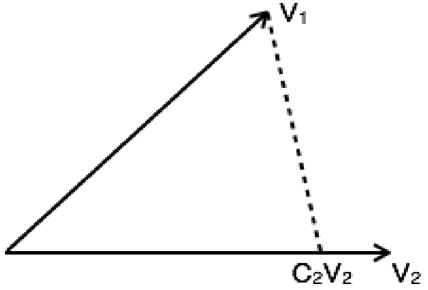
Analogy between vectors and signals

But this is not the only way of expressing vector V1 in terms of V2. The alternate possibilities are:

$V1=C1V2+Ve1$



$V2=C2V2+Ve2$



The error signal is minimum for large component value. If C12=0, then two signals are said to be orthogonal.

Dot Product of Two Vectors $V1 . V2 = V1.V2 \cos\theta$

θ = Angle between V1 and V2 $V1. V2 = V2.V1$

Analogy between vectors and signals

The error signal is minimum for large component value. If C12=0, then two signals are said to be orthogonal.

Dot Product of Two Vectors

V1 . V2 = V1.V2 cosθ

θ = Angle between V1 and V2 V1. V2 =V2.V1

From the diagram, components of V1 a long V2 = C 12 V2

$$\frac{V_1 . V_2}{V_2 = C_1 2\, V_2}$$

$$\Rightarrow C_{12} = \frac{V_1 . V_2}{V_2}$$

Signal

The concept of orthogonality can be applied to signals. Let us consider two signals f1(t) and f2(t).

Similar to vectors, you can approximate f1(t) in terms of f2(t) as

$f1(t) = C12\ f2(t) + fe(t)$ for $(t1 < t < t2)$

$\Rightarrow fe(t) = f1(t) - C12\ f2(t)$

One possible way of minimizing the error is integrating over the interval t1 to t2.

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f_e(t)]dt$$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f_1(t) - C_{12}f_2(t)]dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$$\varepsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f_e(t)]^2dt$$

$$\Rightarrow \frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f_e(t) - C_{12}f_2]^2dt$$

Analogy between vectors and signals

Where ε is the mean square value of error signal. The value of C12 which minimizes the error, you need to calculate *dε/dC*12=0

$$\Rightarrow \frac{d}{dC_{12}}\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f_1(t)-C_{12}f_2(t)]^2dt\right]=0$$

$$\Rightarrow \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[\frac{d}{dC_{12}}f_1^2(t)-\frac{d}{dC_{12}}2f_1(t)C_{12}f_2(t)+\frac{d}{dC_{12}}f_2^2(t)C_{12}^2\right]dt=0$$

Derivative of the terms which do not have C12 term are zero.

$$\Rightarrow \int_{t_1}^{t_2}-2f_1(t)f_2(t)dt+2C_{12}\int_{t_1}^{t_2}[f_2^2(t)]dt=0$$

If $C_{12}=\frac{\int_{t_1}^{t_2}f_1(t)f_2(t)dt}{\int_{t_1}^{t_2}f_2^2(t)dt}$ component is zero, then two signals are said to be orthogonal.

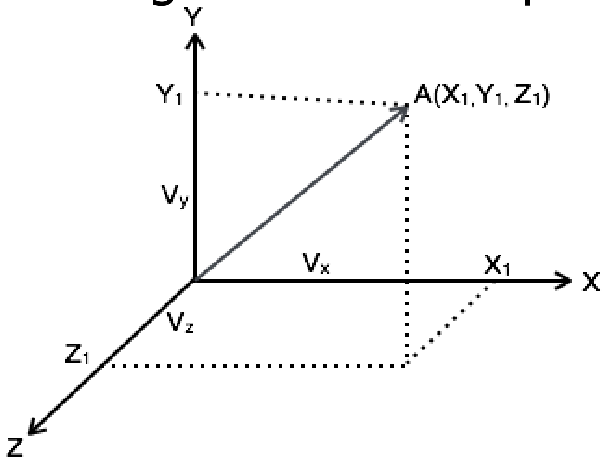
Put C 12 = 0 to get condition for orthogonality.

$$0=\frac{\int_{t_1}^{t_2}f_1(t)f_2(t)dt}{\int_{t_1}^{t_2}f_2^2(t)dt} \qquad \int_{t_1}^{t_2}f_1(t)f_2(t)dt=0$$

Analogy between vectors and signals

Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:



Consider a vector A at a point (X1, Y1, Z1). Consider three unit vectors (VX, VY, VZ) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X.V_X = V_Y.V_Y = V_Z.V_Z = 1$$
$$V_X.V_Y = V_Y.V_Z = V_Z.V_X = 0$$

$$V_a.V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

Analogy between vectors and signals

The vector A can be represented in terms of its components and unit vectors as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + N_1 V_N \ldots\ldots (2)$$

As the magnitude of unit vectors is unity for any vector A

The component of A along x axis = A.VX

The component of A along Y axis = A.VY

The component of A along Z axis = A.VZ

Similarly, for n dimensional space, the component of A along some G axis

$= A.VG$ (3)

Analogy between vectors and signals

Substitute equation 2 in equation 3.

$$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + G_1 V_G \ldots + N_1 V_N) V_G$$

$$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + \ldots + G_1 V_G V_G \ldots + N_1 V_N V_G$$

$$= G_1 \quad \text{since } V_G V_G = 1$$

$$If V_G V_G \neq 1 \text{ i.e.} V_G V_G = k$$

$$A V_G = G_1 V_G V_G = G_1 K$$

$$G_1 = \frac{(A V_G)}{K}$$

Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions x1(t), x2(t)... xn(t) over the interval t1 to t2. As these functions are orthogonal to each other, any two signals xj(t), xk(t) have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t)x_k(t)dt = 0 \text{ where } j \neq k$$

$$\text{Let } \int_{t_1}^{t_2} x_k^2(t)dt = k_k$$

Let a function f(t), it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t) + f_e(t)$$

$$= \Sigma_{r=1}^{n} C_r x_r(t)$$

$$f(t) = f(t) - \Sigma_{r=1}^{n} C_r x_r(t)$$

Analogy between vectors and signals

Mean sqaure error $\varepsilon = \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$$= \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f[t] - \sum_{r=1}^{n} C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = \ldots = \frac{d\varepsilon}{dC_k} = 0$$

Let us consider $\frac{d\varepsilon}{dC_k} = 0$

$$\frac{d}{dC_k}[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt] = 0$$

All terms that do not contain Ck is zero. i.e. in summation, r=k term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k \int_{t_1}^{t_2} [x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

Analogy between vectors and signals

Mean Square Error

The average of square of error function fe(t) is called as mean square error. It is denoted by ε (epsilon).

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_e(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1} [\int_{t_1}^{t_2} [f_e^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt - 2\Sigma_{r=1}^{n} C_r \int_{t_1}^{t_2} x_r(t)f(t)dt$$

You know that $C_r^2 \int_{t_1}^{t_2} x_r^2(t)dt = C_r \int_{t_1}^{t_2} x_r(t)f(d)dt = C_r^2 K_r$

$$\varepsilon = \frac{1}{t_2 - t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 K_r - 2\Sigma_{r=1}^{n} C_r^2 K_r]$$

$$= \frac{1}{t_2 - t_1} [\int_{t_1}^{t_2} [f^2(t)]dt - \Sigma_{r=1}^{n} C_r^2 K_r]$$

$$\therefore \varepsilon = \frac{1}{t_2 - t_1} [\int_{t_1}^{t_2} [f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + \ldots + C_n^2 K_n)]$$

Fourier Series

To represent any periodic signal x(t), Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthoganality condition.

Fourier Series Representation of Continuous Time Periodic Signals

A signal is said to be periodic if it satisfies the condition x (t) = x (t + T) or x (n) = x (n + N). Where T = fundamental time period,

ω0= fundamental frequency = 2π/T

There are two basic periodic signals: $x(t)=\cos\omega 0t$(sinusoidal) & $x(t)=ej\omega 0t$(complex exponential)

These two signals are periodic with period $T=2\pi/\omega 0$

A set of harmonically related complex exponentials can be represented as $\{\phi k(t)\}$

$$\phi_k(t) = \{e^{jk\omega_0 t}\} = \{e^{jk(\frac{2\pi}{T})t}\} \text{where} k = 0 \pm 1, \pm 2..n \; \ldots..(1)$$

All these signals are periodic with period T

Fourier Series

According to orthogonal signal space approximation of a function x (t) with n, mutually orthogonal functions is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(2)$$

Where *ak* = Fourier coefficient = coefficient of approximation.

This signal x(t) is also periodic with period T.

Equation 2 represents Fourier series representation of periodic signal x(t).

The term k = 0 is constant.

- The term $k=\pm1$ having fundamental frequency $\omega 0$, is called as 1st harmonics.
- The term $k=\pm2$ having fundamental frequency $2\omega 0$, is called as 2nd harmonics, and so on...
- The term $k=\pm n$ having fundamental frequency $n\omega 0$, is called as nth harmonics.

Fourier Series

Deriving Fourier Coefficient

We know that

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(1)$$

Multiply $e{-}jn\omega 0t$ on both sides. Then

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}$$

Consider integral on both sides.

$$\int_0^T x(t)e^{jk\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt$$

$$= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} . dt$$

$$\int_0^T x(t)e^{jk\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt. \ \ldots\ldots(2)$$

by Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t} dt. = \int_0^T \cos(k-n)\omega_0 dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt. = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Hence in equation 2, the integral is zero for all values of k except at k = n. Put k = n in equation 2.

$$\Rightarrow \int_0^T x(t) e^{-jn\omega_0 t} dt = a_n T$$

$$\Rightarrow a_n = \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\, dt$$

Replace n by k

$$\Rightarrow a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\, dt$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\text{where} a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\, dt$$

Fourier Series Properties

Properties of Fourier series:

Linearity Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{yn}$

then linearity property states that

$$\text{a}\, x(t) + \text{b}\, y(t) \xleftrightarrow{fourier\ series\ \ coefficient} \text{a}\, f_{xn} + \text{b}\, f_{yn}$$

Time Shifting Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then time shifting property states that

$$x(t - t_0) \xleftrightarrow{fourier\ series\ \ coefficient} e^{-jn\omega_0 t_0} f_{xn}$$

Fourier Series Properties

Frequency Shifting Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then frequency shifting property states that

$e^{jn\omega_0 t_0} . x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{x(n-n_0)}$

Time Reversal Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then time reversal property states that

If $x(-t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{-xn}$

Time Scaling Property

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then time scaling property states that

If $x(at) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

Time scaling property changes frequency components from ω_0 to $a\omega_0$.

Fourier Series Properties

Differentiation and Integration Properties

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

then differentiation property states that

If $\frac{dx(t)}{dt} \xleftrightarrow{fourier\ series\ \ coefficient} jn\omega_0 . f_{xn}$

& integration property states that

If $\int x(t)dt \xleftrightarrow{fourier\ series\ \ coefficient} \frac{f_{xn}}{jn\omega_0}$

Multiplication and Convolution Properties

f $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$ & $y(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{yn}$

:hen multiplication property states that

$x(t). y(t) \xleftrightarrow{fourier\ series\ \ coefficient} T f_{xn} * f_{yn}$

& convolution property states that

$x(t) * y(t) \xleftrightarrow{fourier\ series\ \ coefficient} T f_{xn} . f_{yn}$

Fourier Series Properties

Conjugate and Conjugate Symmetry Properties

If $x(t) \xleftrightarrow{fourier\ series\ \ coefficient} f_{xn}$

Then conjugate property states that

$x*(t) \xleftrightarrow{fourier\ series\ \ coefficient} f*_{xn}$

Conjugate symmetry property for real valued time signal states that

$$f*_{xn} = f_{-xn}$$

& Conjugate symmetry property for imaginary valued time signal states that

$$f*_{xn} = -f_{-xn}$$

Trigonometric Fourier Series

Trigonometric Fourier Series (TFS)

$\sin n\omega_0 t$ and $\sin m\omega_0 t$ are orthogonal over the interval $(t_0, t_0+\frac{2\pi}{\omega_0})$. So $\sin\omega_0 t, \sin 2\omega_0 t$ forms an orthogonal set. This set is not complete without $\{\cos n\omega_0 t\}$ because this cosine set is also orthogonal to sine set. So to complete this set we must include both cosine and sine terms. Now the complete orthogonal set contains all cosine and sine terms i.e. $\{\sin n\omega_0 t, \cos n\omega_0 t\}$ where n=0, 1, 2...

$\therefore$ Any function xt in the interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ can be represented as

$$
\begin{aligned}
x(t) &= a_0 \cos 0\omega_0 t + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\
&\quad + b_0 \sin 0\omega_0 t + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots \\
&= a_0 + a_1 \cos 1\omega_0 t + a_2 \cos 2\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots \\
&\quad + b_1 \sin 1\omega_0 t + \ldots + b_n \sin n\omega_0 t + \ldots
\end{aligned}
$$

$$\therefore x(t) = a_0 + \sum^{\infty} (a_n \cos n\omega_0 t + b_n \sin \ldots$$

The above equation represents trigonometric Fourier series representation of x(t).

Trigonometric Fourier Series

$$\text{Where } a_0 = \frac{\int_{t_0}^{t_0+T} x(t)\cdot 1dt}{\int_{t_0}^{t_0+T} 1^2 dt} = \frac{1}{T}\cdot\int_{t_0}^{t_0+T} x(t)dt$$

$$a_n = \frac{\int_{t_0}^{t_0+T} x(t)\cdot \cos n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt}$$

$$b_n = \frac{\int_{t_0}^{t_0+T} x(t)\cdot \sin n\omega_0 t\, dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\, dt}$$

$$\text{Here } \int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\, dt = \int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\, dt = \frac{T}{2}$$

$$\therefore a_n = \frac{2}{T}\cdot\int_{t_0}^{t_0+T} x(t)\cdot \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\cdot\int_{t_0}^{t_0+T} x(t)\cdot \sin n\omega_0 t\, dt$$

Exponential Fourier Series

Exponential Fourier Series (EFS):

Consider a set of complex exponential functions $\left\{e^{jn\omega_0 t}\right\} (n = 0, \pm 1, \pm 2...)$ which is orthogonal over the interval ($t0$,$t0$+T). Where T=2π/ω0 . This is a complete set so it is possible to represent any function f(t) as shown below

$$f(t) = F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$\boxed{\therefore f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad (t_0 < t < t_0 + T) \ldots\ldots (1)}$$

Equation 1 represents exponential Fourier series representation of a signal f(t) over the interval (t0, t0+T).

Exponential Fourier Series

$$F_n = \frac{\int_{t_0}^{t_0+T} f(t)(e^{jn\omega_0 t})^* dt}{\int_{t_0}^{t_0+T} e^{jn\omega_0 t}(e^{jn\omega_0 t})^* dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} e^{-jn\omega_0 t} e^{jn\omega_0 t} dt}$$

$$= \frac{\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t} dt}{\int_{t_0}^{t_0+T} 1\, dt} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t} dt$$

$$\boxed{\therefore F_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t} dt}$$

Relation Between Trigonometric and Exponential Fourier Series:

Consider a periodic signal x(t), the TFS & EFS representations are given below respectively

$$x(t) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\ldots\ldots(1)$$

$$x(t) = \Sigma_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$= F_0 + F_1 e^{j\omega_0 t} + F_2 e^{j2\omega_0 t} + \ldots + F_n e^{jn\omega_0 t} + \ldots$$

$$F_{-1} e^{-j\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

$$= F_0 + F_1(\cos\omega_0 t + j\sin\omega_0 t) + F_2(cos2\omega_0 t + j\sin 2\omega_0 t) + \ldots + F_n(\cos n\omega_0 t + j\sin n\omega_0 t) + .$$

$$. + F_{-1}(\cos\omega_0 t - j\sin\omega_0 t) + F_{-2}(\cos 2\omega_0 t - j\sin 2\omega_0 t) + \ldots + F_{-n}(\cos n\omega_0 t - j\sin n\omega_0 t) + \ldots$$

$$= F_0 + (F_1 + F_{-1})\cos\omega_0 t + (F_2 + F_{-2})\cos 2\omega_0 t + \ldots + j(F_1 - F_{-1})\sin\omega_0 t + j(F_2 - F_{-2})\sin 2\omega_0 t + \ldots$$

$$\therefore x(t) = F_0 + \Sigma_{n=1}^{\infty}((F_n + F_{-n})\cos n\omega_0 t + j(F_n - F_{-n})\sin n\omega_0 t)\ldots\ldots(2)$$

$$a_0 = F_0$$

$$a_n = F_n + F_{-n}$$

$$b_n = j(F_n - F_{-n})$$

$$F_n = \frac{1}{2}(a_n - jb_n)$$

$$F_{-n} = \frac{1}{2}(a_n + jb_n)$$

Continuous Time Fourier Transform

INTRODUCTION:

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

Deriving Fourier transform from Fourier series:

Consider a periodic signal f(t) with period T. The complex Fourier series representation of f(t) is given as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots(1)$$

Continuous Time Fourier Transform

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \Delta f t} \ldots\ldots (2)$$

but you know that

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t}\, dt$$

Substitute in equation 2.

$$2 \Rightarrow f(t) = \Sigma_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t}\, dt\, e^{j2\pi k \Delta f t}$$

Let $t_0 = \frac{T}{2}$

$$= \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k \Delta f t}\, dt]\, e^{j2\pi k \Delta f t} . \Delta f$$

Continuous Time Fourier Transform

In the limit as T→∞,Δf approaches differential df, kΔf becomes a continuous variable f, and summation becomes integration

$$f(t) = lim_{T\to\infty}\left\{\Sigma_{k=-\infty}^{\infty}\left[\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft}\,dt\right]e^{j2\pi k\Delta ft}\,.\,\Delta f\right\}$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\,dt\right]e^{j2\pi ft}\,df$$

$$\textbf{Where } F[\omega] = \left[\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\,dt\right]$$

Fourier transform of a signal

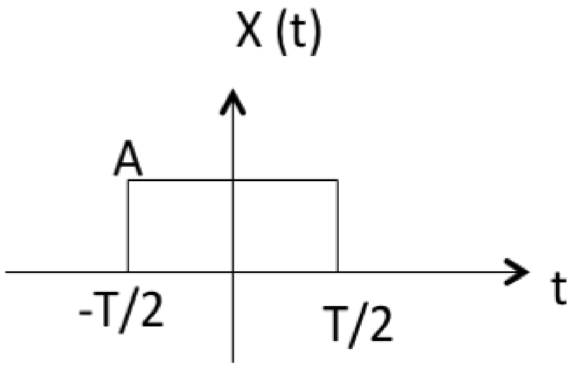
$$f(t) = F[\omega] = \left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right]$$

Inverse Fourier Transform is

$$f(t) = \int_{-\infty}^{\infty} F[\omega]e^{j\omega t}\,d\omega$$

Fourier Transform of Basic functions

FT of GATE Function



$$F[\omega] = AT\,Sa(\frac{\omega T}{2})$$

FT of Impulse Function:

$$FT[\omega(t)] = [\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt]$$

$$= e^{-j\omega t} \mid t = 0$$

$$= e^{0} = 1$$

$$\therefore \delta(\omega) = 1$$

Fourier Transform of Basic functions

FT of Unit Step Function:

$$U(\omega) = \pi\delta(\omega) + 1/j\omega$$

FT of Exponentials:

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-a|t|} \xleftrightarrow{\text{F.T}} \frac{2a}{a^2+\omega^2}$$

$$e^{j\omega_0 t} \xleftrightarrow{\text{F.T}} \delta(\omega - \omega_0)$$

FT of Signum Function :

$$sgn(t) \xleftrightarrow{\text{F.T}} \frac{2}{j\omega}$$

Continuous Time Fourier Transform

Conditions for Existence of Fourier Transform:

Any function f(t) can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t),in the given interval of time.
- It must be absolutely integrable in the given interval of time i.e.

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

Fourier Transform Properties

Linearity Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

$$\& \ y(t) \overset{\text{F.T}}{\longleftrightarrow} Y(\omega)$$

Then linearity property states that

$$ax(t) + by(t) \overset{\text{F.T}}{\longleftrightarrow} aX(\omega) + bY(\omega)$$

Time Shifting Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time shifting property states that

$$x(t - t_0) \overset{\text{F.T}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

Fourier Transform Properties

Frequency Shifting Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then frequency shifting property states that

$$e^{j\omega_0 t}.\, x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega - \omega_0)$$

Time Reversal Property:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Time reversal property states that

$$x(-t) \overset{\text{F.T}}{\longleftrightarrow} X(-\omega)$$

Fourier Transform Properties

Time Scaling Property: If $x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$

Then Time scaling property states that $x(at)\frac{1}{|a|}X\frac{\omega}{a}$

Differentiation and Integration Properties:

$$\text{If } x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega)$$

Then Differentiation property states that

$$\frac{dx(t)}{dt} \overset{\text{F.T}}{\longleftrightarrow} j\omega.X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \overset{\text{F.T}}{\longleftrightarrow} (j\omega)^n.X(\omega)$$

and integration property states that

$$\int x(t)\,dt \overset{\text{F.T}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega)$$

$$\iiint \ldots \int x(t)\,dt \overset{\text{F.T}}{\longleftrightarrow} \frac{1}{(j\omega)^n}X(\omega)$$

Fourier Transform Properties

Multiplication and Convolution Properties:

$$\text{If } x(t) \xleftrightarrow{\text{F.T}} X(\omega)$$

$$\& \; y(t) \xleftrightarrow{\text{F.T}} Y(\omega)$$

Then multiplication property states that

$$x(t).\,y(t) \xleftrightarrow{\text{F.T}} X(\omega) * Y(\omega)$$

and convolution property states that

$$x(t) * y(t) \xleftrightarrow{\text{F.T}} \frac{1}{2\pi} X(\omega).\,Y(\omega)$$

Sampling theorem of low pass signals

Statement of Sampling Theorem:

A band limited signal can be reconstructed exactly if it is sampled at a rate atleast twice the maximum frequency component in it."

The following figure shows a signal

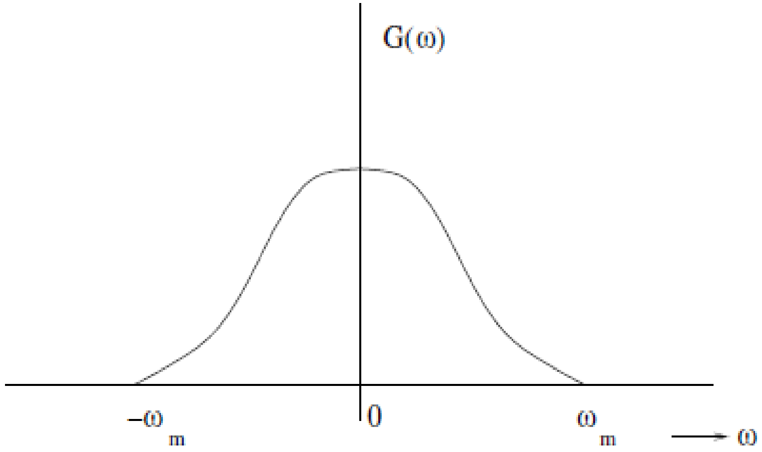


Figure1: Spectrum of band limited signal g(t)

The maximum frequency component of g(t) is fm. To recover the signal g(t) exactly from its samples it has to be sampled at a rate fs ≥2fm.

The minimum required sampling rate fs = 2fm is called "Nyquist rate".

Sampling theorem of low pass signals

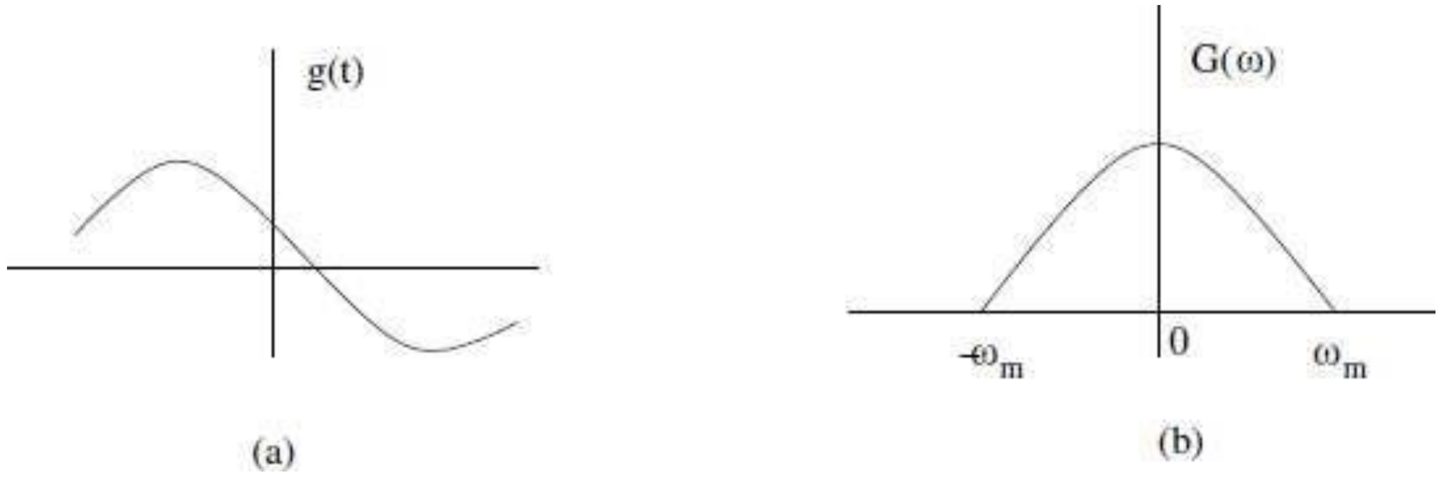
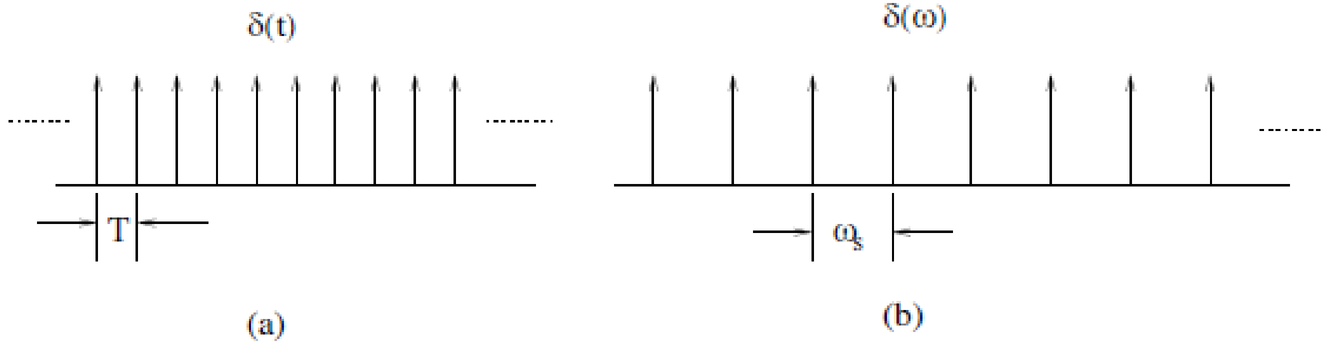


Figure 2: (a) Original signal g(t) (b) Spectrum G(ω)

$\delta_T(t)$ is the sampling signal with $f_s = 1/T > 2f_m$.



(a) sampling signal $\delta_T(t)$ (b) Spectrum $\delta_T(\omega)$

Sampling theorem of low pass signals

Let g s(t) be the sampled signal. Its Fourier Transform Gs(ω) is given by

$$
\begin{aligned}
\mathcal{F}(g_s(t)) &= \mathcal{F}\left[g(t)\delta_T(t)\right] \\
&= \mathcal{F}\left[g(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right] \\
&= \frac{1}{2\pi}\left[G(\omega) * \omega_0 \sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega) * \delta(\omega-n\omega_0) \\
G_s(\omega) &= \mathcal{F}\left[g(t) + 2g(t)\cos(\omega_0 t) + 2g(t)\cos(2\omega_0 t) + \cdots\right] \\
G_s(\omega) &= \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega-n\omega_0)
\end{aligned}
$$

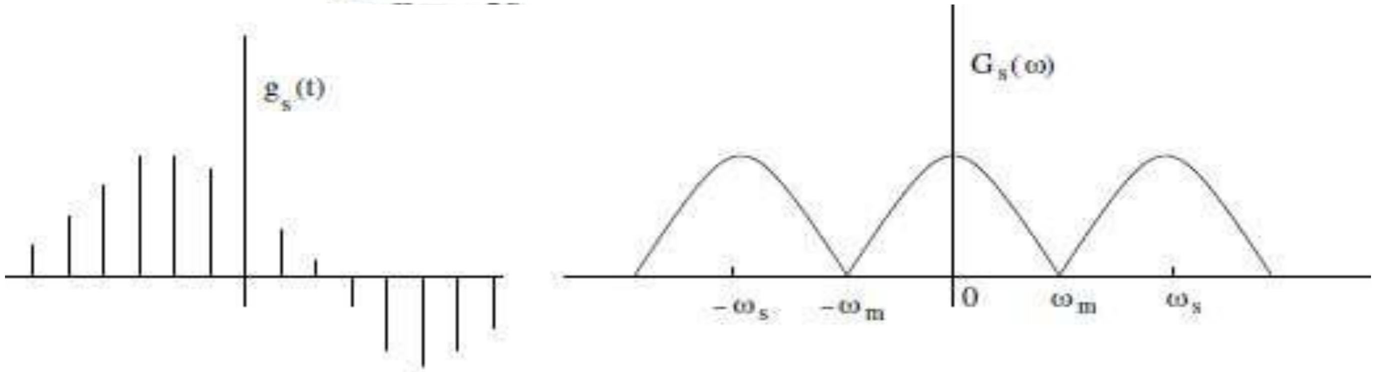


Figure 4: (a) sampled signal $g_s(t)$ (b) Spectrum $G_s(\omega)$

Sampling theorem of low pass signals

If $\omega_s = 2\omega_m$, i.e., $T = 1/2f_m$. Therefore, $G_s(\omega)$ is given by

$$G_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(\omega - n\omega_m)$$

To recover the original signal $G(\omega)$:

1. Filter with a Gate function, $H_{2\omega_m}(\omega)$ of width $2\omega_m$.

2. Scale it by T.

$$G(\omega) = TG_s(\omega)H_{2\omega_m}(\omega).$$

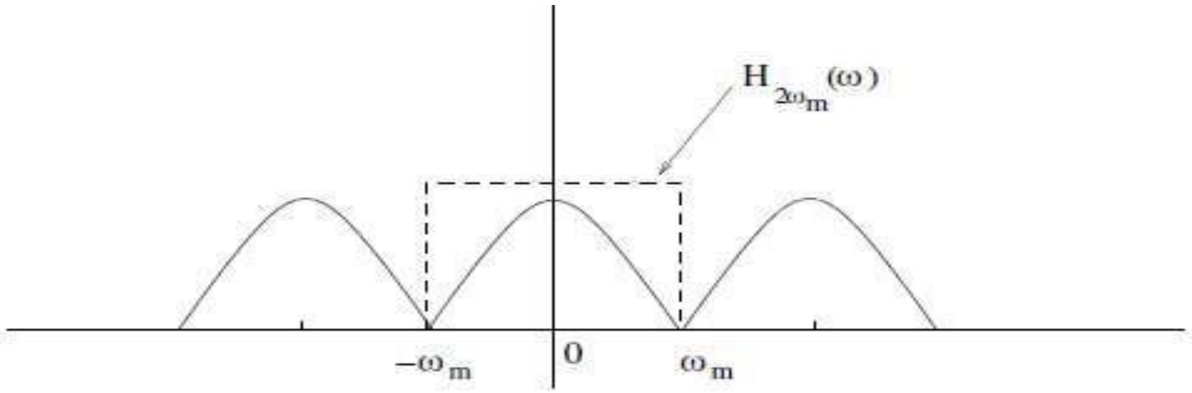


Figure 5: Recovery of signal by filtering with a filter of width $2\omega_m$

Sampling theorem of low pass signals

Aliasing:

Aliasing is a phenomenon where the high frequency components of the sampled signal interfere with each other because of inadequate sampling $\omega s < \omega m$

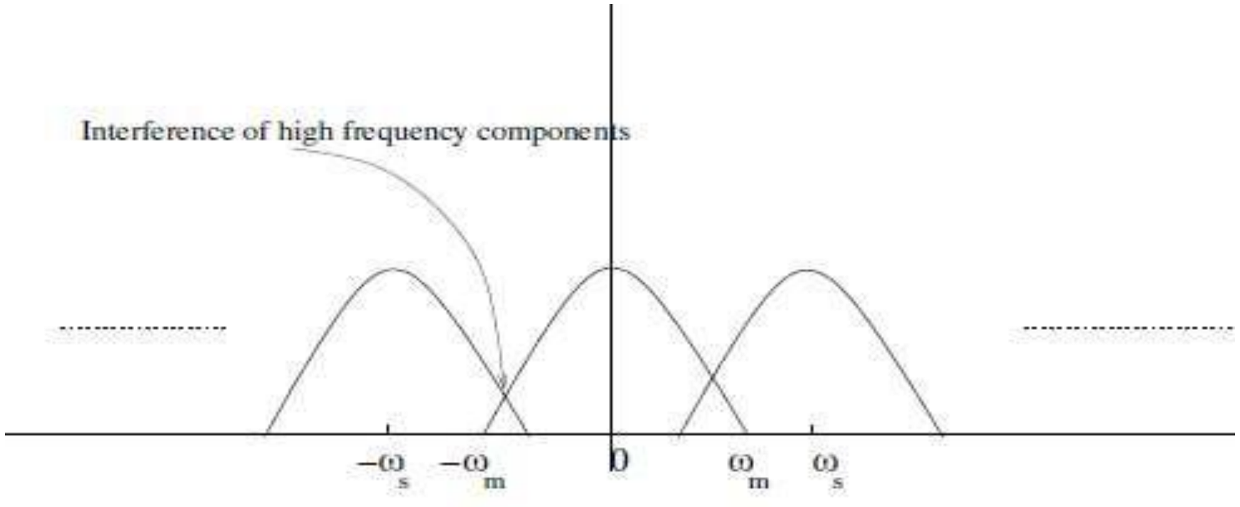


Figure 6: Aliasing due to inadequate sampling

Aliasing leads to distortion in recovered signal. This is the reason why sampling frequency should be atleast twice the bandwidth of the signal.

Sampling theorem of low pass signals

Oversampling:

In practice signal are oversampled, where fs is significantly higher than Nyquist rate to avoid aliasing.

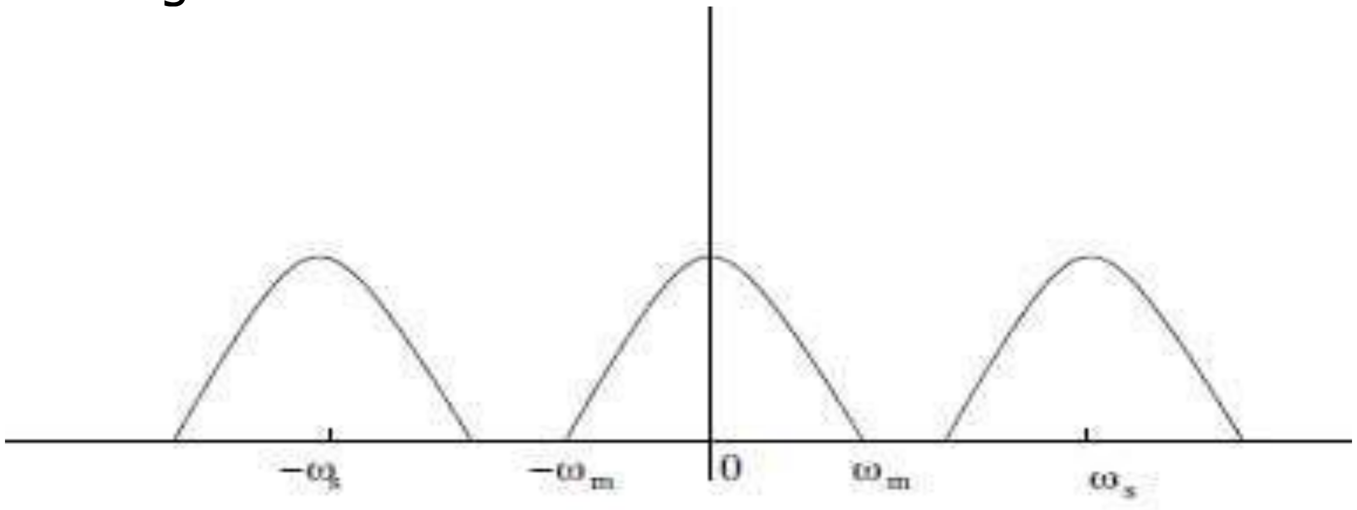


Figure 7: Oversampled signal-avoids aliasing

Discrete Time Fourier Transforms (DTFT)

Here we take the exponential signals to be $\{e^{j\omega n}\}$ where 'w'is a real number. The representation is motivated by the Harmonic analysis, but instead of following the historical development of the representation we give directly the defining equation.

Let {x[n]} be discrete time signal such that $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, that is sequence is absolutely summable.

The sequence {x[n]} can be represented by a Fourier integral of the form,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Where,

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Discrete Time Fourier Transform

Equation (1) and (2) give the Fourier representation of the signal.

Equation (1) is referred as synthesis equation or the inverse discrete time Fourier transform (IDTFT) and equation (2)is Fourier transform in the analysis equation.

Fourier transform of a signal in general is a complex valued function, we can write,

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where $|X(e^{j\omega})|$ is magnitude and $\angle X(e^{j\omega})$ is the phase.

We also use the term Fourier spectrum or simply, the spectrum to refer to. Thus $|X(e^{j\omega})|$ is called the magnitude spectrum and $\angle X(e^{j\omega})$ is called the phase spectrum.

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{+j\omega n}d\omega$$

Interchanging the order of integration,

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+j\omega(n-m)}d\omega\right)$$

Discrete Time Fourier Transform

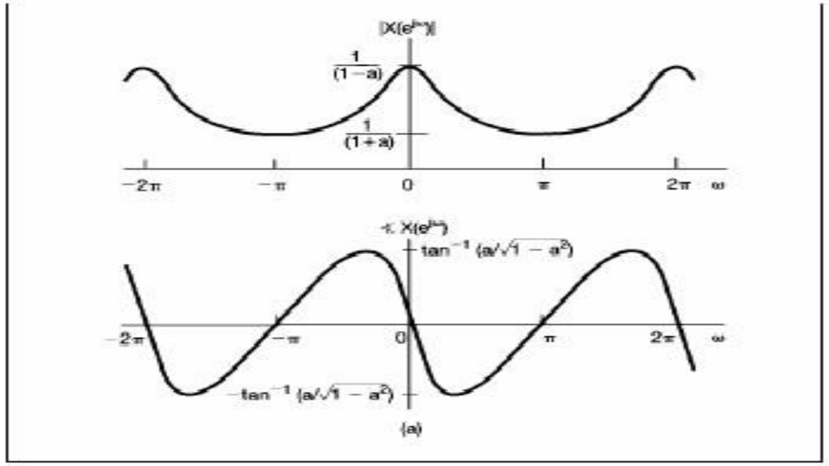
Example: Let $\{x[n]\} = \{a^n u[n]\}$

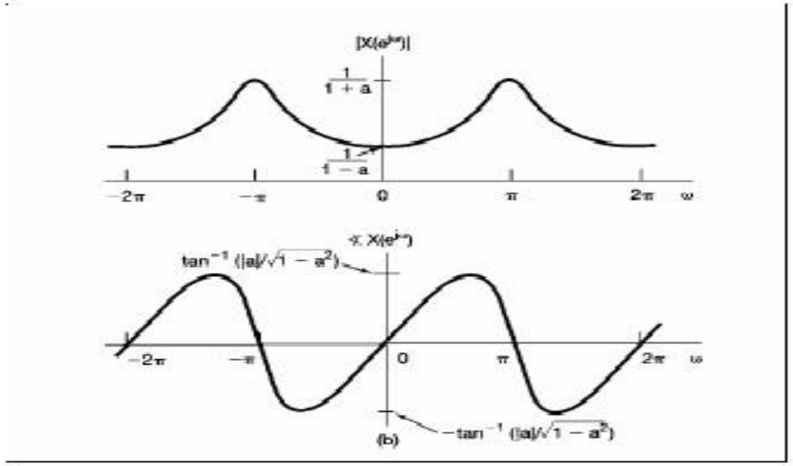
Fourier transform of this sequence will exist if it is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

The magnitude and phase for this example are show in the figure below, where $a > 0$ and $a < 0$ are shown in (a) and (b).





Discrete Time Fourier Transform

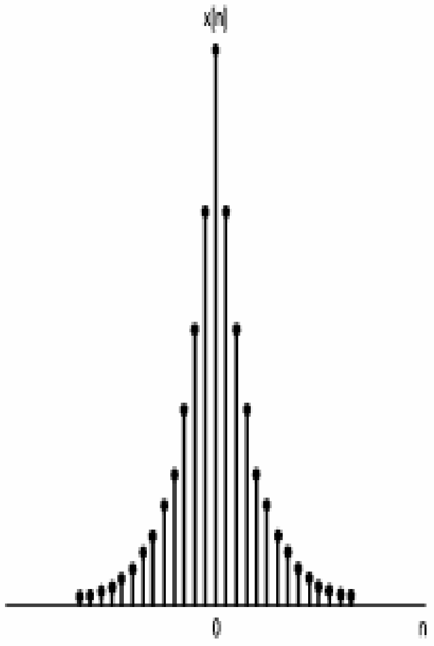
Example: $x[n] = a^{|n|}, \ |a| < 1$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

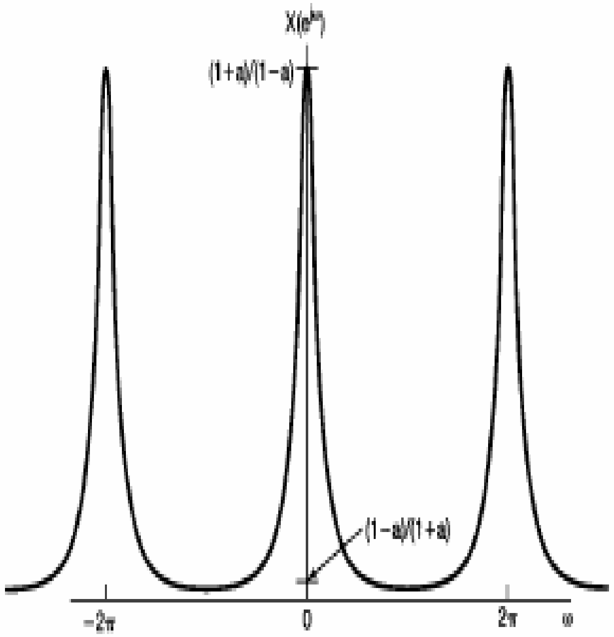
Let $m = -n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{m=1}^{\infty} a^{m} e^{j\omega m} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$



(a)



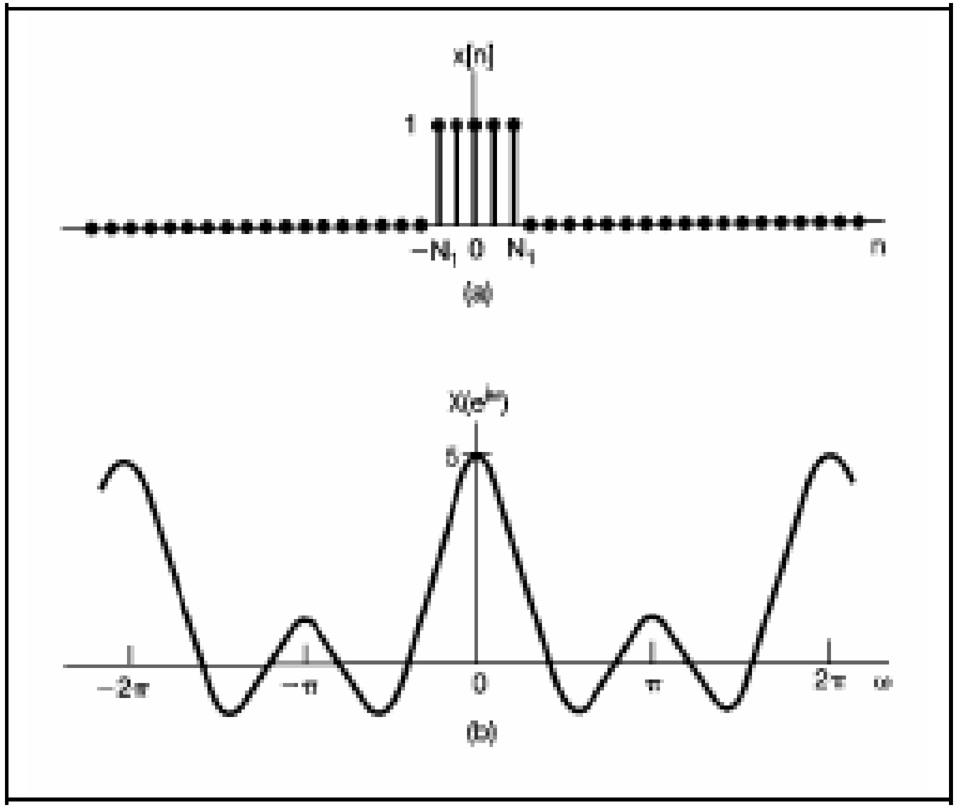
(b)

Discrete Time Fourier Transform

Example: Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & |n| > 2 \end{cases}, \tag{5.14}$$

$$X(j\omega) = \sum_{n=-2}^{2} e^{-j\omega n} = \frac{\sin\omega(N_1 + 1/2)}{\sin(\omega/2)}. \tag{5.15}$$



This function is the discrete counterpart of the sic function, which appears in the Fourier transform of the continuous-time pulse.

The difference between these two functions is that the discrete one is periodic (see figure) with period of 2π, whereas the sinc function is aperiodic.

Fourier transform of Periodic Signals

For a periodic discrete-time signal,

$$x[n] = e^{j\omega_0 n},$$

its Fourier transform of this signal is periodic in w with period 2∏ , and is given

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

The Fourier transform is,

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N}).$$

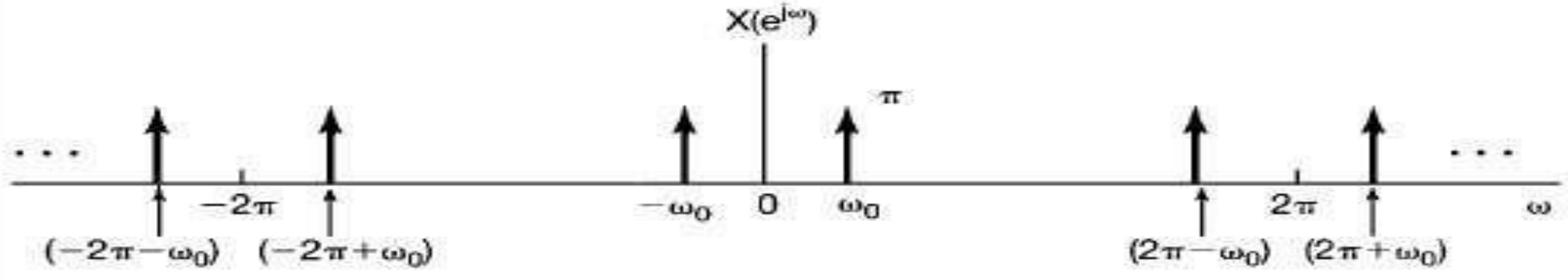
Discrete Time Fourier Transform

Example: The Fourier transform of the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \text{ with } \omega_0 = \frac{2\pi}{3}.$$

is given as

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{3}\right) + \pi\delta\left(\omega + \frac{2\pi}{3}\right), \qquad -\pi \le \omega < \pi .$$



Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

Discrete Time Fourier Transform

Example : The periodic impulse train

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN].$$

The Fourier series coefficients for this signal can be calculated

$$a_k = \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

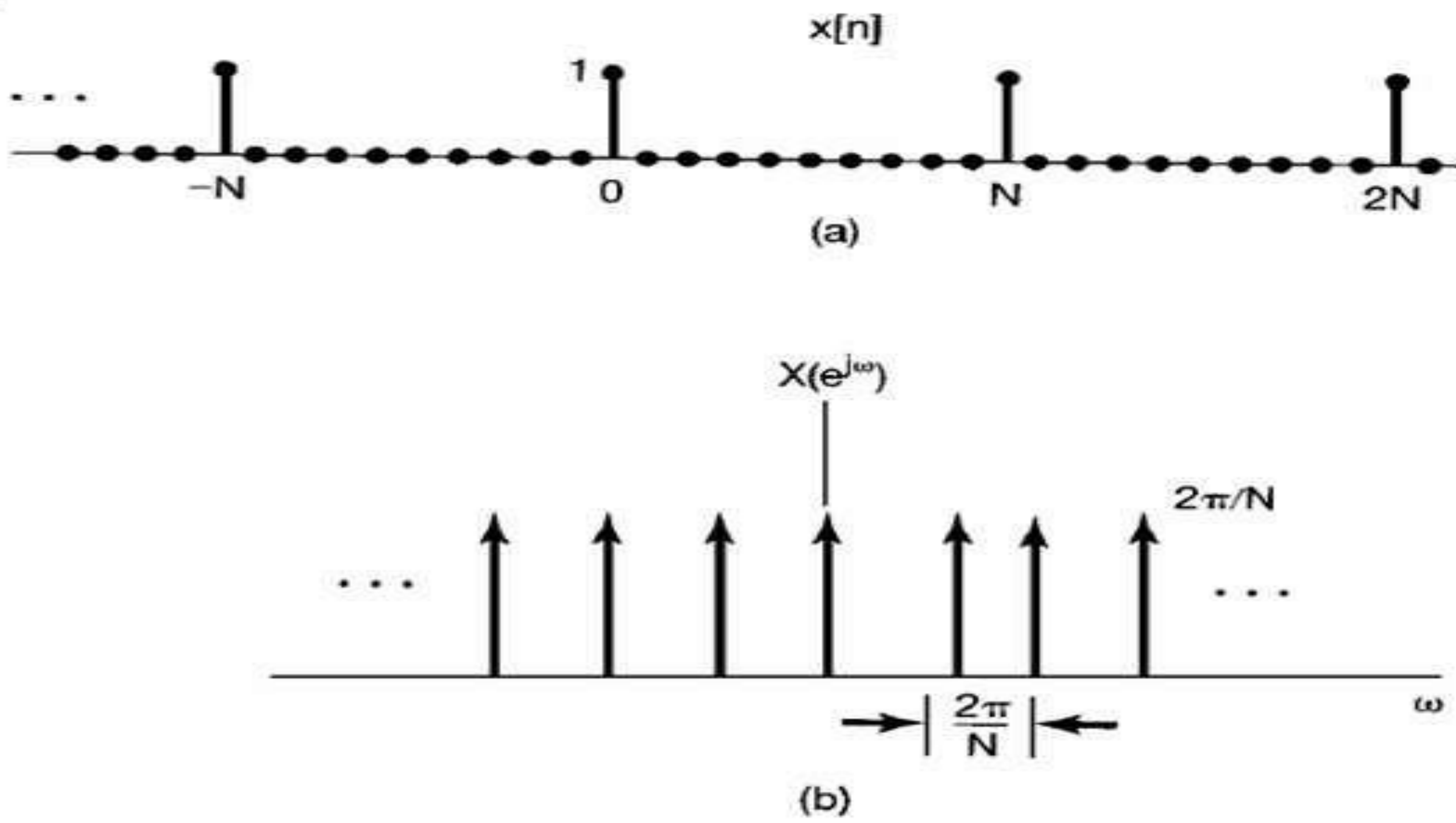
Choosing the interval of summation as $0 \le n \le N-1$, we have

$$a_k = \frac{1}{N}.$$

The Fourier transform is

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right).$$

Discrete Time Fourier Transform



(a) Discrete-time periodic impulse train; (b) its Fourier transform.

Properties of the Discrete Time Fourier Transform:

Let {x[n]}and {y[n]} be two signal, then their DTFT is denoted by $X(e^{j\omega})$and. The notation

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

is used to say that left hand side is the signal x[n] whose DTFT $X(e^{j\omega})$ is given at right hand side.

1.Periodicity of the DTFT:

The discrete-time Fourier transform is always periodic in ω with period 2π, i.e.,

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right).$$

Discrete Time Fourier Transform

2. Linearity of the DTFT:

If $x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$, and $x_2[n] \xleftrightarrow{F} X_2(e^{j\omega})$,

then

$$\boxed{ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega}) + bX_2(e^{j\omega})}$$

3.Time Shifting and Frequency Shifting:

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})}$$

and

$$\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})}$$

4.Conjugation and Conjugate Symmetry:

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega})}$$

If $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega})}$$

From this, it follows that Re{$X(e^{jw})$} is an even function of w and Im{$X(e^{jw})$} is an odd function of w. Similarly, the **magnitude** of $X(e^{jw})$ is an even function and the phase angle is an odd function. Furthermore,

$$Ev\{x[n]\} \xleftrightarrow{F} \mathrm{Re}\{X(e^{j\omega})\}.$$

and

$$Od\{x[n]\} \xleftrightarrow{F} j\,\mathrm{Im}\{X(e^{j\omega})\}.$$

Discrete Time Fourier Transform

5.Differencing and Accumulation

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[n] - x[n-1] \xleftrightarrow{F} \left(1 - e^{-j\omega}\right) X(e^{j\omega})}$$

For signal

$$y[n] = \sum_{m=-\infty}^{n} x[m],$$

its Fourier transform is given as

$$\boxed{\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi k)}$$

The impulse train on the right-hand side reflects the dc or average value that can result from summation.

Discrete Time Fourier Transform

6.Time Reversal

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$,

then

$$\boxed{x[-n] \xleftrightarrow{F} X(-e^{j\omega})}.$$

7.Time Expansion

For continuous-time signal, we have

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

For discrete-time signals, however, *a* should be an integer. Let us define a signal with *k* a positive integer,

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}.$$

Discrete Time Fourier Transform

$x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal.

The Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk} = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$\boxed{x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})}.$$

For $k > 1$, the signal is spread out and slowed down in time, while its Fourier transform is compressed.

8.Differentiation in Frequency

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$.

Differentiate both sides of the analysis equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of the above equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$\boxed{nx[n] \xleftrightarrow{F} j\frac{dX(e^{j\omega})}{d\omega}}.$$

9.Parseval's Relation

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$, then we have

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}$$

Linear Systems:

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x1(t)$, $x2(t)$, and outputs as $y1(t)$, $y2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T[a_1 x1(t) + a_2 x2(t)] = a_1 T[x1(t)] + a_2 T[x2(t)]$$

$$\therefore T[a_1 x1(t) + a_2 x2(t)] = a_1 y1(t) + a_2 y2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example: $y(t) = 2x(t)$

Solution:

$$y_1(t) = T[x1(t)] = 2x1(t)$$

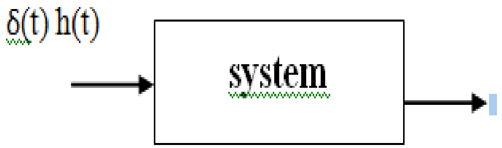
$$y_2(t) = T[x2(t)] = 2x2(t)$$

$$T[a_1 x1(t) + a_2 x2(t)] = 2[a_1 x1(t) + a_2 x2(t)]$$

Which is equal to $a1y1(t) + a2\ y2(t)$. Hence the system is said to be linear

Impulse Response:

The impulse response of a system is its response to the input δ(t) when the system is initially at rest. The impulse response is usually denoted h(t). In other words, if the input to an initially at rest system is δ(t) then the output is named h(t).



Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

- If a system is both liner and time variant, then it is called liner time variant (LTV) system.

- If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Response of a continuous-time LTI system and the convolution integral

Impulse Response:

The ***impulse response*** h(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is δ(t), that is,

$$h(t) = T\{\delta(t)\} \quad (1)$$

Response to an Arbitrary Input:

- The input ***x(*** t) can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \quad \_\_\_\_(2)$$

Since the system is linear, the response ***y(*** t of the system to an arbitrary input x(t) can be expressed as

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \quad \text{------}(3)$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathrm{T}\{\delta(t-\tau)\} \quad \text{———(4)}$$

Substituting **Eq.** (4) into **Eq.** (3), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{———(5)}$$

Equation *(5)* indicates that a continuous-time LTI system is completely characterized by its impulse response *h(t)*.

Convolution Integral:

Equation (5) defines the convolution of two continuous-time signals x (t) and h(t) denoted

By

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{-----(6)}$$

Equation (6) is commonly called the convolution integral.

Thus, we have the fundamental result that the output of any continuous-time LTI system is the convolution of the input x (t) with the impulse response h(t) of the system.

The following figure illustrates the definition of the impulse response h(t) and the relationship of Eq. (6).

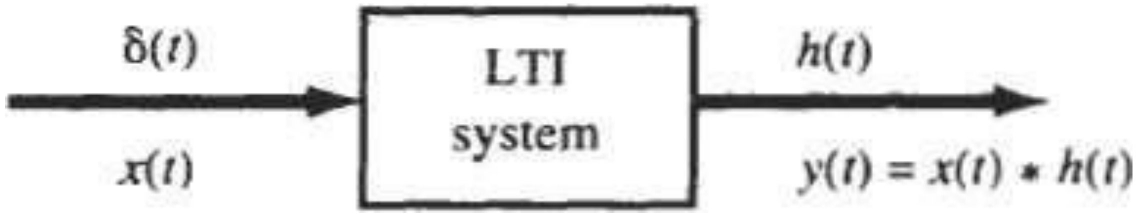


Fig. : Continuous-time LTI system.

Properties of the Convolution Integral:

The convolution integral has the following properties.

1. *Commutative:*

$$x(t) * h(t) = h(t) * x(t)$$

2. *Associative:*

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. *Distributive:*

$$x(t) * \{h_1(t)\} + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

Step Response:

The step response s(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is u(t); that is,

S(t)= T{u(t)}

In many applications, the step response s(t) is also a useful characterization of the system.

The step response s(t) can be easily determined by,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus, the step response s(t) can be obtained by integrating the impulse response h(t). Differentiating the above equation with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

Distortion less transmission through a system:

Transmission is said to be distortion-less if the input and output have identical wave shapes. i.e., in distortion-less transmission, the input x(t) and output y(t) satisfy the condition:

$$y(t) = Kx(t - td)$$

Where td = delay time and

k = constant.

Take Fourier transform on both sides

$$FT[y(t)] = FT[Kx(t - td)]$$

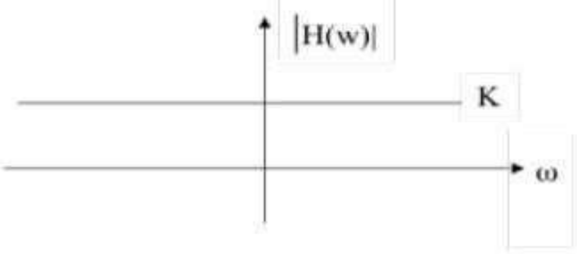
$$= K\,FT[x(t - td)]$$

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

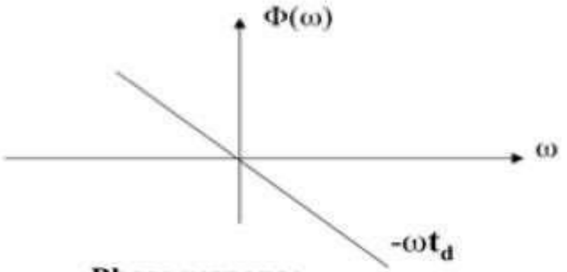
According to time shifting property, $Y(w) = KX(w)e^{-j\omega t_d}$

Thus, distortion less transmission of a signal x(t) through a system with impulse response h(t) is achieved when

|H(ω)|=K and (amplitude response) $\Phi(\omega) = -\omega t_d = -2\pi f t_d \quad phaseresponse$

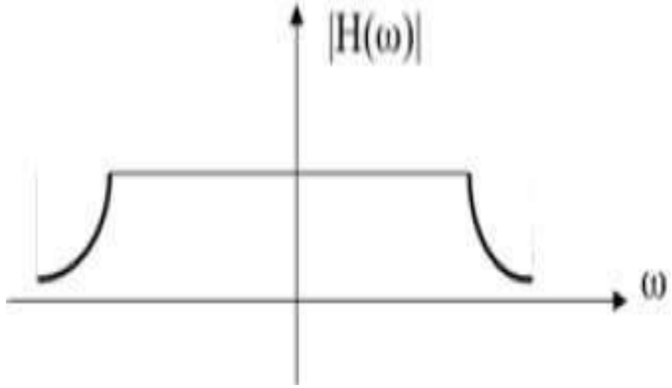


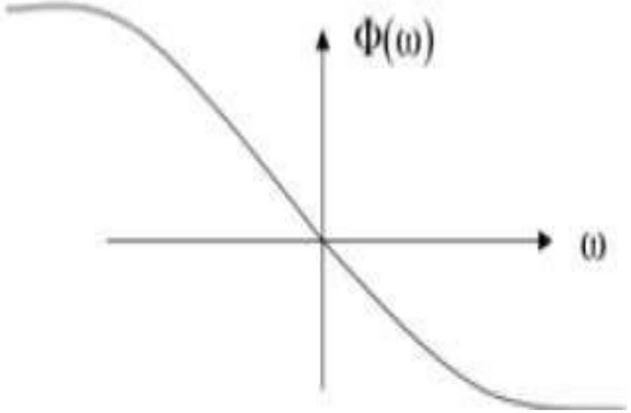
Amplitude response



Phase response

A physical transmission system may have amplitude and phase responses as shown below:





Bandwidth

- Theorems of communication and information theory are based on the assumption of *strictly band limited* channels
- The mathematical description of a real signal does not permit the signal to be strictly duration limited and strictly band limited.

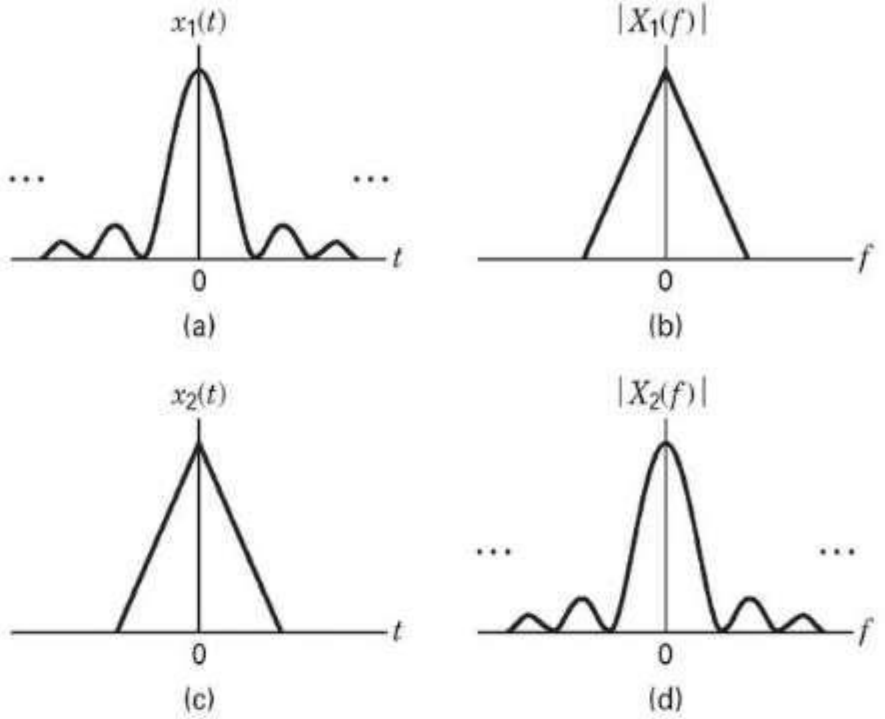


Figure 1.19 (a) Strictly bandlimited signal in the time domain. (b) In the frequency domain. (c) Strictly time limited signal in the time domain. (d) In the frequency domain.

Different Bandwidth Criteria

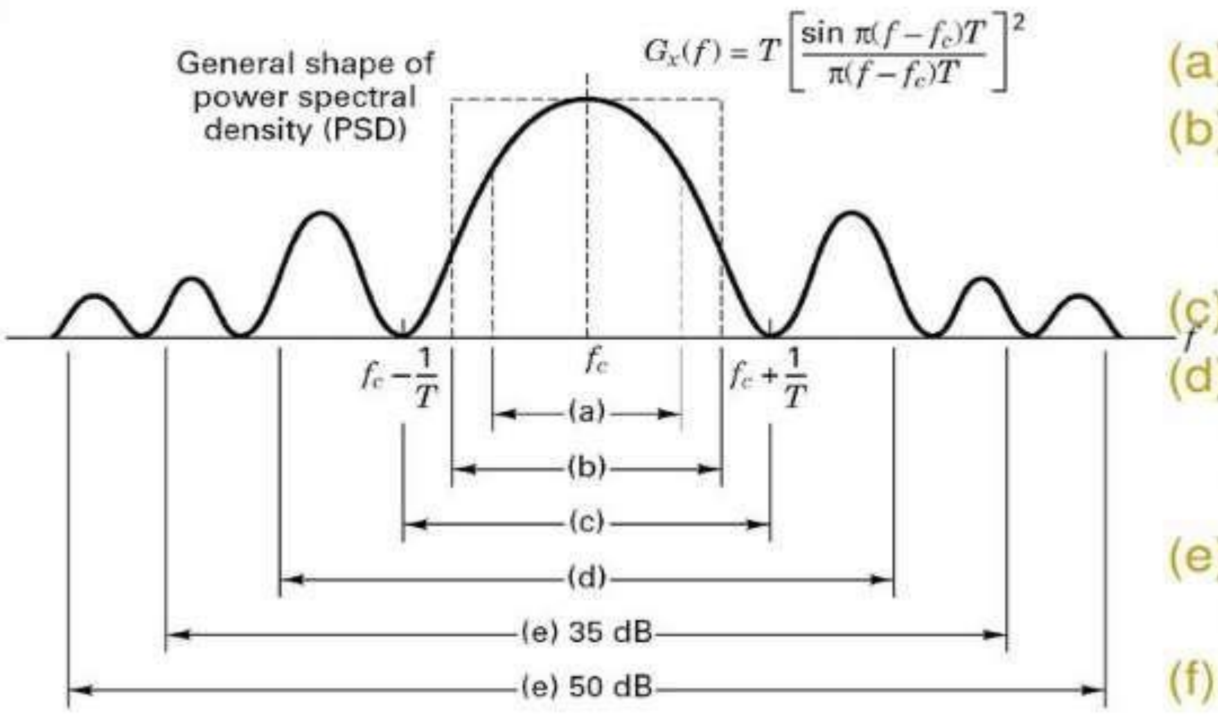


Figure 1.20 Bandwidth of digital data. (a) Half-power. (b) Noise equivalent. (c) Null to null. (d) 99% of power. (e) Bounded PSD (defines attenuation outside bandwidth) at 35 and 50 dB.

(a) Half-power bandwidth.
(b) Equivalent rectangular or noise equivalent bandwidth.
(c) Null-to-null bandwidth.
(d) Fractional power containment bandwidth.
(e) Bounded power spectral density.
(f) Absolute bandwidth.

FILTERING

One of the most basic operations in any signal processing system is filtering.

Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or perhaps some frequency components are suppressed.

For continuous-time LTI systems, the spectrum of the output is that of the input multiplied by the frequency response of the system.

Therefore, an LTI system acts as a filter on the input signal. Here the word "filter" is used to denote a system that exhibits some sort of frequency-selective behavior.

Ideal Frequency-Selective Filters:

An ideal frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest.

The band of frequencies passed by the filter is referred to as the pass band, and the band of frequencies rejected by the filter is called the stop band.

The most common types of ideal frequency-selective filters are the following.

Ideal Low-Pass Filter:

An ideal low-pass filter (LPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The frequency ***wc*** is called the cutoff frequency.

Ideal High-Pass Filter:

An ideal high-pass filter (HPF) is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

Ideal Bandpass Filter:

An ideal bandpass filter (BPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Ideal Bandstop Filter:

An ideal bandstop filter (BSF) is specified by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

The following figures shows the magnitude responses of ideal filters

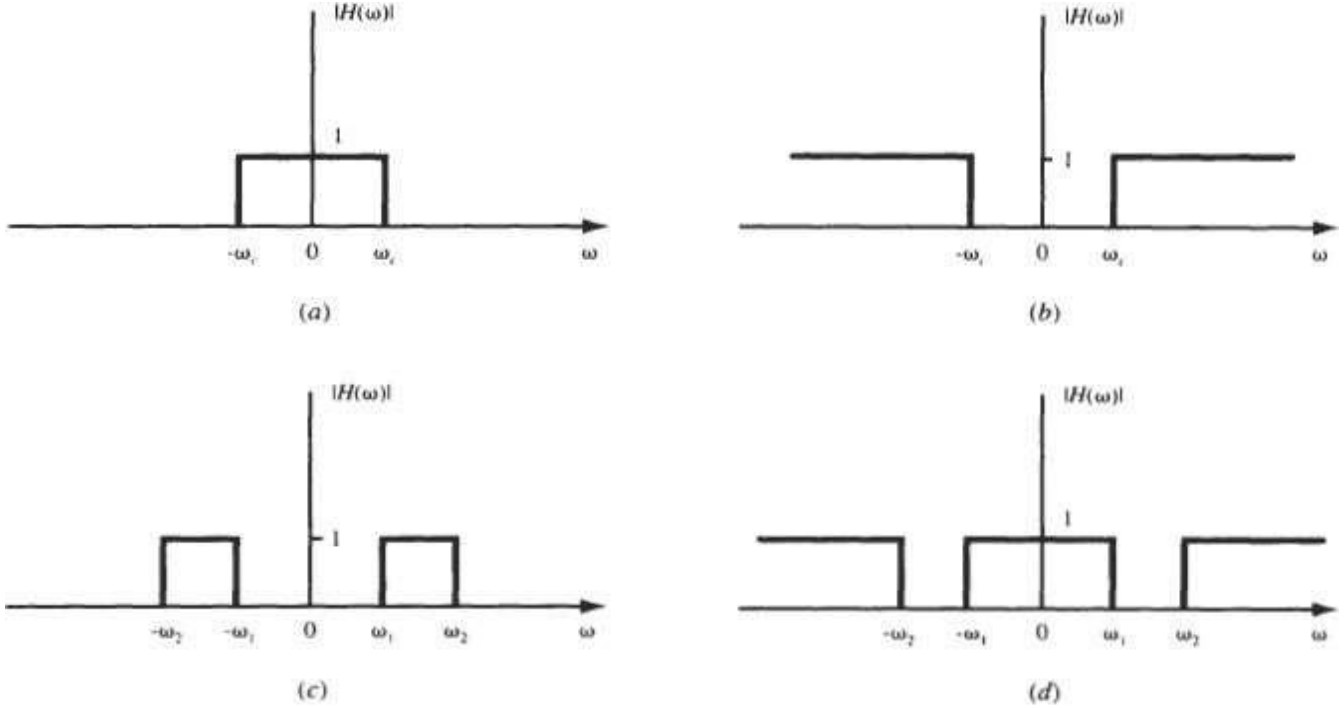


Fig: Magnitude responses of ideal filters (a) Ideal Low-Pass Filter (b)Ideal High-Pass Filter ©Ideal Bandpass Filter (d) Ideal Bandstop Filter

Causality and Stability

$$h(t) = 0, \quad t < 0$$

- Causality : It does not respond before the excitation is applied
- Stability
  - The output signal is bounded for all bounded input signals (BIBO)

$$|x(t)| < M \quad \text{for all } t$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right|$$

$$\left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau) x(t-\tau)| d\tau$$

$$= M \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

$$|y(t)| \le M \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

  - An LTI system to be stable
    - The impulse response h(t) must be absolutely integrable
    - The necessary and sufficient condition for BIBO stability of a linear time-invariant system

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (2.100)$$

Paley-Wiener Criterion

- The frequency-domain equivalent of the causality requirement

$$\int_{-\infty}^{\infty}\left(\frac{|\alpha(f)|}{1+f^2}\right)df < \infty$$

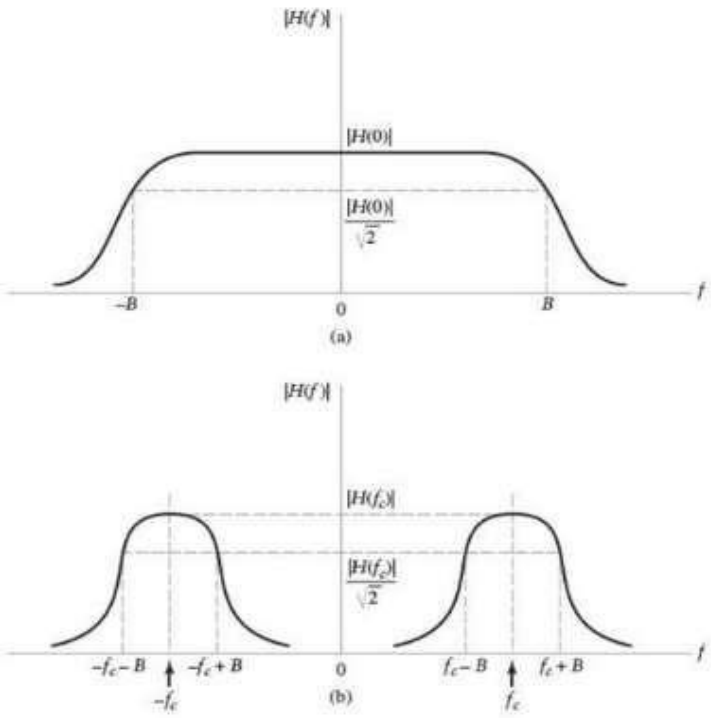


FIGURE 2.22 Illustration of the definition of system bandwidth. (*a*) Low-pass system. (*b*) Band-pass system.

Spectral Density

- The *spectral density* of a signal characterizes the distribution of the signal's energy or power in the frequency domain.
- This concept is particularly important when considering filtering in communication systems while evaluating the signal and noise at the filter output.
- The energy spectral density (ESD) or the power spectral density (PSD) is used in the evaluation.

Energy Spectral Density (ESD)

- Energy spectral density describes the signal energy per unit bandwidth measured in joules/hertz.
- Represented as $\psi_x(f)$, the squared magnitude spectrum

$$\psi_x(f) = |X(f)|^2$$

- According to Parseval's theorem, the energy of x(t):

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

- Therefore:

$$E_x = \int_{-\infty}^{\infty} \psi_x(f)\, df$$

- The Energy spectral density is symmetrical in frequency about origin and total energy of the signal x(t) can be expressed as:

$$E_x = 2\int_{0}^{\infty} \psi_x(f)\, df$$

Power Spectral Density (PSD)

- The *power spectral density* (PSD) function $G_x(f)$ of the periodic signal $x(t)$ is a real, even, and nonnegative function of frequency that gives the distribution of the power of $x(t)$ in the frequency domain.
- PSD is represented as: $$G_x(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$
- Whereas the average power of a periodic signal x(t) is represented as: $$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$
- Using PSD, the average normalized power of a real-valued signal is represented as:

$$P_x = \int_{-\infty}^{\infty} G_x(f)\,df = 2\int_{0}^{\infty} G_x(f)\,df$$

THE LAPLACE TRANSFORM

We know that for a continuous-time LTI system with impulse response h(t), the output **y(t)** of the system to the complex exponential input of the form e^{st} is,

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

Definition:

The function H(s) is referred to as the Laplace transform of h(t). For a general continuous-time signal x(t), the Laplace transform X(s) is defined as,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

The variable s is generally complex-valued and is expressed as,

$$s = \sigma + j\omega$$

LAPLACE TRANSFORM

Relation between Laplace and Fourier transforms:

Laplace transform of x(t)

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Substitute s= σ + jω in above equation.

$$\rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt$$

$$\therefore X(S) = F.T[x(t)e^{-\sigma t}]$$

$$X(S) = X(\omega) \qquad for\ s = j\omega$$

Inverse Laplace Transform:

We know that

$$X(S) = F.T[x(t)e^{-\sigma t}]$$

$$\to x(t)e^{-\sigma t} = F.T^{-1}[X(S)] = F.T^{-1}[X(\sigma + j\omega)]$$

$$= \frac{1}{2}\pi \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$x(t) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t} d\omega.$$

Here, $\sigma + j\omega = s$

$$jd\omega = ds \to d\omega = ds/j$$

$$\therefore x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st} ds \ldots\ldots$$

Conditions for Existence of Laplace Transform:

Dirichlet's conditions are used to define the existence of Laplace transform. i.e.

- The function f has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f ,in the given interval of time.
- It must be absolutely integrable in the given interval of time. i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$

Initial and Final Value Theorems

If the Laplace transform of an unknown function x(t) is known, then it is possible to determine the initial and the final values of that unknown signal i.e. x(t) at $t=0^+$ and $t=\infty$.

Initial Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the initial value of x(t) is given by $x(0^+) = \lim_{s \to \infty} SX(S)$

Final Value Theorem

Statement: If x(t) and its 1st derivative is Laplace transformable, then the final value of x(t) is given by,

$$x(\infty) = \lim_{s \to \infty} SX(S)$$

Properties of Laplace transform:

The properties of Laplace transform are:

Linearity Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

& $y(t) \xleftrightarrow{\text{L.T}} Y(s)$

Then linearity property states that

$$ax(t) + by(t) \xleftrightarrow{\text{L.T}} aX(s) + bY(s)$$

LAPLACE TRANSFORM

Time Shifting Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then time shifting property states that

$$x(t-t_0) \xleftrightarrow{\text{L.T}} e^{-st_0} X(s)$$

Frequency Shifting Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then frequency shifting property states that

$$e^{s_0 t} . x(t) \xleftrightarrow{\text{L.T}} X(s-s_0)$$

Time Reversal Property

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

Then time reversal property states that

$$x(-t) \xleftrightarrow{\text{L.T}} X(-s)$$

Time Scaling Property

If $x(t) \xleftrightarrow{\mathrm{L.T}} X(s)$

Then time scaling property states that

$$x(at) \xleftrightarrow{\mathrm{L.T}} \frac{1}{|a|}X(\frac{s}{a})$$

Differentiation and Integration Properties

If $x(t) \xleftrightarrow{\mathrm{L.T}} X(s)$

Then differentiation property states that

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathrm{L.T}} s.X(s)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{\mathrm{L.T}} (s)^n.X(s)$$

The integration property states that

$$\int x(t)dt \xleftrightarrow{\mathrm{L.T}} \frac{1}{s}X(s)$$

$$\iiint \ldots \int x(t)dt \xleftrightarrow{\mathrm{L.T}} \frac{1}{s^n}X(s)$$

Multiplication and Convolution Properties

If $x(t) \xleftrightarrow{\text{L.T}} X(s)$

and $y(t) \xleftrightarrow{\text{L.T}} Y(s)$

Then multiplication property states that

$$x(t).\,y(t) \xleftrightarrow{\text{L.T}} \frac{1}{2\pi j} X(s) * Y(s)$$

The convolution property states that

$$x(t) * y(t) \xleftrightarrow{\text{L.T}} X(s).\,Y(s)$$

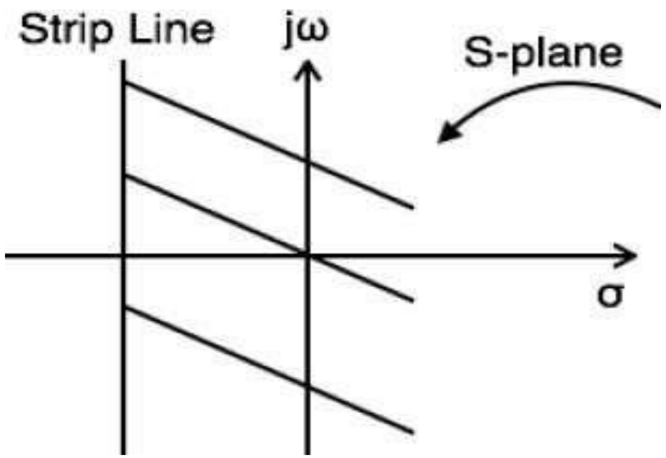
LAPLACE TRANSFORM

Region of convergence

The range variation of σ for which the Laplace transform converges is called region of convergence.

Properties of ROC of Laplace Transform

- ROC contains strip lines parallel to jω axis in s-plane.



- If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x(t) is a right sided sequence then ROC : Re{s} > σo.
- If x(t) is a left sided sequence then ROC : Re{s} < σo.
- If x(t) is a two sided sequence then ROC is the combination of two regions.

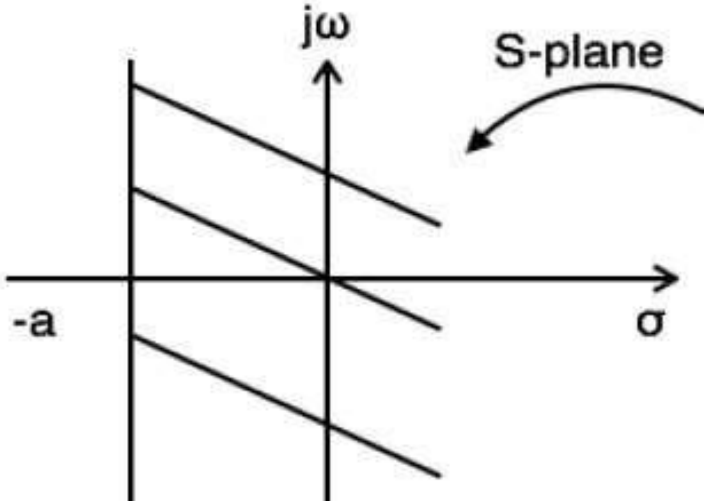
LAPLACE TRANSFORM

Example 1: Find the Laplace transform and ROC of x(t)=e– at u(t) $x(t)=e^{-at}u(t)$

$$L.T[x(t)] = L.T[e^{-at}\,u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$



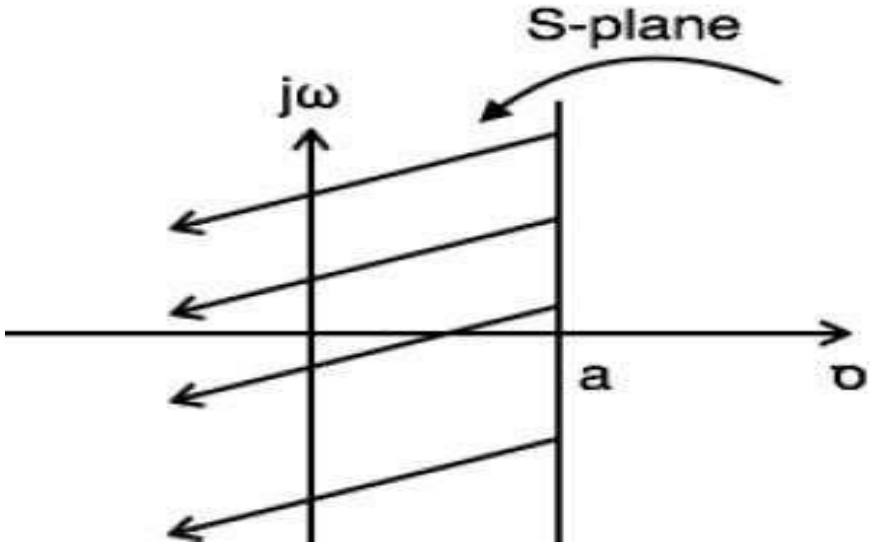
LAPLACE TRANSFORM

Example 2: Find the Laplace transform and ROC of x(t)=e at u(−t) x(t)=e^{at}u(−t)

$$L.T[x(t)] = L.T[e^{at}u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$



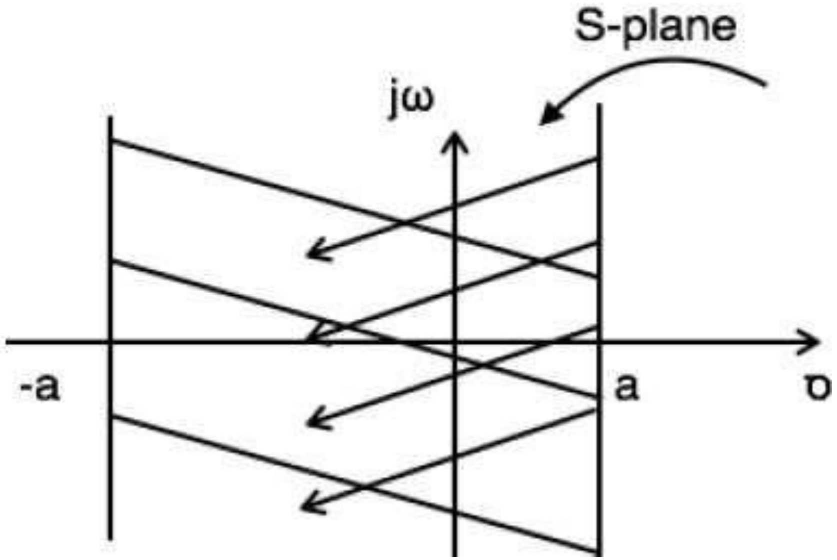
Example 3: Find the Laplace transform and ROC of x(t)=e −at u(t)+e at u(−t)

$x(t)=e^{-at}u(t)+e^{at}u(-t)$

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a} Re\{s\} > -a$

For $\frac{1}{S-a} Re\{s\} < a$

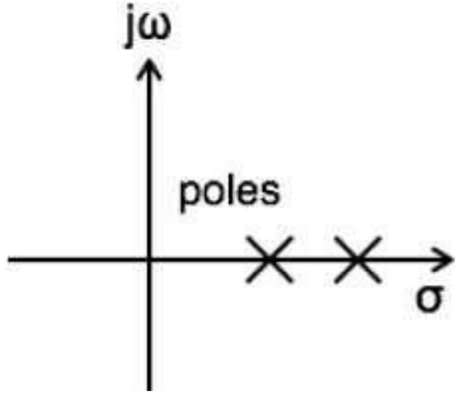


Referring to the above diagram, combination region lies from −a to a. Hence, ROC:

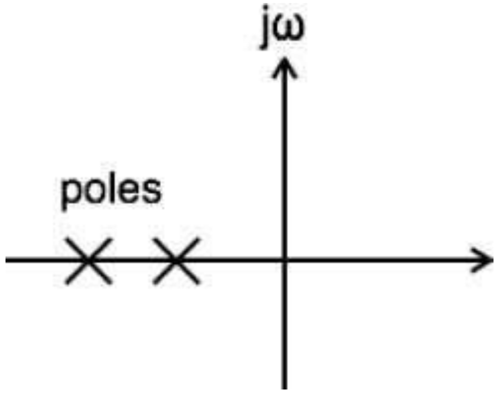
−a <Res<a

Causality and Stability

For a system to be causal, all poles of its transfer function must be right half of s-plane.

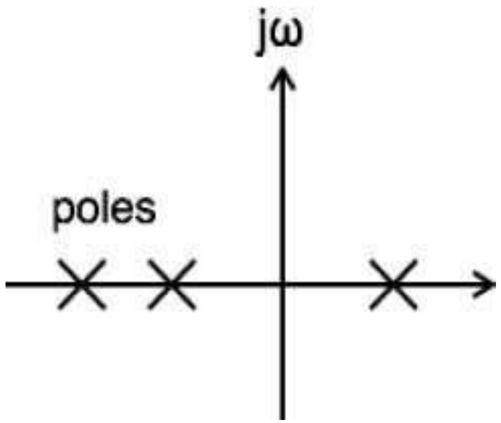


A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.

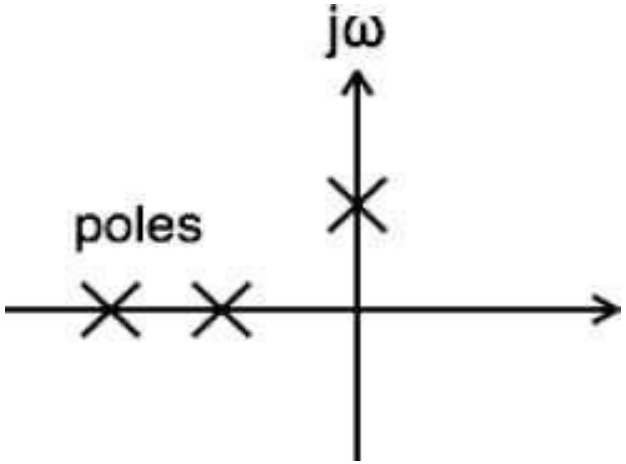


LAPLACE TRANSFORM

A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



A system is said to be marginally stable when at least one pole of its transfer function lies on the jω axis of s-plane



LAPLACE TRANSFORMS OF SOME COMMON SIGNALS

Unit Impulse Function δ(***t***):

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad \text{all } s$$

Unit Step Function ***u(t)*:**

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \mathrm{Re}(s) > 0$$

where $0^+ = \lim_{\varepsilon \to 0}(0 + \varepsilon)$.

LAPLACE TRANSFORM

Some Laplace Transforms Pairs:

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

Z-Transform

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

Z-TRANSFORM

Concept of Z-Transform and Inverse Z-Transform

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots \ldots (1)$$

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n) [re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n) [r^{-n}] e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n) r^{-n}] \ldots \ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

Inverse Z-transform:

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \frac{1}{2\pi} \int X(re^j\omega)e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jzd\omega$$

$$d\omega = \frac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \frac{1}{2\pi} \int X(z)z^n \frac{1}{j} z^{-1} dz = \frac{1}{2\pi j} \int X(z)z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z)z^{n-1} dz$$

Z-Transform Properties:

Z-Transform has following properties:

Linearity Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \xleftrightarrow{\text{Z.T}} a\,X(Z) + b\,Y(Z)$$

Time Shifting Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then Time shifting property states that

$$x(n-m) \xleftrightarrow{\text{Z.T}} z^{-m} X(Z)$$

Multiplication by Exponential Sequence Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \,.\, x(n) \xleftrightarrow{\text{Z.T}} X(Z/a)$$

Time Reversal Property:

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{Z.T}{\longleftrightarrow} X(1/Z)$$

Differentiation in Z-Domain OR Multiplication by n Property:

If $x(n) \overset{Z.T}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{Z.T}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

Z-TRANSFORM

Convolution Property: If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z)$$

Correlation Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z^{-1})$$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z\to\infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1} [z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z- transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at $z = 0$.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at $z = \infty$.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a.

  i.e. $|z| > a$.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius

  a. i.e. $|z| < a$.
- f x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at $z = 0$ & $z = \infty$.

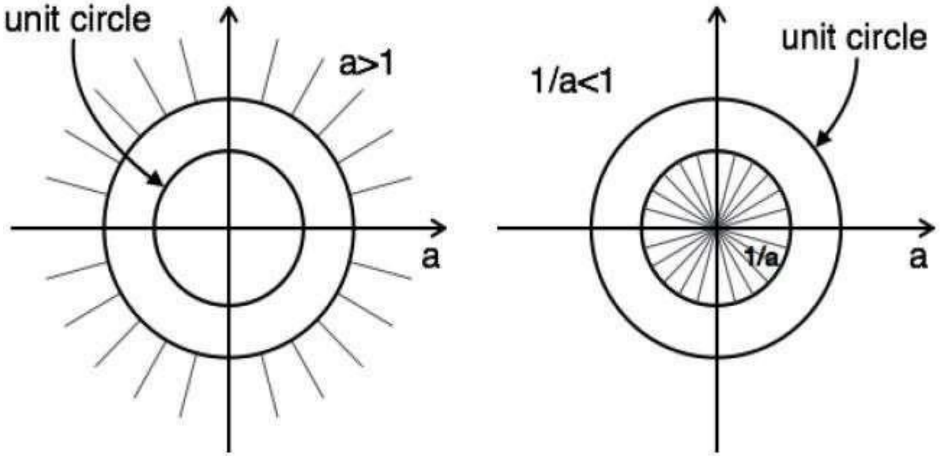
Z-TRANSFORM

Example 1: Find z-transform and ROC of $a^{n}u[n]+a^{-n}u[-n-1]$ $a^{n}u[n]+a^{-n}u[-n-1]$

$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

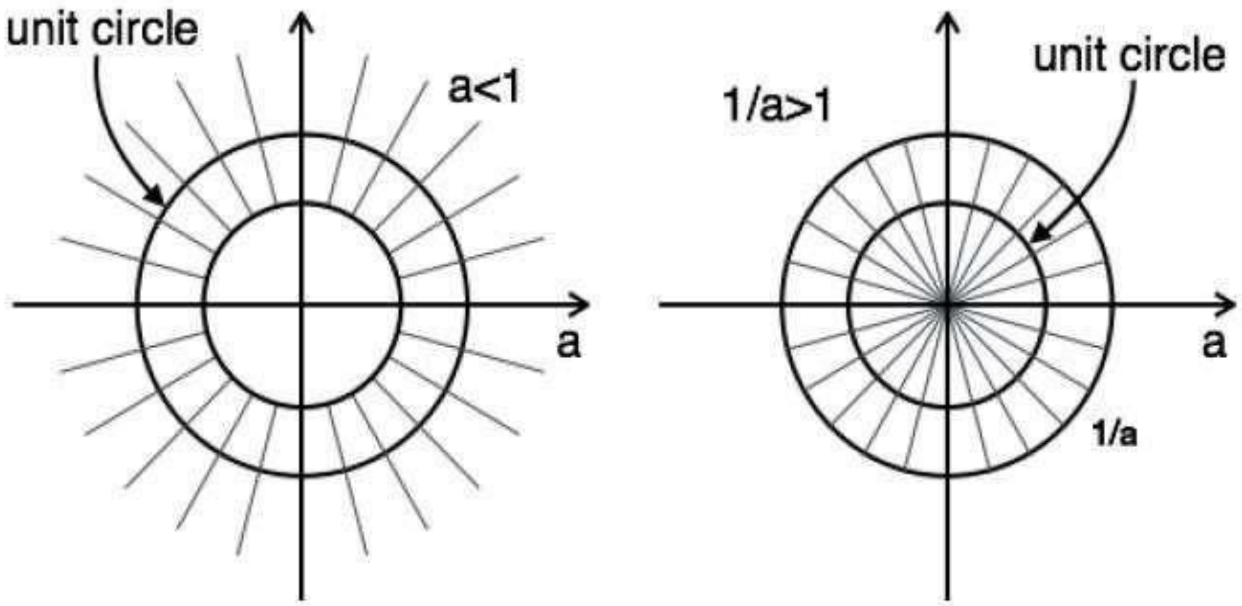
$$ROC: |z| > a \qquad ROC: |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1, as we do not know a.



In this case, there is no combination ROC.

Z-TRANSFORM



Here, the combination of ROC is from $a<|z|<1/a$

Hence for this problem, z-transform is possible when $a < 1$.

Causality and Stability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when,

- ROC is outside the outermost pole.
- In The transfer function H[Z], the order of numerator cannot be grater than the order of denominator.

Stability Condition for Discrete Time LTI Systems:

A discrete time LTI system is stable when

- its system function H[Z] include unit circle $|z|=1$.
- all poles of the transfer function lay inside the unit circle $|z|=1$.

Z-TRANSFORM

Some Properties of the Z- Transform:

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\lvert z\rvert<\infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \lvert z_0\rvert R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\lvert z\rvert>1\}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

Inverse Z transform:

Three different methods are:

- Partial fraction method
- Power series method
- Long division method

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

Usually $M < N$

Z-TRANSFORM

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{\tilde{B}(z)}{A(z)}$$

where $\tilde{B}(z)$ now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \xleftrightarrow{z} \delta[n]$$

$$z^{-n_o} \xleftrightarrow{z} \delta[n - n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair
- We get

$$A_k (d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1 - d_k z^{-1}},$$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial-fraction expansion associated with that pole

$$\frac{A_{i_1}}{1-d_i z^{-1}}, \frac{A_{i_2}}{(1-d_i z^{-1})^2}, \ldots, \frac{A_{i_r}}{(1-d_i z^{-1})^r}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[n] \xleftrightarrow{z} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC} |z| > d_i$$

- If the ROC is of the form $|z| < d_i$, the left-sided inverse z-transform is chosen, ie.

$$-A\frac{(n+1)\ldots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \xleftrightarrow{z} \frac{A}{(1-d_i z^{-1})^m}, \qquad \text{with ROC} |z| < d_i$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.

- In order to chose the correct inverse z-transform, we must infer the ROC of each term from the ROC of $X(z)$.

- By comparing the location of each pole with the ROC of $X(z)$.

- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term

- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued
- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will be complex conjugate of each other

Z-TRANSFORM

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
  - If the signal is stable, then t is absolutely summable and has DTFT
  - Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. $|z| = 1$
  - The inverse z-transform is determined by comparing the poles and the unit circle
  - If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
  - If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

Power series expansion method

- Express $X(z)$ as a power series in z^{-1} or z as given in z-transform equation
- The values of the signal $x[n]$ are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals
- Signals with ROCs of the form $|z| > a$ or $|z| < a$
- If the ROC is $|z| > a$, then express $X(z)$ as a power series in z^{-1} and we get right sided signal
- If the ROC is $|z| < a$, then express $X(z)$ as a power series in z and we get left sided signal

Long division method:

- Find the z-transform of

$$X(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2}z^{-1}}, \text{ with ROC } \quad |z| > \frac{1}{2}$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence
- We get

$$X(z) = 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \dots$$

- Compare with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Z-TRANSFORM

- We get

$$x[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand $X(z)$ as a power series in z using long division method
- We get

$$X(z) = -2 - 8z - 16z^2 - 32z^3 + \ldots$$

- We can write $x[n]$ as

$$x[n] = -2\delta[n] - 8\delta[n+1] - 16\delta[n+2] - 32\delta[n+3] + \ldots$$

Z-TRANSFORM

- Find the z-transform of

$$X(z) = e^{z^2}, \text{with ROC all } z \text{ except } |z| = \infty$$

- Solution is: use power series expansion for e^a and is given by

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!}$$

$$X(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

- We can write $x[n]$ as

$$x[n] = \begin{cases} 0 & n > 0 \;\; \text{or } n \text{ is odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

Z-TRANSFORM

Example: A finite sequence *x [n]* is defined as $x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$

Find X(z) and its ROC.

Sol: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-2}^{3} x[n] z^{-n}$$

$$= x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$= 5z^{2} + 3z - 2 + 4z^{-2} - 3z^{-3}$$

For *z* not equal to zero or infinity, each term in *X(z)* will be finite and consequently *X(z)* will converge. Note that *X (z)* includes both positive powers of *z* and negative powers of *z*. Thus, from the result we conclude that the ROC of *X (z)* is **0** < |*z*| < m.

Example: Consider the sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X(z)$ **and plot the poles and zeros of X(z).**

Sol:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

From the above equation we see that there is a pole of $(N-1)^{th}$ order at $z = 0$ and a pole at $z = a$. Since x[n] is a finite sequence and is zero for $n < 0$, the ROC is $|z| > 0$. The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 0, 1, \ldots, N-1$$

Z-TRANSFORM

The root at k = **0** cancels the pole at $z = a$. The remaining zeros of $X(z)$ are at

$$z_k = ae^{j(2\pi k/N)} \qquad k = 1, \ldots, N-1$$

The pole-zero plot is shown in the following figure with N=8