STRENGTH OF MATERIALS

THEORY OF Elasticity

Energy Methods Applications

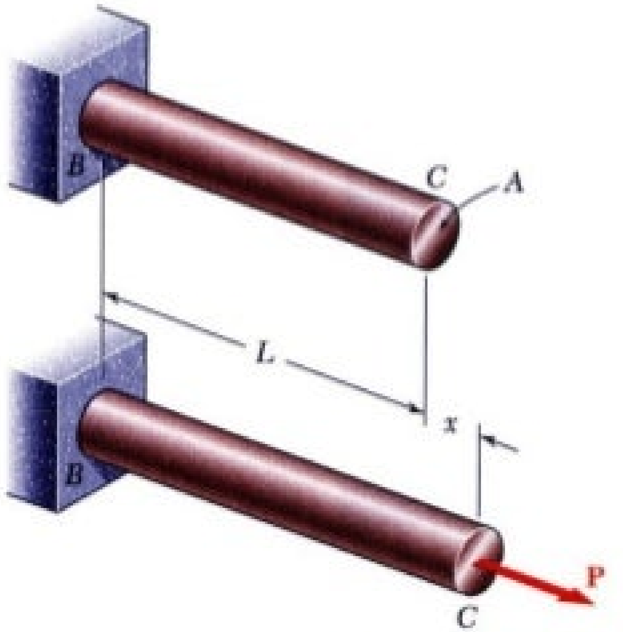
Introduction

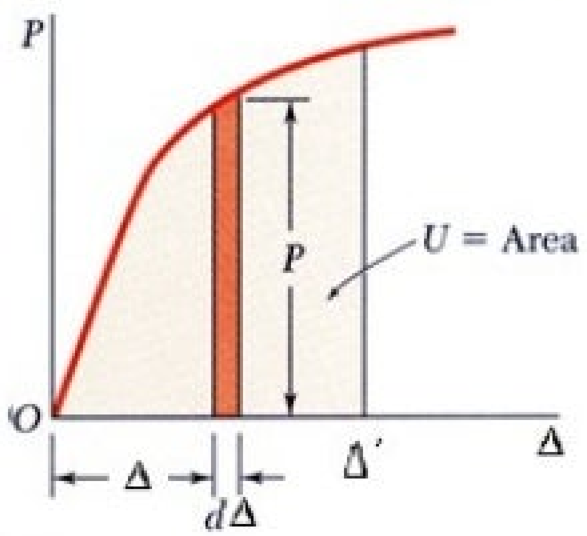
- Energy methods are used widely to obtain solutions to elasticity problems and determine deflections of structures.

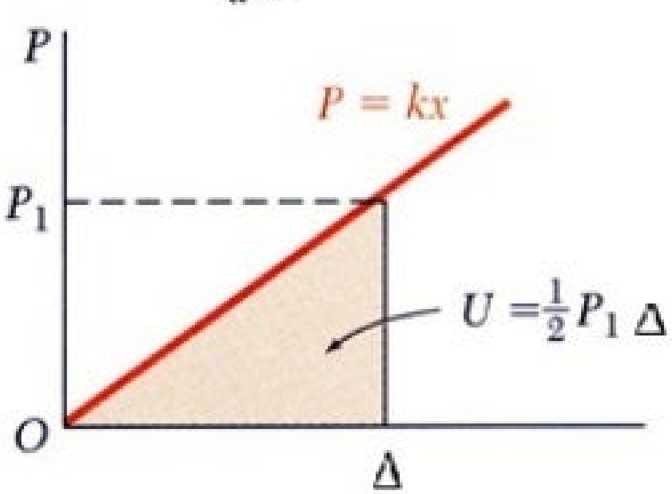
- The deflection of joints on a truss or points on a beam or shaft can be determined using energy methods. we will first define the work caused by an external force and couple moment and show how to express this work in terms of a body's strain energy.

- In mechanics, a force does work when it undergoes a displacement dx that is in the same direction as the force.

External Force and Strain Energy







- A uniform rod is subjected to a slowly increasing load
- The *elementary work* done by the load P as the rod elongates by a small $d\Delta$ is

$$dU_e = P\,d\Delta = elementary\ work$$

which is equal to the area of width $d\Delta$ under the load-deformation diagram.

- The *total work* done by the load for a deformation Δ,

$$U_e = \int_0^{\Delta} P\,d\Delta = total\ work = strain\ energy$$

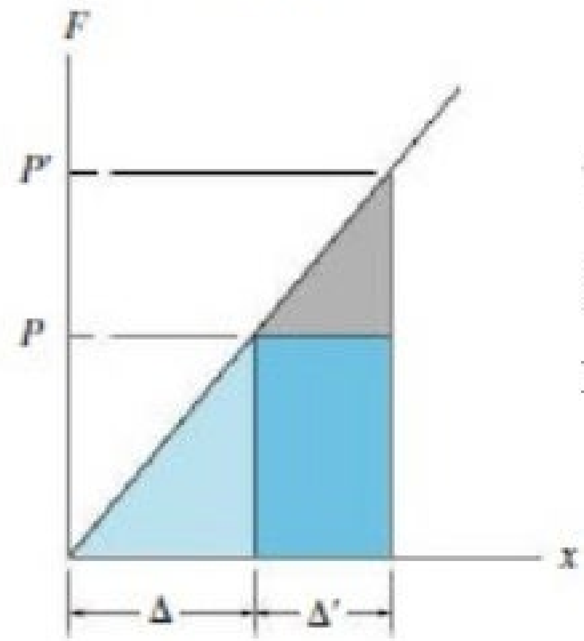
which results in an increase of *strain energy* in the rod.

- In the case of a linear elastic deformation,

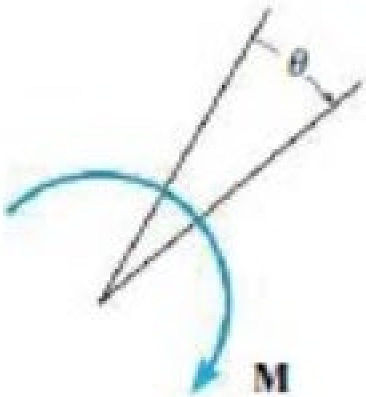
$$U_e = \int_0^{\Delta} k\Delta\,d\Delta = \tfrac{1}{2}k\Delta_1^2 = \tfrac{1}{2}P_1\Delta$$

- A load P is already applied to the bar and that *another force* P' is now applied, so that the end of the bar is displaced *further* by an amount Δ'

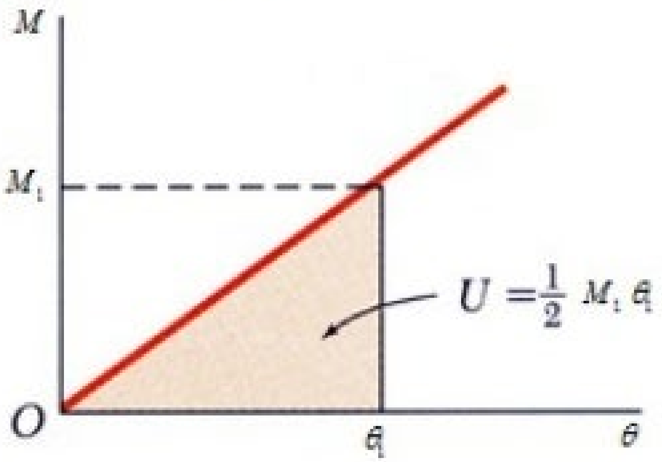
$$U_e = \tfrac{1}{2} P\Delta + P\Delta' + \tfrac{1}{2} P'\Delta'$$



The work done by P′ is equal to the gray shaded triangular area and the work done by P represents the dark-blue shaded rectangular area

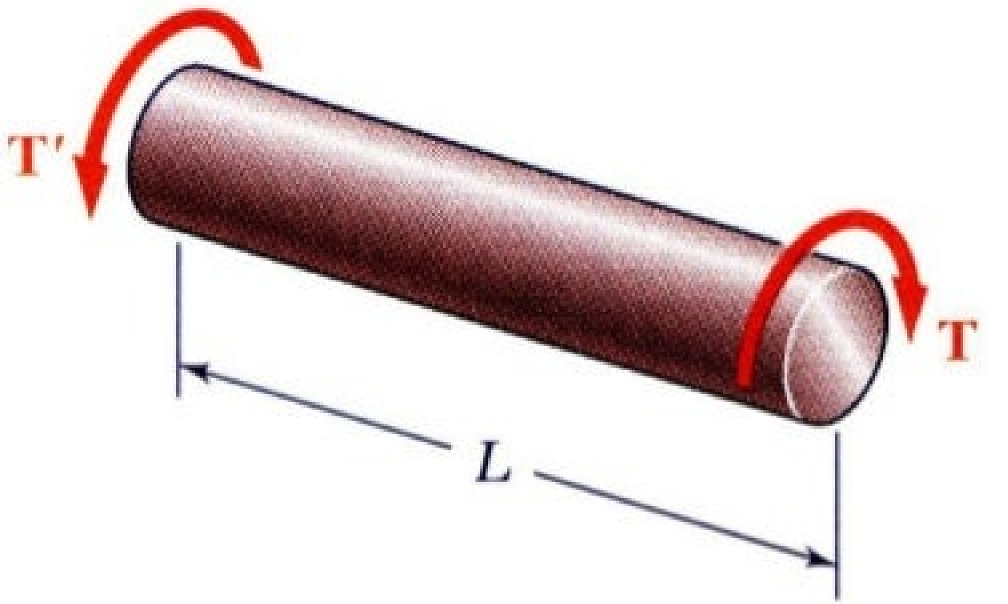


- A couple moment $\boldsymbol{M}$ does work when it undergoes an angular displacement $d\theta$ along its line of action. If the total angular displacement is θ rad, the work becomes



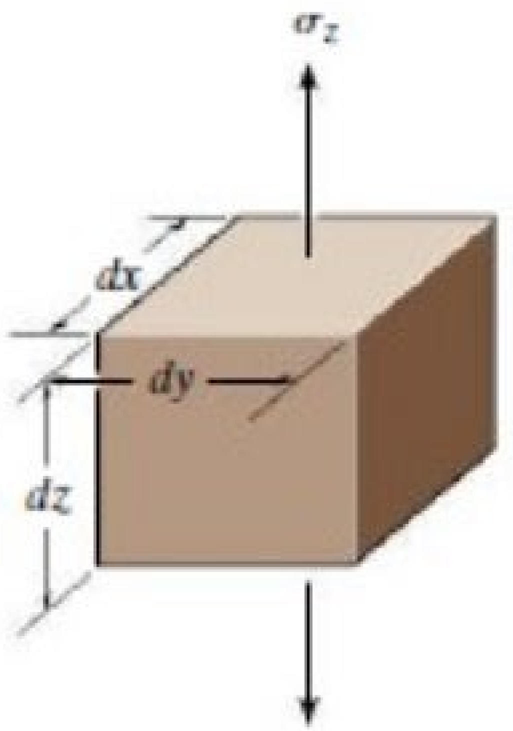
$$U_e = \int_0^{\theta_1} M\, d\theta = \frac{1}{2} M_1 \theta_1$$

- For a shaft subjected to a torsional load,



$$U_e = \int_0^{\phi_1} T\, d\phi = \frac{1}{2} T_1 \phi_1$$

Internal Force and Strain Energy Density



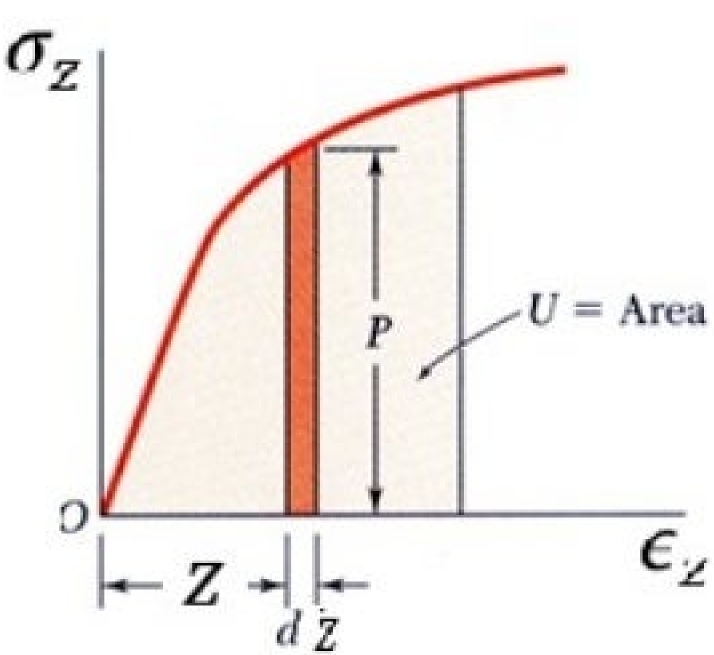
- A body is subjected only to a uniaxial *normal stress* σ_z, then the force created on the element's top and bottom faces is

$$dF = \sigma_z dA = \sigma_z dxdy$$

while the element undergoes an elongation $d\Delta_z = \epsilon_z dz$, The work done by dF is

$$dU_i = \frac{1}{2} dF d\Delta_z = \frac{1}{2}(\sigma_z dxdy)(\epsilon_z dz)$$

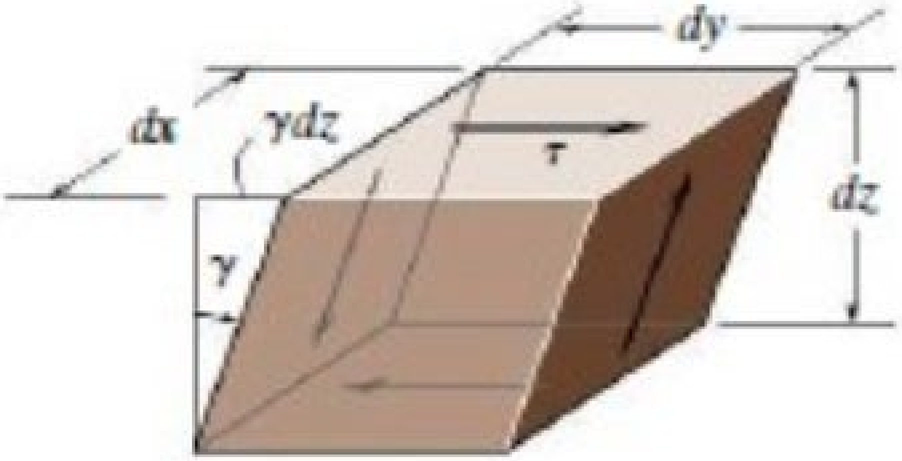
therefore the strain energy in the body is then



$$U_i = \int dU_i = \frac{1}{2}\int \sigma_z \epsilon_z dxdydz$$

$$= \frac{1}{2}\int \sigma_z \frac{\sigma_z}{E} dV$$

$$= \frac{1}{2}\int \frac{{\sigma_z}^2}{E} dV$$



- When a body is subjected to shear stress,

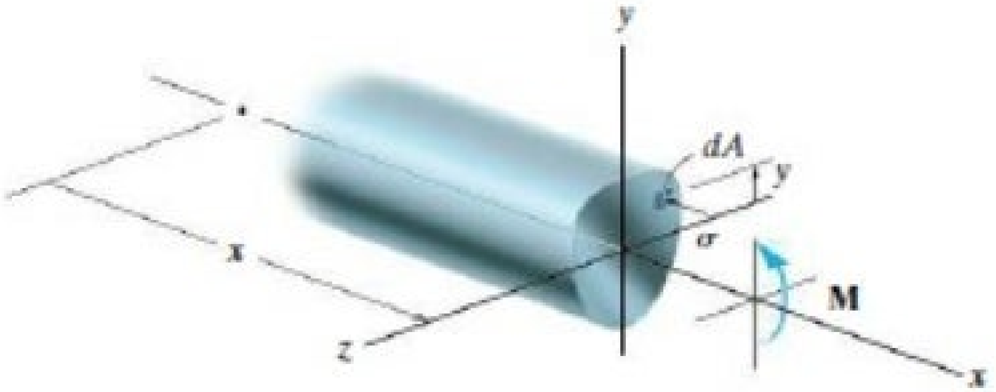
$dF = \tau dA = \tau dxdy$ = shear force

$d\Delta = \gamma dz$ = displacement

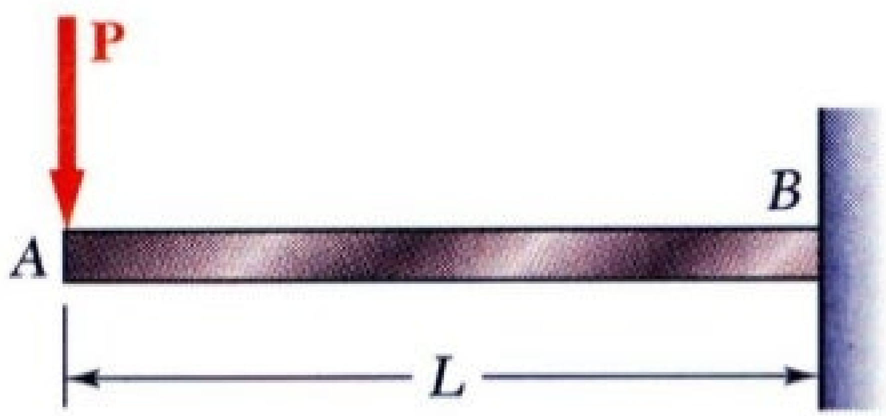
$$dU_i = \frac{1}{2} dF d\Delta = \frac{1}{2} (\tau dxdy)(\gamma dz)$$

$$U_i = \int dU_i = \frac{1}{2} \int \tau \gamma dxdydz$$

$$= \frac{1}{2} \int \tau \frac{\tau}{G} dV = \frac{1}{2} \int \frac{\tau^2}{G} dV$$



$$\sigma_x = \frac{M\,y}{I}$$



- For a beam subjected to a bending load,

$$U_i = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

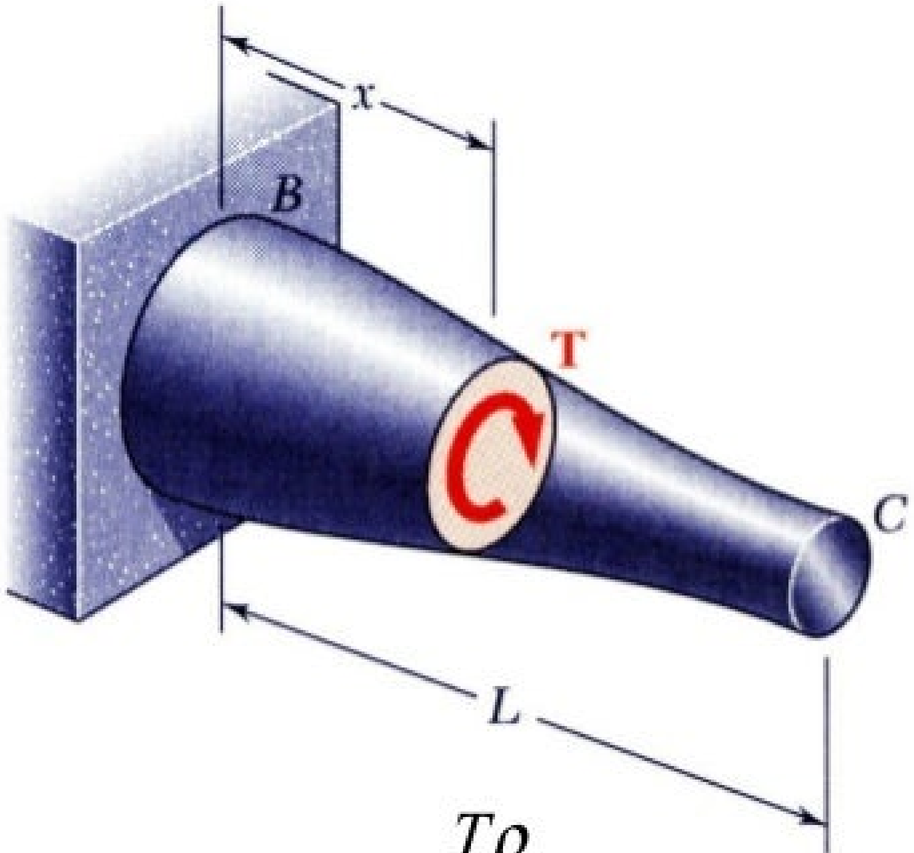
$$U_i = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right)dx$$

$$= \int_0^L \frac{M^2}{2EI} dx$$

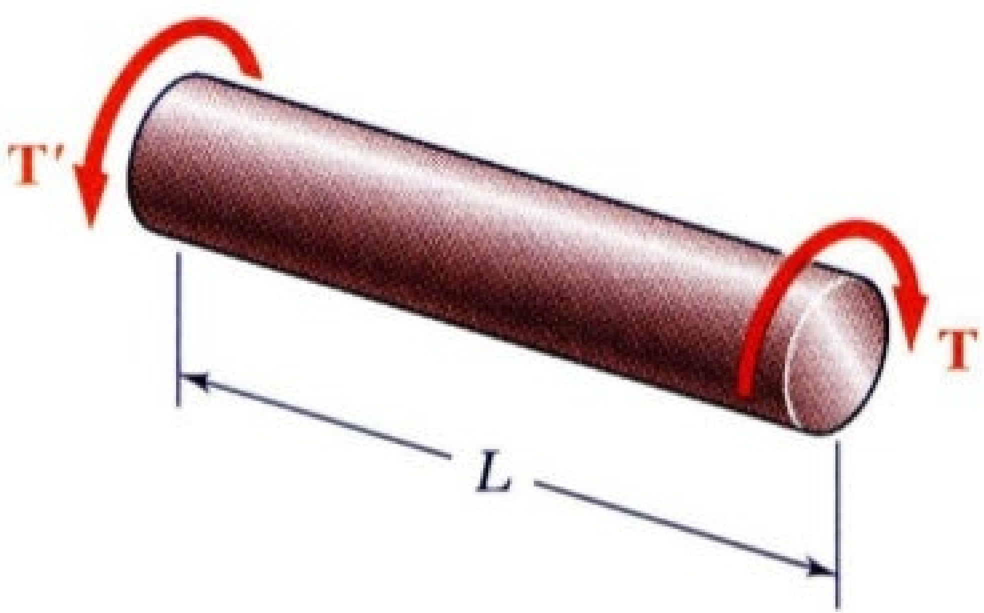
- For an end-loaded cantilever beam,

$$M = -Px$$

$$U_i = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$



$$\tau_{xy} = \frac{T\rho}{J}$$



- For a shaft subjected to a torsional load,

$$U_i = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

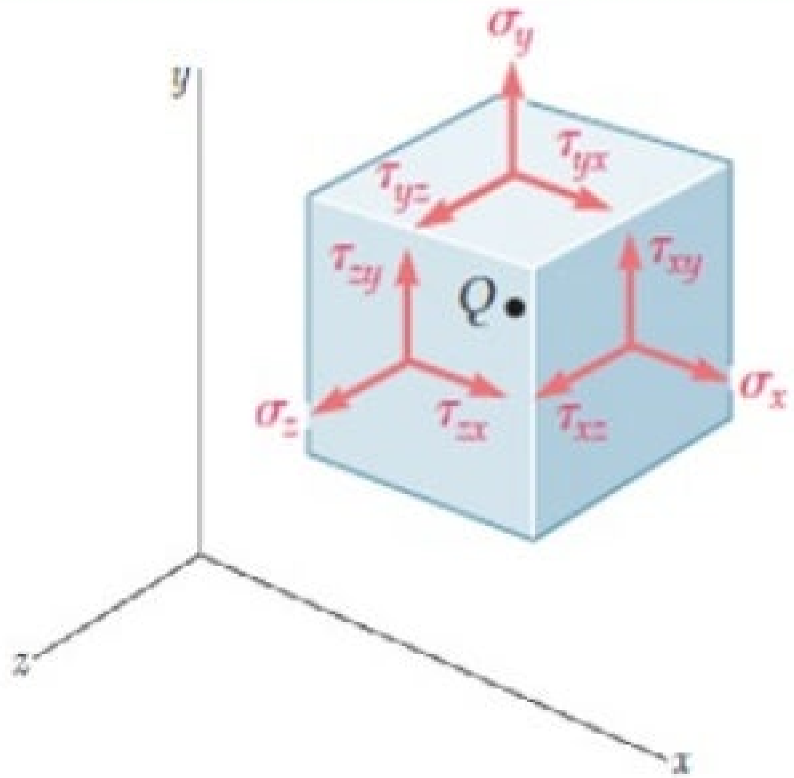
- Setting $dV = dA\,dx$,

$$U_i = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\,dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GJ} dx$$

- In the case of a uniform shaft,

$$U_i = \frac{T^2 L}{2GJ}$$



- The strain energy stored in members subjected to several types of loads, the normal and shear stress components, can be obtained from

$$dU_i = \frac{1}{2}\sigma\epsilon dV \quad , \quad dU_i = \frac{1}{2}\tau\gamma dV$$

- The total strain energy in the body is therefore

$$U_i = \int \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})dV$$

- The strains can be eliminated by using the generalized form of Hooke's law given by

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

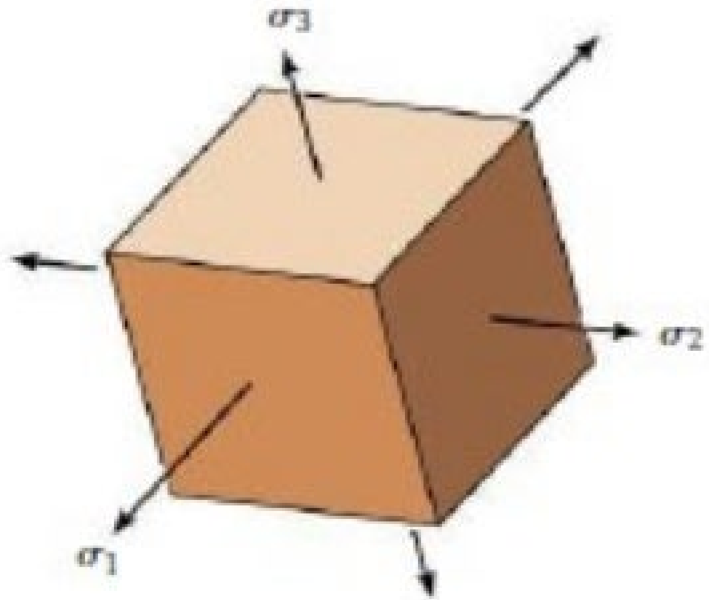
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} , \quad \gamma_{xz} = \frac{\tau_{xz}}{G} , \quad \gamma_{zy} = \frac{\tau_{zy}}{G}$$

- After substituting and combining terms, we have

$$U_i = \int \left[\frac{1}{2E}({\sigma_x}^2+{\sigma_y}^2+{\sigma_z}^2)-\frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_x\sigma_z+\sigma_z\sigma_y\right) + \frac{1}{G}({\tau_{xy}}^2+ {\tau_{xz}}^2+{\tau_{yz}}^2)\right] dV$$

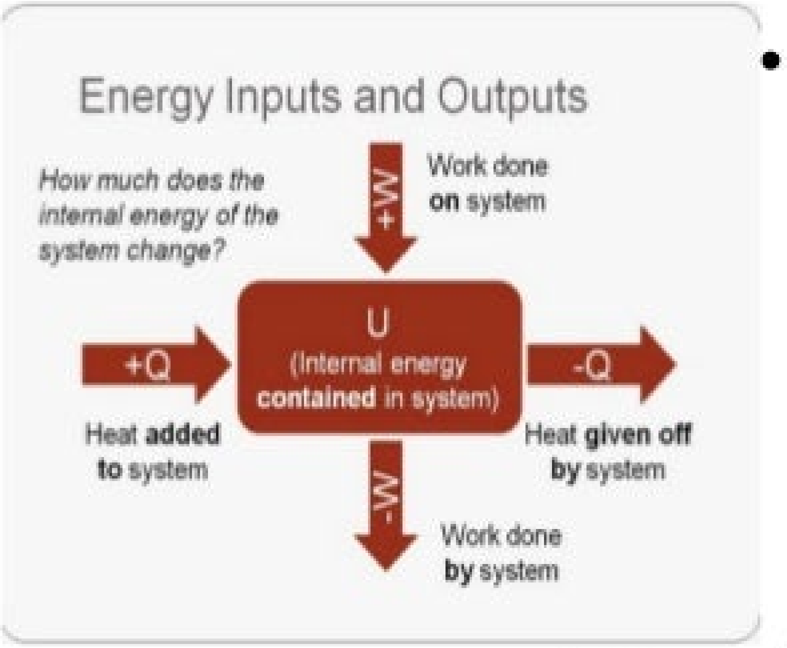
- If only the principal stresses $\sigma_1, \sigma_2, \sigma_3$ act on the element



$$U_i = \int \left[\frac{1}{2E}({\sigma_1}^2+{\sigma_2}^2+{\sigma_3}^2) -\frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_1\sigma_3+\sigma_2\sigma_3)\right] dV$$

Energy Methods in Elasticity

The work performed on a mechanical system by external forces plus the heat that flaws into the system from the outside equals the increase in internal energy plus the increase in kinetic energy



- For adiabatic conditions (no heat flow) and static equilibrium (kinetic energy=0), the variation in work of the external force is equal to the variation of internal energy

$$\delta W = \delta U$$

- The external force δW is equal to

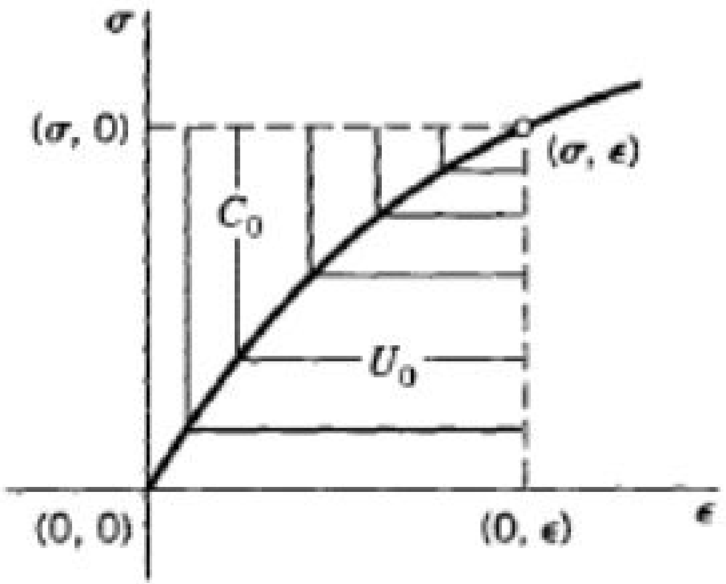
$$\delta W = \int (\sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \sigma_z \delta\epsilon_z + 2\tau_{xy}\delta\gamma_{xy} + 2\delta\tau_{xz}\gamma_{xz} + 2\tau_{yz}\delta\gamma_{yz})\, dV$$

- the variation of the internal energy δU in terms of internal-energy density δU_0

$$\delta U = \int \delta U_0 \, dV$$

- Since $\delta W = \delta U$, the internal energy density becomes

$$\delta U_0 = \sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \sigma_z \delta\epsilon_z + 2\tau_{xy}\delta\gamma_{xy} + 2\delta\tau_{xz}\gamma_{xz} + 2\tau_{yz}\delta\gamma_{yz}$$



- For elastic material behaviour, strain-energy density function U_0 generally depends on the strain components and the coordinates.

$$U_0 = U_0(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{zy}, x, y, z)$$

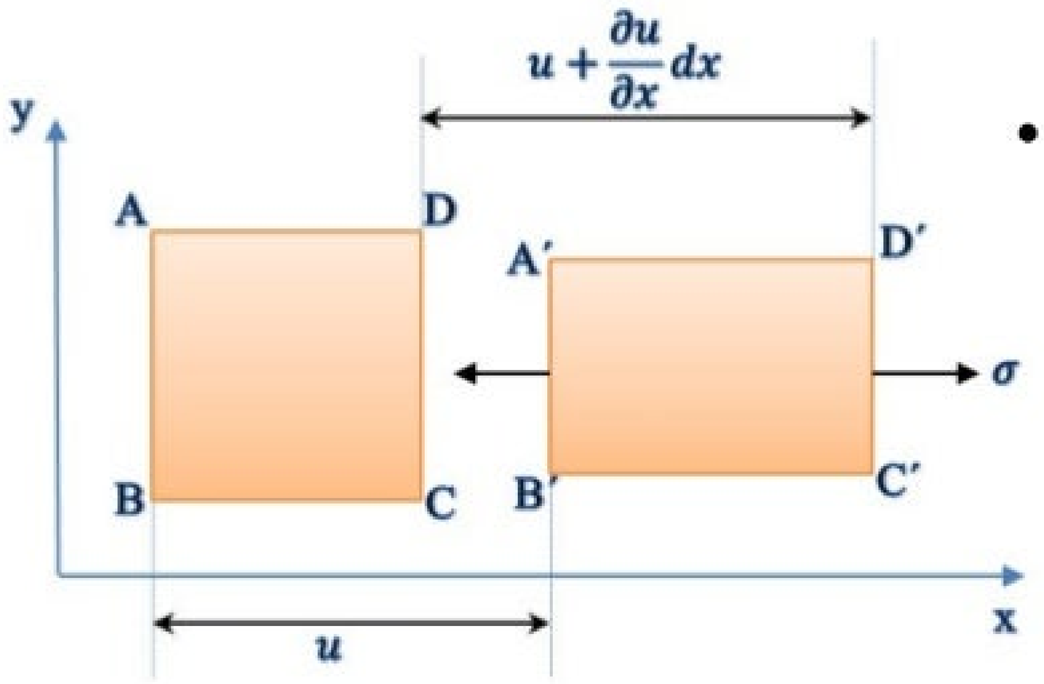
- For small displacement (u,v,w), the variation of the function U_0 becomes

$$\delta U_0 = \frac{\partial U_0}{\partial \epsilon_x}\delta\epsilon_x + \frac{\partial U_0}{\partial \epsilon_y}\delta\epsilon_y + \frac{\partial U_0}{\partial \epsilon_z}\delta\epsilon_z + \frac{\partial U_0}{\partial \gamma_{xy}}\delta\gamma_{xy} + \frac{\partial U_0}{\partial \gamma_{xz}}\gamma_{xz} + \frac{\partial U_0}{\partial \gamma_{yz}}\delta\gamma_{yz}$$

- From above two equation, we obtain

$$\sigma_x = \frac{\partial U_0}{\partial \epsilon_x} \qquad \sigma_y = \frac{\partial U_0}{\partial \epsilon_y} \qquad \sigma_z = \frac{\partial U_0}{\partial \epsilon_z}$$

$$\tau_{xy} = \frac{1}{2}\frac{\partial U_0}{\partial \gamma_{xy}} \qquad \tau_{xz} = \frac{1}{2}\frac{\partial U_0}{\partial \gamma_{xz}} \qquad \tau_{yz} = \frac{1}{2}\frac{\partial U_0}{\partial \gamma_{yz}}$$



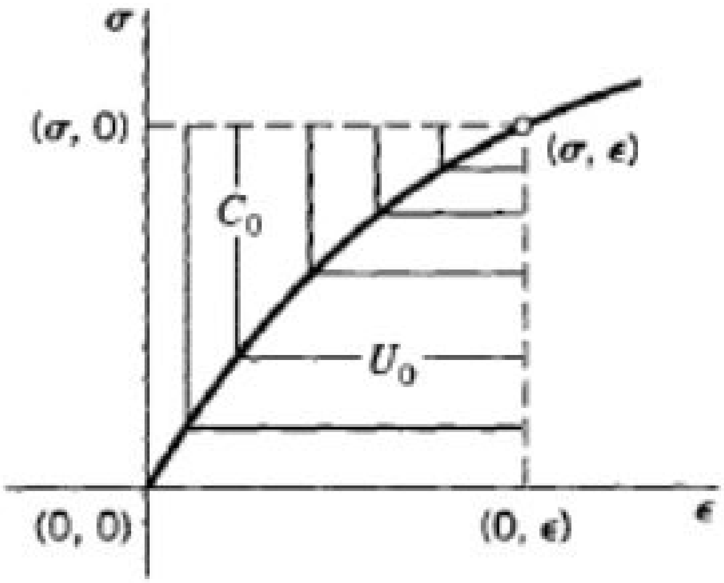
- If we apply a normal stress σ_x, the square undergoes an elongation $d\Delta_x = \epsilon_x dx$, the work done by the stress on CD and AB is:

$$U_0 = \int dU_0 = \int \sigma_x \, d(u + \frac{\partial u}{\partial x}dx)dydz \text{ -}\int \sigma_x \, dudydz$$
$$= \int \sigma_x \frac{\partial u}{\partial x} dxdydz$$

- Since $\frac{\partial u}{\partial x} = \epsilon_x = \frac{\sigma}{E}$, we can integrate the expression to get

$$U = \frac{{\sigma_x}^2}{2E}$$

Elasticity and Complementary Energy Density



- The rectangular area (0,0), (0,ϵ), (σ, ϵ), (σ,0) is represented by the product σ_ϵ, this area is given by

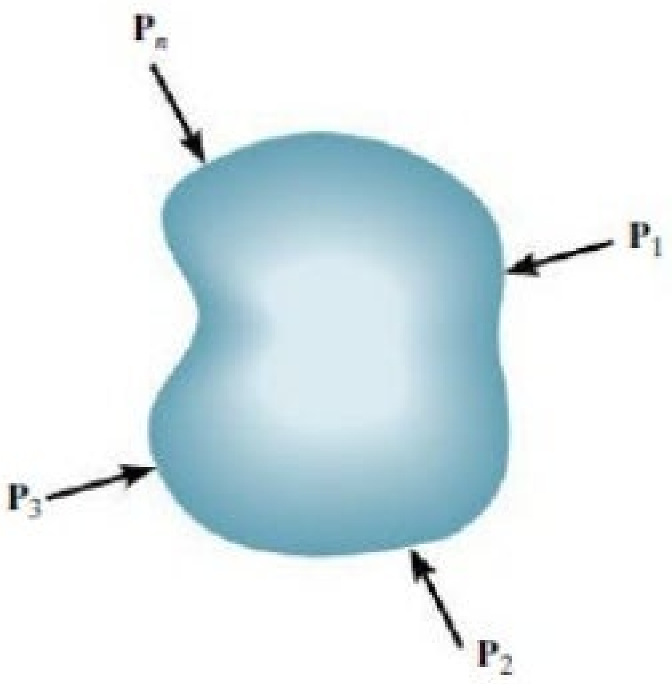
$$\sigma_\epsilon = C_0 + U_0$$

Where C_0 is called the complementary internal energy density

$$C_0 = \int \epsilon d\sigma$$

$$\epsilon = \frac{\partial C_0}{\partial \sigma}$$

Castigliano's Theorem



- This method, which is referred to as Castigliano's theorem, applies only to bodies that have constant temperature and material with linear-elastic behaviour
- Strain energy for a body subjected to to a series of *n* forces, $P_1, P_2, \ldots \ldots, P_n$

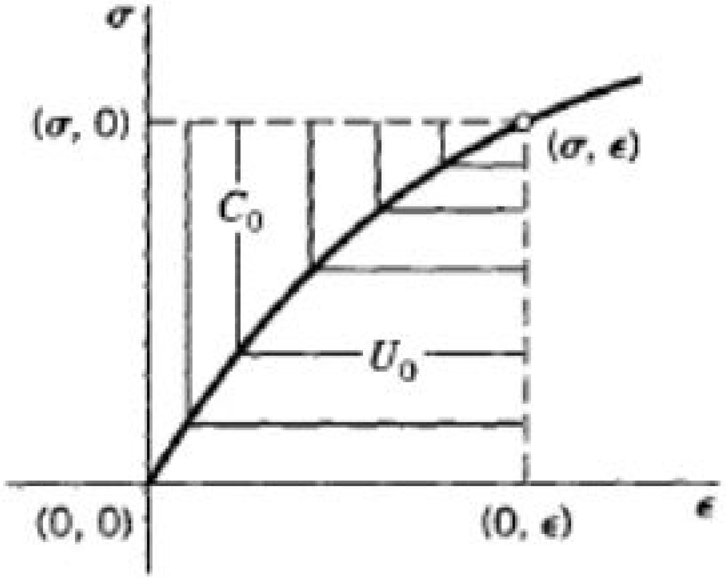
$$U_i = U_e = f(P_1, P_2, \ldots\ldots, P_n)$$

- If any one of the external forces, say P_j, is increased by a differential amount dP_j, the internal work will also be increased, such that the strain energy becomes

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_j} dP_j$$

- application of the loads $P_1, P_2, \ldots \ldots, P_n$ causes dP_j to move through the displacement Δ_j so that now the strain energy becomes

$$U_i + dU_i = U_i + dP_j \Delta_j$$



$$\epsilon = \frac{\partial C_0}{\partial \sigma} \qquad \sigma = \frac{\partial U_0}{\partial \epsilon}$$

The displacement Δ_j in the direction of P_j is equal to the first partial derivative of the strain energy with respect to P_j (**Castigliano's second theorem**)

$$\Delta_j = \frac{\partial U_i}{\partial P_j}$$

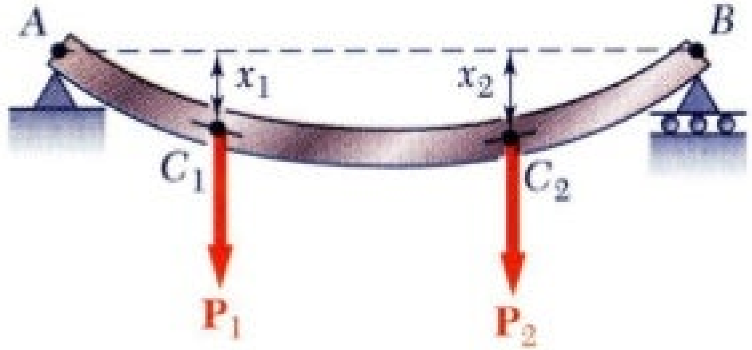
- **Castigliano's first theorem** relates the load P_j to the partial derivative of the strain energy with respect to the corresponding displacement, that is

$$P_j = \frac{\partial U_i}{\partial \Delta_j}$$

- *Castigliano's second theorem*: For an elastic structure subjected to *n* loads, the deflection Δ_j of the point of application of P_j can be expressed as

$$x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \quad \phi_j = \frac{\partial U}{\partial T_j}$$

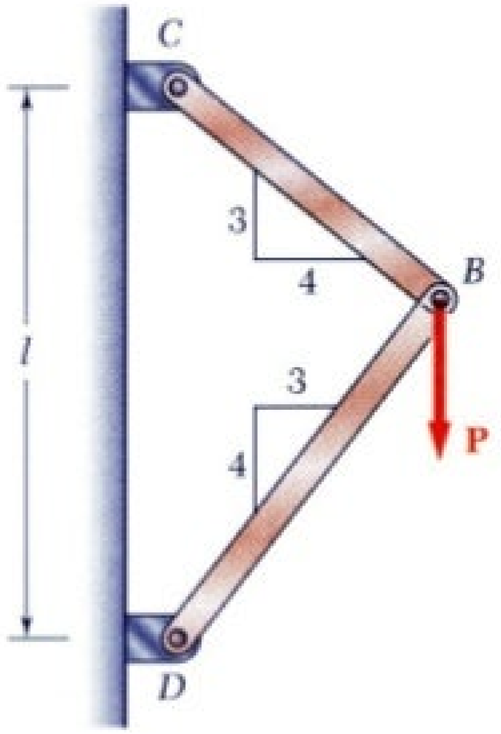
Deflections By Castigliano's Second Theorem



- In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad \Delta_j = \frac{\partial}{\partial P_j} \int_0^L \frac{M^2}{2EI} dx$$

$$\Delta_j = \int_0^L M \left(\frac{\partial M}{\partial P_j} \right) \frac{dx}{EI}$$

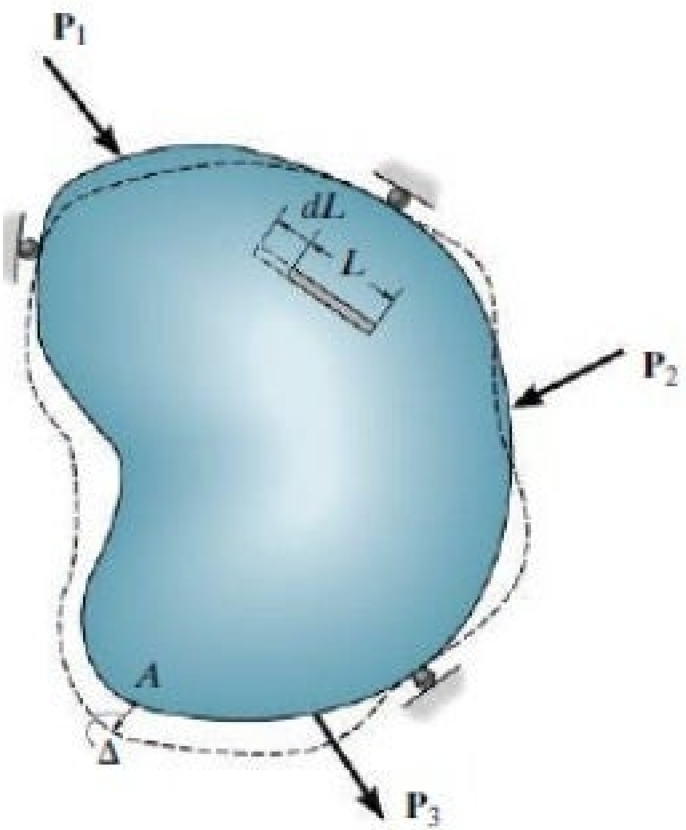


- For a truss,

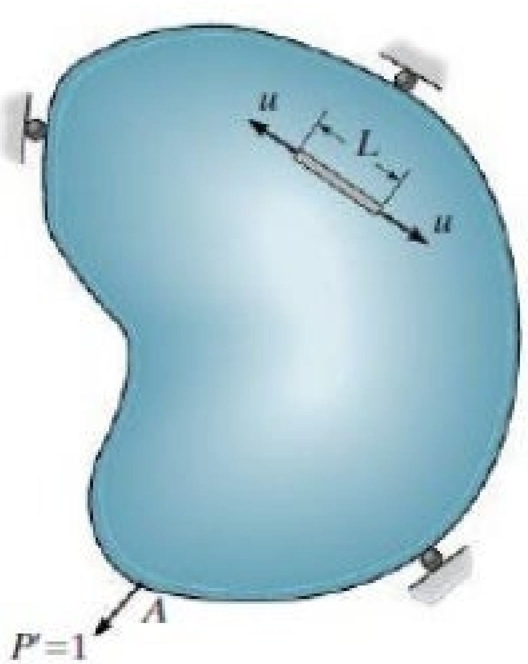
$$U_i = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E} \qquad \Delta_j = \frac{\partial}{\partial P_j} \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E}$$

$$\Delta_j = \sum_{i=1}^{n} F \left(\frac{\partial F_i}{\partial P_j} \right) \frac{L_i}{A_i E}$$

Principle of Virtual Work



- The principle of virtual work was developed by John Bernoulli in 1717,
- We will use it to obtain the displacement and slope at a point on a deformable body
- Consider the body to be of arbitrary shape and to be subjected to the "real loads" $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$
- It is assumed that these loads cause no movement of the supports
- To determine the displacement Δ of point *A* on the body, place an *imaginary* or "virtual" force **P′** on the body at point *A*, such that **P′** acts in the *same direction* as Δ
- **P′** to have a "unit" magnitude; that is, **P′** = 1 and create an internal virtual load **u** in a representative element or fiber of the body

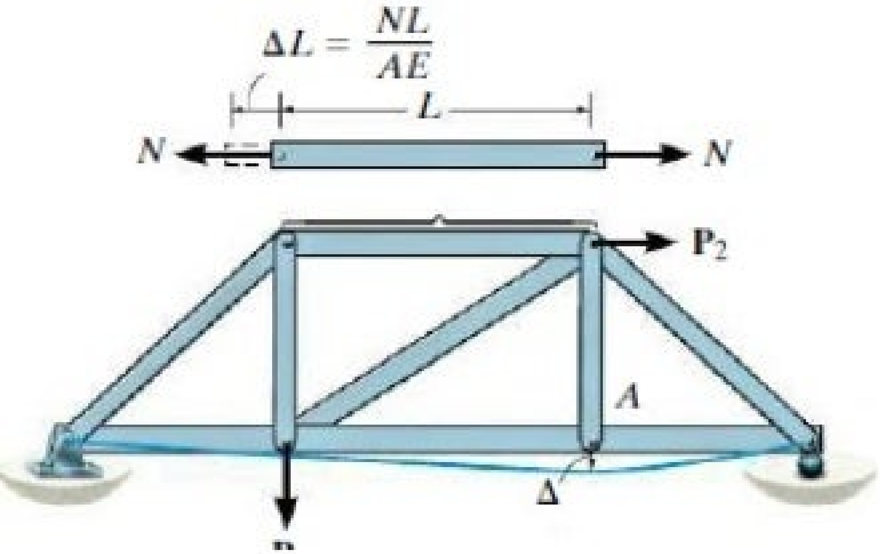


- The external virtual work is then equal to the internal virtual work done on all the elements of the body.

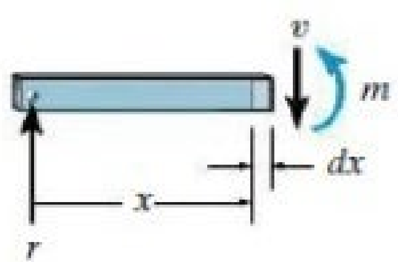
virtual loadings

$$1 \cdot \Delta = \Sigma u \cdot dL$$

real displacements



Deformation caused by	Strain energy	Internal virtual work
Axial load N	$\int_0^L \frac{N^2}{2EA} dx$	$\int_0^L \frac{nN}{EA} dx$
Bending moment M	$\int_0^L \frac{M^2}{2EI} dx$	$\int_0^L \frac{mM}{EI} dx$
Torsional moment T	$\int_0^L \frac{T^2}{2GJ} dx$	$\int_0^L \frac{tT}{GJ} dx$



- Virtual wok can be applied to elasticity problem. We can represent a small virtual change in quantity Q equal to δQ with the virtual displacement $\delta u, \delta v \; and \; \delta w$

- These virtual displacement rise a virtual strain $\delta \epsilon$ with the virtual displacement $\delta u, \delta v \; and \; \delta w$

$$\delta \epsilon = \frac{\partial}{\partial x} \delta(u, v, \mathrm{w})$$

- We can express strain energy density as

$$\delta U_0 dxdydz = \sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_z \delta \epsilon_z + \tau_{xy} \delta \gamma_{xy} + \delta \tau_{xz} \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}$$

$$\int \delta U_0 dV = \delta \int U_0 dV = \delta U$$

References

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6. Lecture Note(Energy methods)

BEAM AND ITS TYPES

BY

ARAVINDKUMAR B

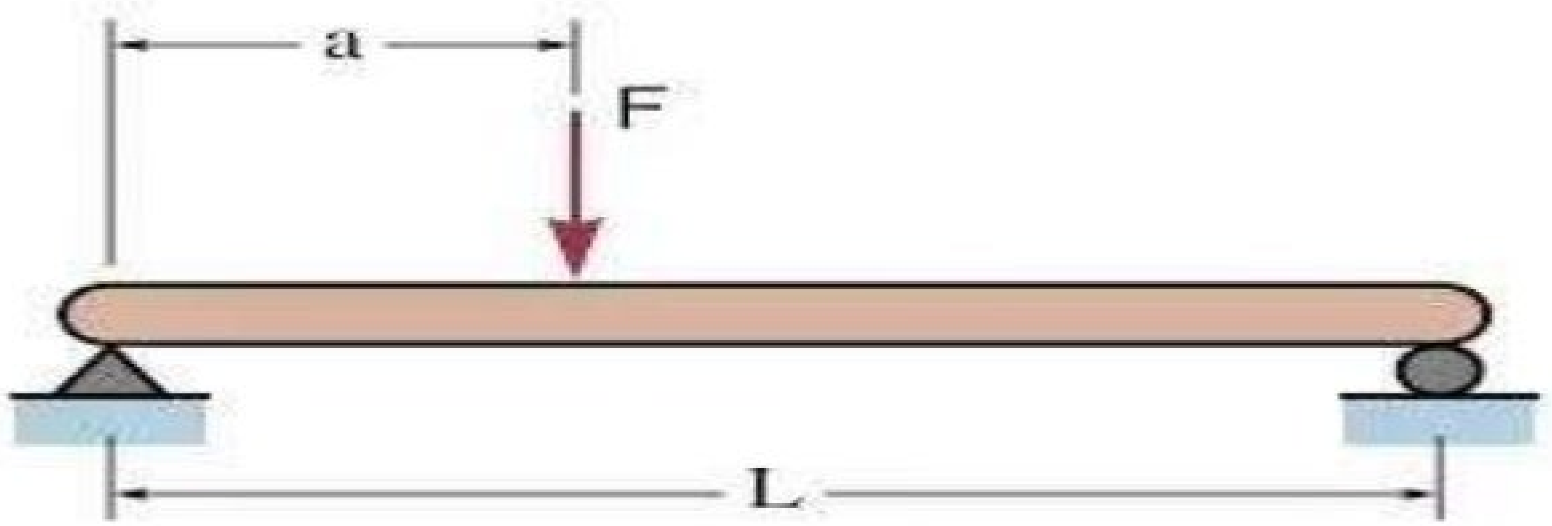
INTRODUCTION

- A beam is a structural element that primarily resists loads applied laterally to the beam's axis.
- Its mode of deflection is primarily by bending. The loads applied to the beam result in reaction forces at the beam's support points.
- The total effect of all the forces acting on the beam is to produce shear forces and bending moments within the beam, that in turn induce internal stresses, strains and deflections of the beam.
- Beams are characterized by their manner of support, profile (shape of cross-section), length, and their material.

Types of beams:
According to end support

Simply supported beam

- Simply supported beam is supported at both end. One end of the beam is supported by hinge support and other one by roller support.
- This support allow to horizontal movement of beam. It beam type undergoes both shear stress and bending moment.



Continuous beam

- This beam is similar to simply supported beam except more than two support are used on it.
- One end of it is supported by hinged support and other one is roller support.
- One or more supports are use between these beams.
  It is used in long concrete bridges where length of bridge is too large.

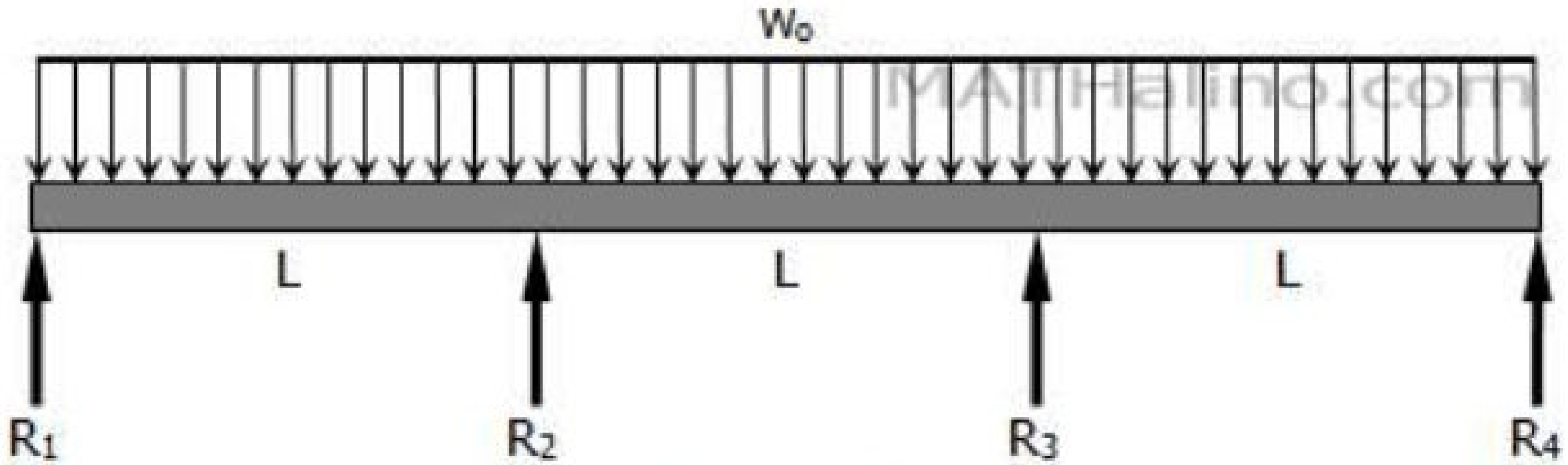
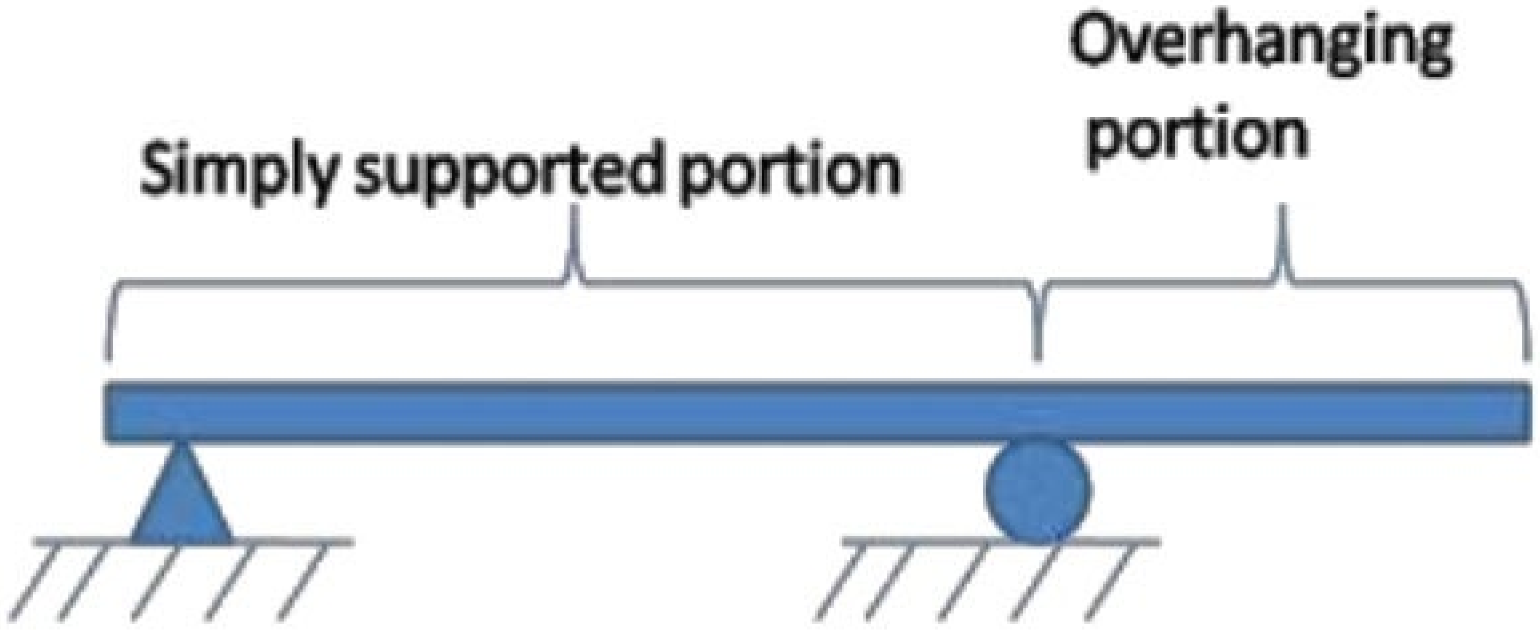


Figure P-829

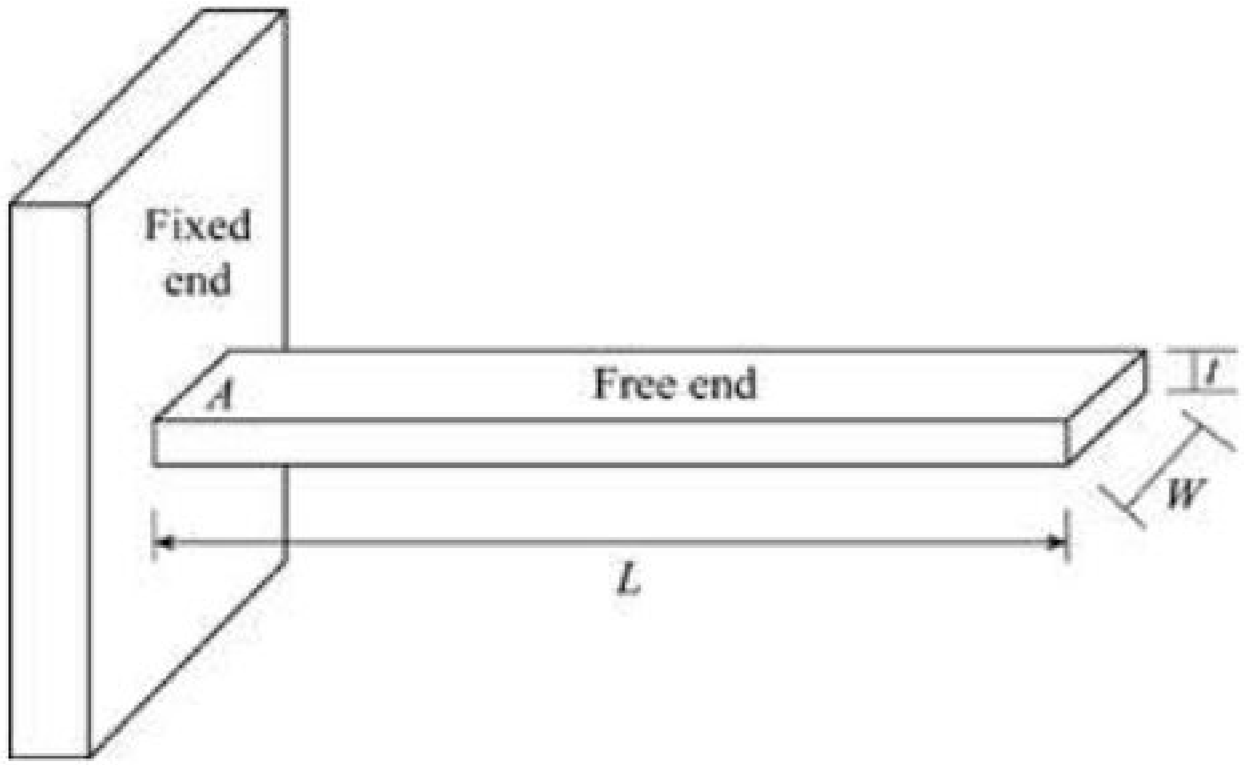
Over hanging beam

- Overhanging beam is combination of simply supported beam and cantilever beam.
- One or both of end overhang of this beam.
- This beam is supported by roller support between two ends.
- This type of beam has heritage properties of cantilever and simply supported beam.



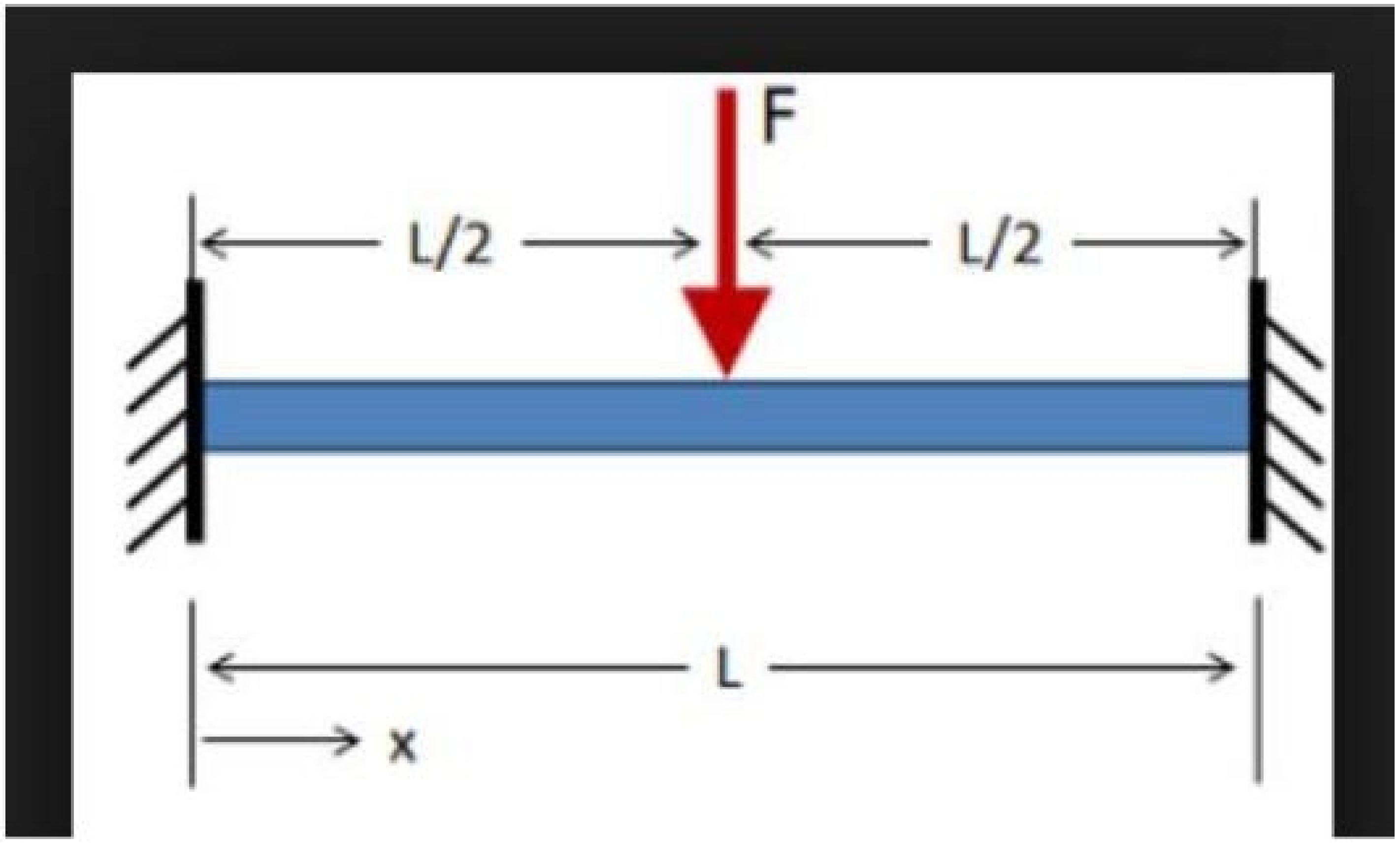
Cantilever beam

- Cantilever beams a structure member of which one end is fixed and other is free.
- This is one of the famous type of beam use in trusses, bridges and other structure member.
- This beam carry load over the span which undergoes both shear stress and bending moment.



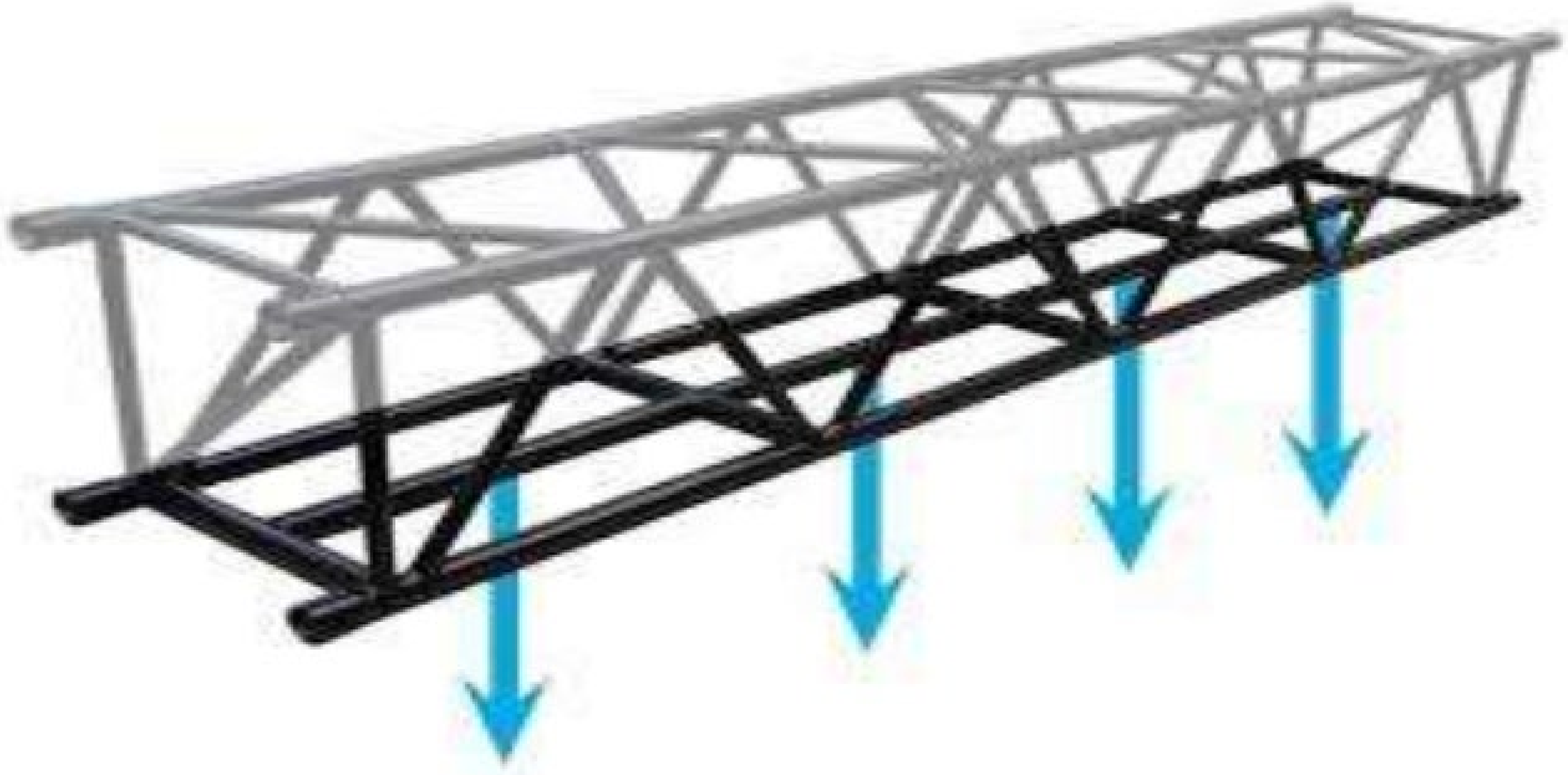
Fixed beam

- This beam is fixed from both ends.
- It does not allow vertical movement and rotation of the beam.
- It is only under shear stress and no moment produces in this beams.
- It is used in trusses, and other structure.



Trussed beam

- A beam strengthened by adding a cable or rod to form a Truss



According to shape of cross section

I-beam

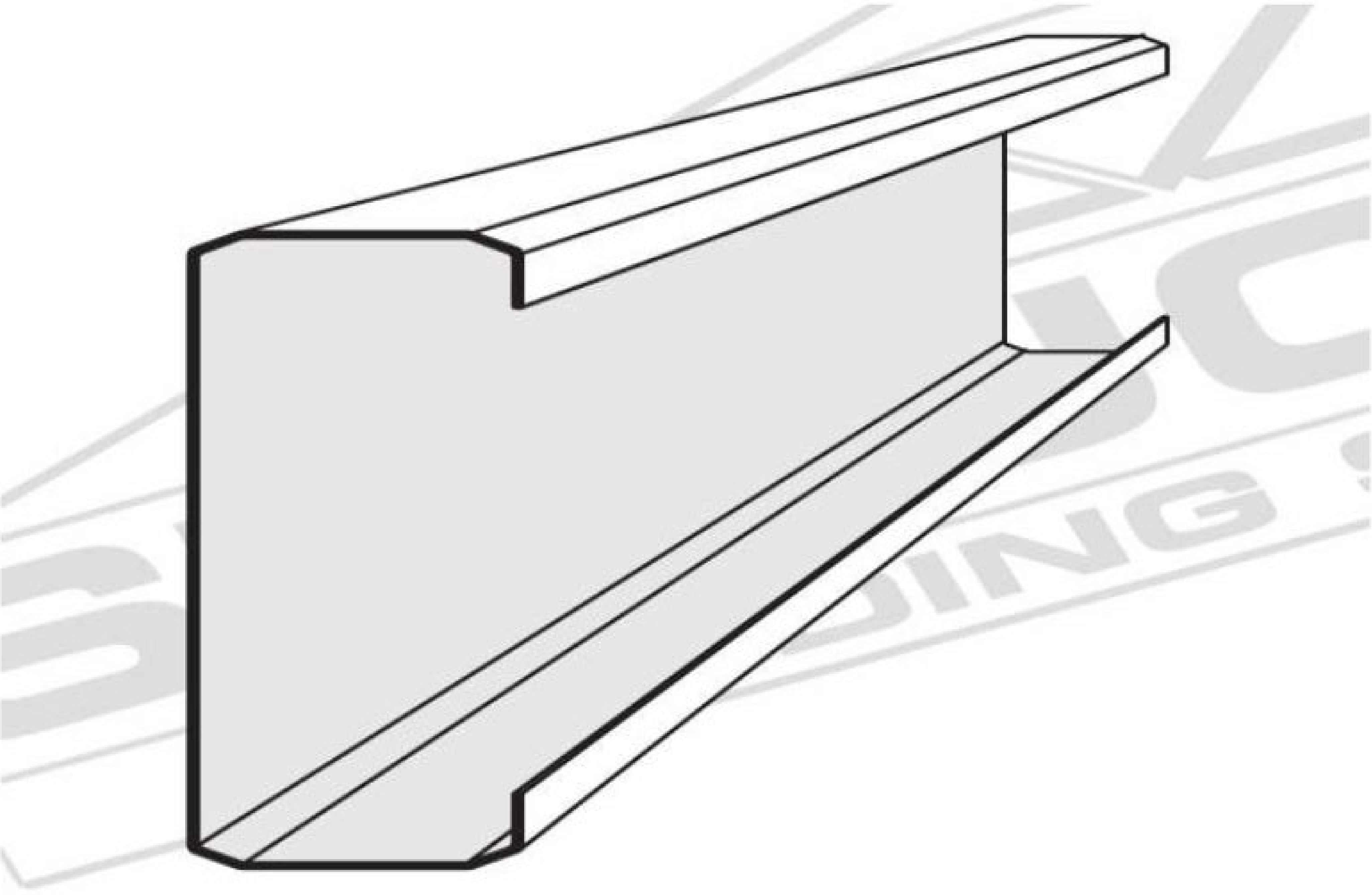
- This beam types have I cross section as shown in figure.
- It has high resistance of bending.

T-beam

- Beam with ″ T ″cross section

C-beam

- Beam with ″ C ″ cross section



According to equilibrium condition

Statically determinate beam

- A beam is called determinate beam if it can be analyze by the basic equilibrium condition.
- The support reaction can be found by using basic equilibrium condition.
- These conditions are Summation of all horizontal forces is zero.
- Summation of all vertical forces is zero.
- Summation of all moments is zero.
- Example: Simply supported beam, Cantilever beam etc.

Statically indeterminate beam

- If the beam cannot be analysis by using basic equilibrium condition, known as statically indeterminate beam.
- The end reaction find out by using basic equilibrium condition with combination of other conditions like strain energy method, virtual work method etc.
- Example: Continuous beam, fixed beam.

According to Geometry

- **Straight beam:** Beam with straight profile
- **Curved beam:** Beam with curved profile
- **Tapered beam:** Beam with tapered cross section

THANK YOU

Columns and thin cylinders

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AMIT
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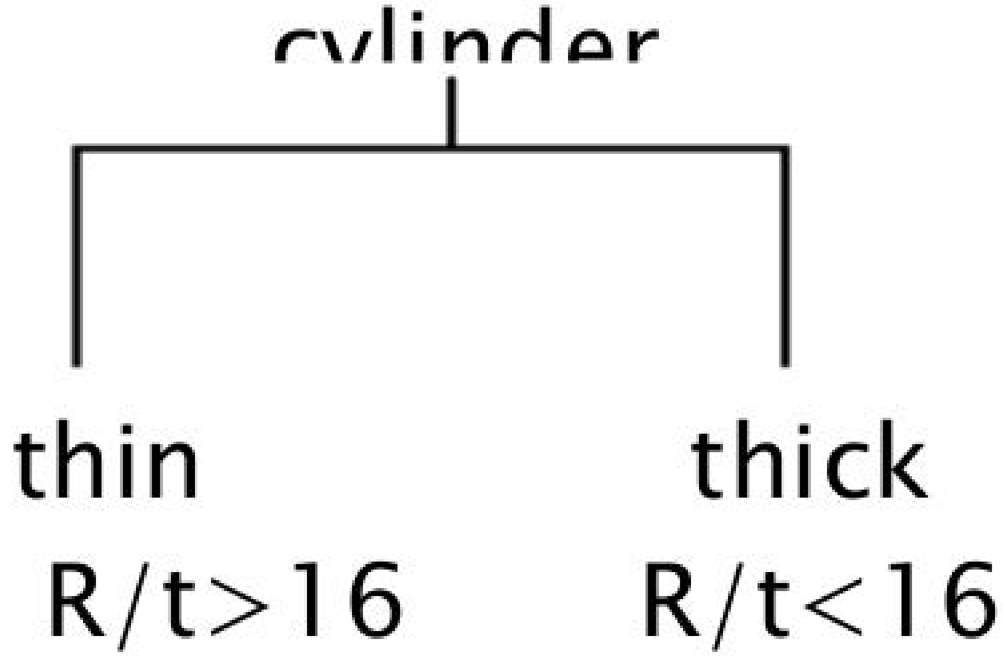

cylinder
thin
R/t>16
thick
R/t<16

Introduction

- The vessel such as boilers, compressed air receivers etc. are of cylindrical and spherical forms. These vessels are generally used for storing fluids (liquid or gas) under pressure.

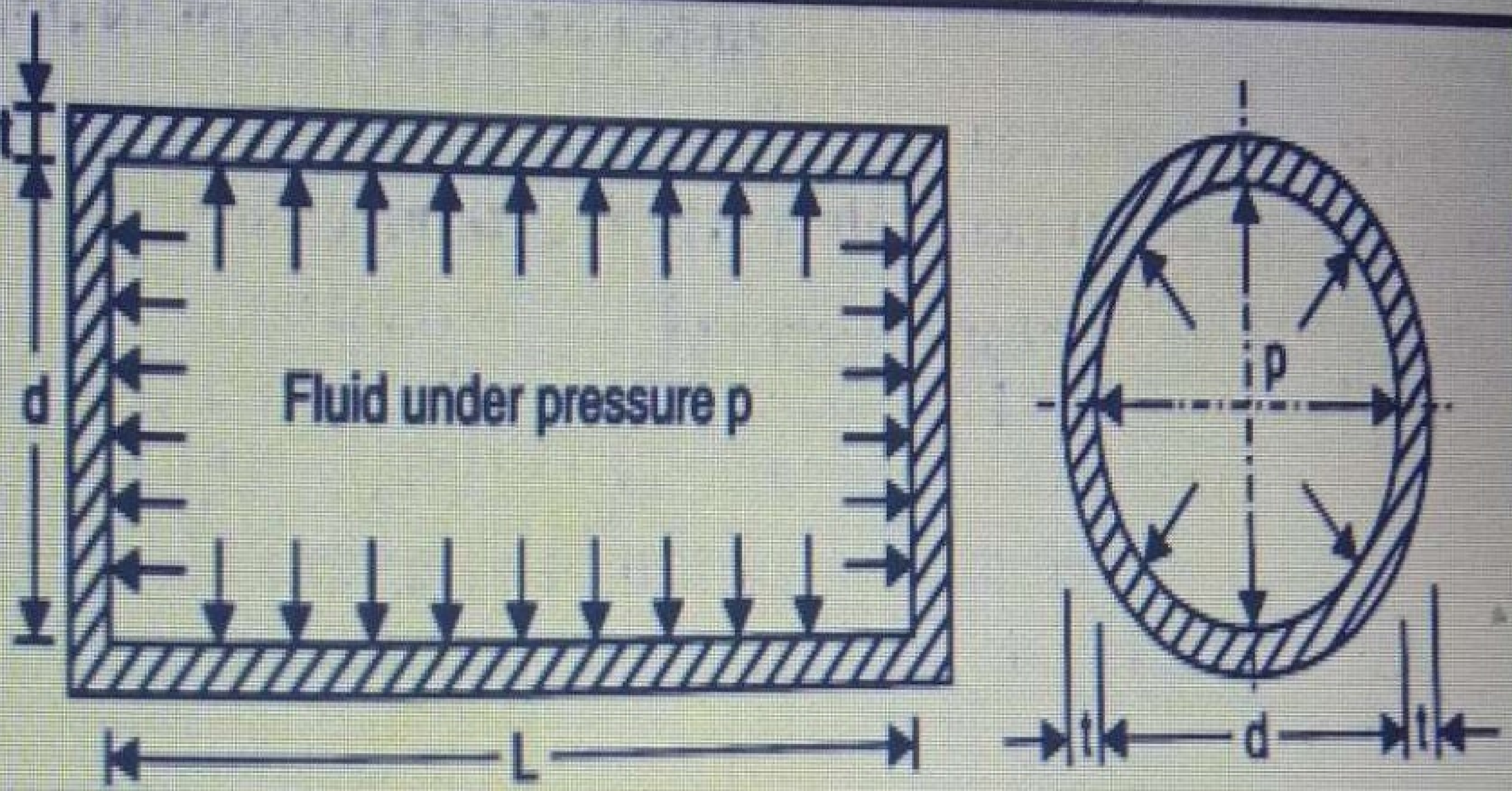


Fig. 17.1

Stresses in thin cylinders

1. Circumferential stress (or hoop stress)

    hoop stress=P*d/2t

2. Longitudinal stress (axial stress)

    axial stress=P*d/4t

Maximum shear stress

max. shear stress=P*d/8t

Q-1) a cylindrical pipe of dia. 1.5m & t=1.5cm is subjected to an internal fluid stress of 1.2N/mm^2. Determine 1.longotudinal stress

2.circumferential stress

3.max. shear stress

I= Πd^4/64

Euler's theory of columns

- Assumptions

1. Perfectly straight column and the axial load applied.
2. Uniform cross sectional of the column throughout its length.
3. Perfectly elastic, homogenous & isotropic material
4. The length of column is large as compared to its cross sectional dimension.

Euler's Crippling Load

$$P_E = \Pi^2 EI / l_e^2$$

S/No.	End Conditions of Column.	Crippling load in terms of.		Relation b/w effective length / Actual length
		Actual length	Effective length	
(1)	Both end hinged.	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$L_e = l.$
(2)	One end fixed & other is free.	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$L_e = 2l.$
(3)	Both end fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$L_e = l/2.$
(4)	One end fixed and other is hinged.	$\frac{2\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{l_e^2}$	$L_e = l/\sqrt{2}$

Q-2) A solid round bar 3m long & 5cm in dia. Is used as a strut with both ends hinged determining the crippling or collapsing load takes E=2*10^5N/mm^2

determining crippling load with the following condition 1.)one end is fixed and another is free

2.) both end are fixed

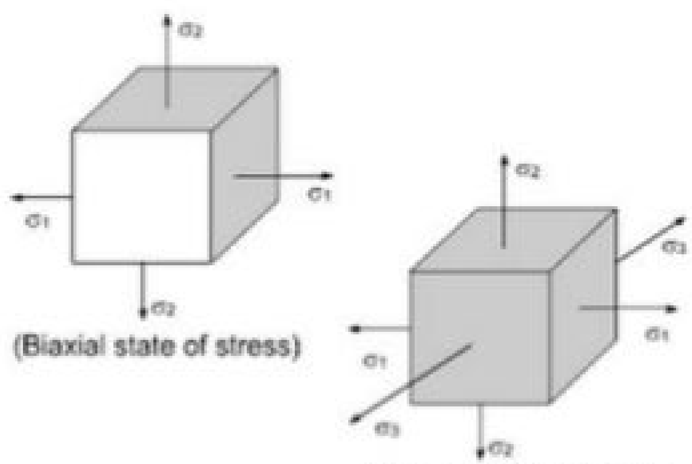
3.) one fixed and other is hinged

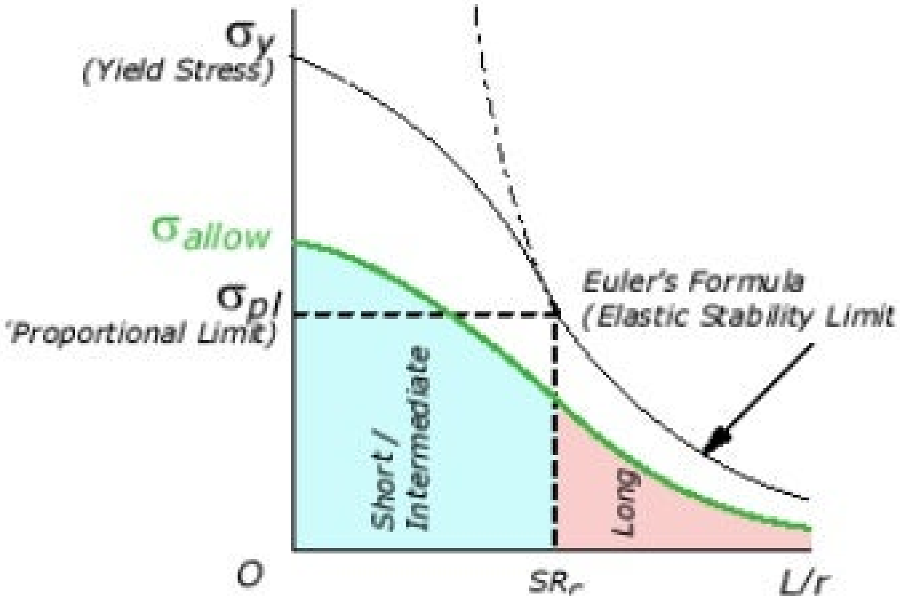
Slenderness ratio

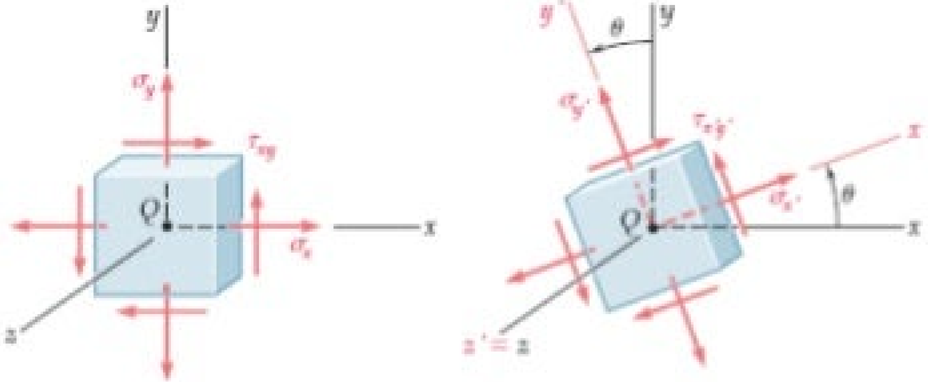
- The ratio of actual length of column to the least radius of gyration of the column is known as slenderness ratio

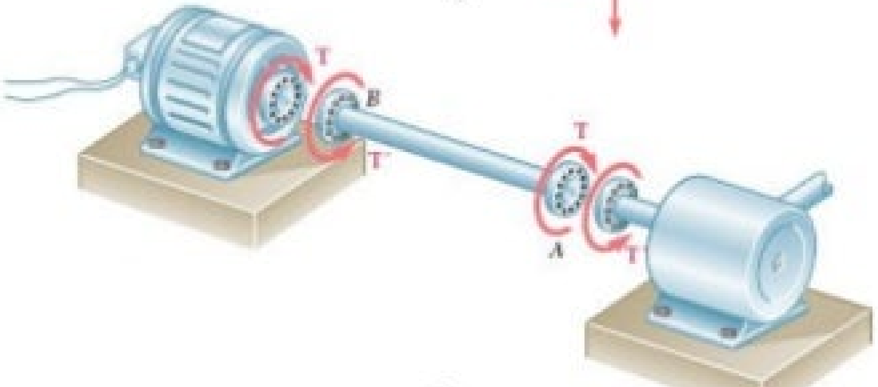
$$SR = \frac{\text{actual length}(L)}{\text{least radius of gyration}(K)}$$

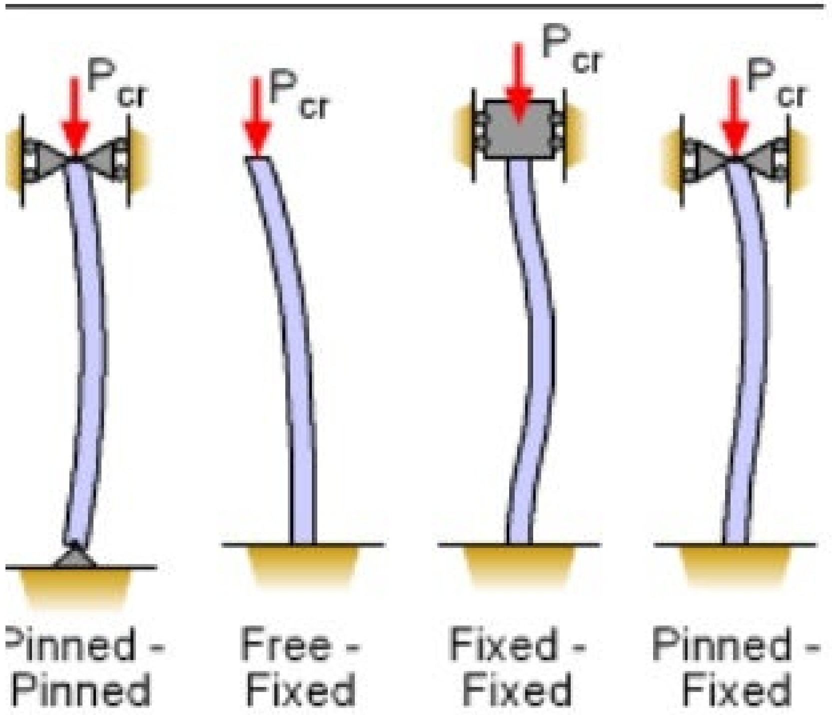
Mechanics of Solids-II (CE-301)


(Biaxial state of stress)
(Triaxial state of stress)






(b)



Dr. Aslam Faqeer Mohammad

Academic Calendar 2015

FALL SEMESTER / TERM	
Classes Start	27th July, 2015,
Classes End	13th November, 2015
Examinations Start	16th November, 2015
Examinations End	5th December, 2015+
Announcement of Results	Latest by 24th December, 2015
Winter Vacations	7th to 27th December, 2015

Time Table for Section-A

TIME TABLE FOR BACHELORS DEGREE PROGRAMMES

F/QSP 11/04/05

DEPARTMENT OF CIVIL ENGINEERING
TIME TABLE FOR: T.E. (Civil) BATCH: 2012-2013 Fall 2015
EFFECTIVE DATE: 27th July 2015 (Version 1.0)

SECTION –A1						Room No:
DAYS/ PERIODS	Monday (R6)	Tuesday (R8)	Wednesday (R2)	Thursday	Friday* (R8)	Sat/ Spare day
08:30 – 09:20	CE-305-A	CE-301-B	MT-443	--	CE-302 (PR)	
09:20 – 10:10	CE-305-A	CE-301-A	CE-316-B	--		
10:10 – 11:00	MT-443	CE-305-B	CE-302-A	--	CE-302-B	
11:00 – 11:30*	Recess					
11:30 – 12:20	CE-301-A	MT-443	CE-316 (DC)	--	CE-316-A	
12:20 – 13:10	CE-316-A	CE-302-B		--	--	
13:10 – 14:00*	Interval for Lunch / Prayer					
14:00 – 14:50	CE-301 (DC)	CE-305 (PR)	--	--	--	
14:50 – 15:40			--	--	--	
15:40 –16:30						

Time Table for Section-B

F/QSP 11/04/05

TIME TABLE FOR BACHELORS DEGREE PROGRAMMES

DEPARTMENT OF CIVIL ENGINEERING
TIME TABLE FOR: T.E. (Civil) BATCH: 2012-2013 Fall 2015
EFFECTIVE DATE: 27th July 2015 (Version 1.0)

SECTION –B1 DAYS/ PERIODS	Monday	Tuesday	Wednesday	Thursday	Friday*	Room No: **R4** Sat/ Spare day
08:30 – 09:20	CE-302-B	CE-305-A	CE-302-B	--	MT-443	
09:20 – 10:10	CE-316-A	CE-305-A	CE-302-A	--	MT-443	
10:10 – 11:00	CE-301-A	CE-301-A	CE-316-B	--	CE-305-B	
11:00 – 11:30*	Recess					
11:30 – 12:20	CE-316 (DC)	CE-301-B	CE-301 (DC)	--	CE-302 (PR)	
12:20 – 13:10		CE-316-A				
13:10 – 14:00*	Interval for Lunch / Prayer					
14:00 – 14:50	MT-443		CE-305 (PR)			
14:50 – 15:40						
15:40 –16:30						

Time Table for Section-C

F/QSP 11/04/05

TIME TABLE FOR BACHELORS DEGREE PROGRAMMES

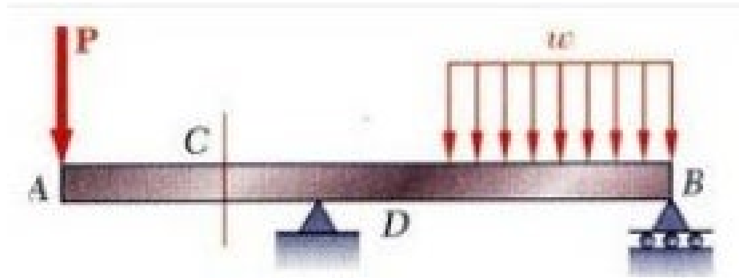
DEPARTMENT OF CIVIL ENGINEERING
TIME TABLE FOR: T.E. (Civil) BATCH: 2012-2013 Fall 2015
EFFECTIVE DATE: 27th July 2015 (Version 1.0)

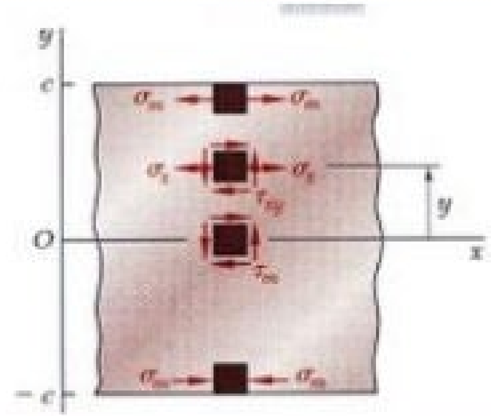
SECTION –C1						Room No: N1
DAYS/ PERIODS	Monday	Tuesday	Wednesday	Thursday	Friday*	Sat/ Spare day
08:30 – 09:20	--	MT-443	CE-301-B	--	CE-305-A	
09:20 – 10:10	CE-301-A	MT-443	CE-301-A	--	CE-305-A	
10:10 – 11:00	CE-302-B	CE-316-B	MT-443	--	CE-302-A	
11:00 – 11:30*	Recess					
11:30 – 12:20	CE-302 (PR)	CE-301 (DC)	CE-302-B	--	CE-305-B	
12:20 – 13:10			CE-316-A	--	CE-316-A	
13:10 – 14:00*	Interval for Lunch / Prayer					
14:00 – 14:50	CE-305 (PR)	CE-316 (DC)				
14:50 – 15:40						
15:40 –16:30						

Note: * For Friday only: **Recess:** 11:00 hours to 11:20 hours **Lunch/ Prayer Time**: 13:00 hours to 14:30 hours

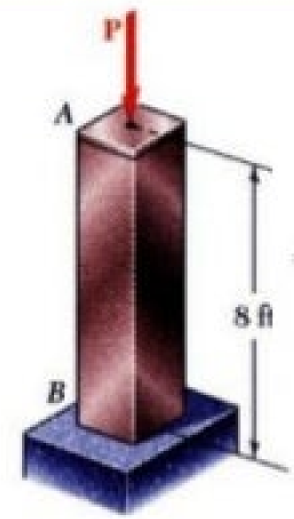
Mechanics of Solids

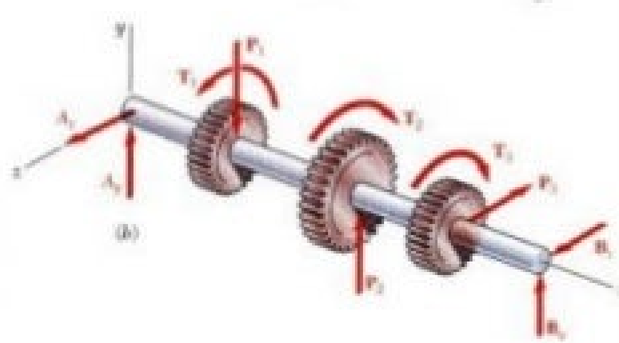
Mechanics of Solids-I

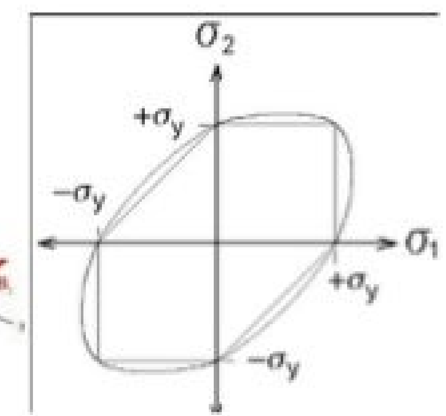




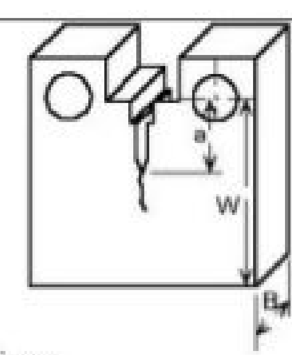
Mechanics of Solids-II



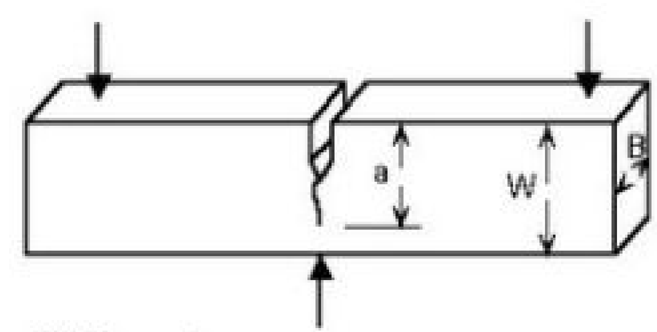




Advance Mechanics of Solids



CT-specimen



SENB-specimen

CE-301-Mechanics of Solids-II

Tentative Course Outlines

1. **INTRODUCTION** (Basic Concepts and Terminologies)
   I. Overview
   II. Types of stresses
   III. Determinate/Indeterminate Structures

2. **ANALYSIS OF STRESSES AND STRAINS DUE TO**
   I. Axial/Bending/Twisting and Shear Forces
   II. Combine effect of axial, bending and twisting forces

3. **STRUTS AND COLUMNS**
   I. Euler, Rankine and other formulae for buckling loads
   II. Stability analysis of column under eccentric loading

4. **ELEMENTARY THEORY OF ELASTICITY**
   I. Equilibrium and compatibility equations
   II. Biaxial Stresses
   III. Stress and deformation relationship
   IV. Stress transformation

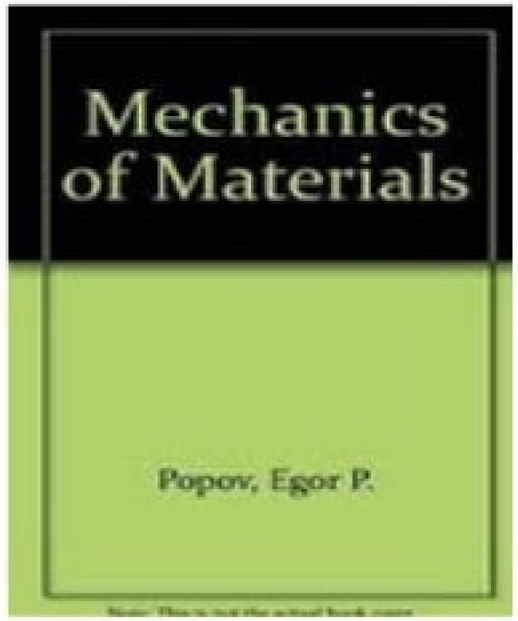
5. **THEORIES OF FAILURES**
   I. Tresca's Yield Criterion
   II. W.Rankine's Criterion
   III. Von Mises Yield Criterion

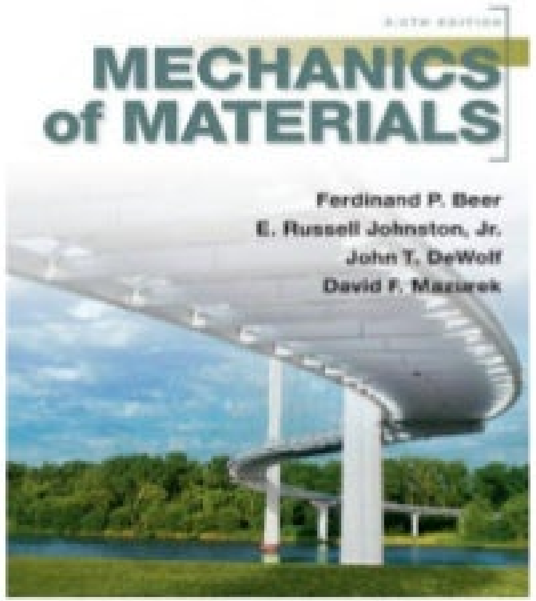
6. **COMPUTER BASED ANALYSIS**

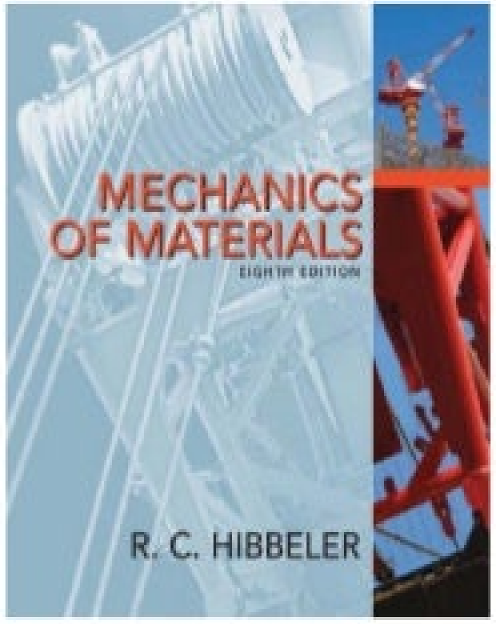
Grading

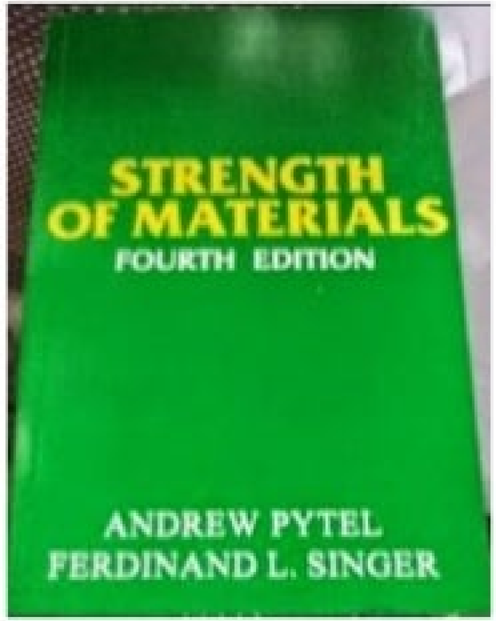
- Homework/class assignments
- E-Tests
- Midterm (9th week)
- Final Exam

Text/Ref Books









Course Learning Outcomes (CLO’s)

1. To get an exposure of biaxial bending and its application
2. To understand the problem of stability of column and design of steel column
3. To get the knowledge of different types of failure theories and its applications in mechanics of solids

Terminologies

Stress: Stress is the intensity of the internal force on a specific plane passing through a point. Mathematically, stress can be expressed as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

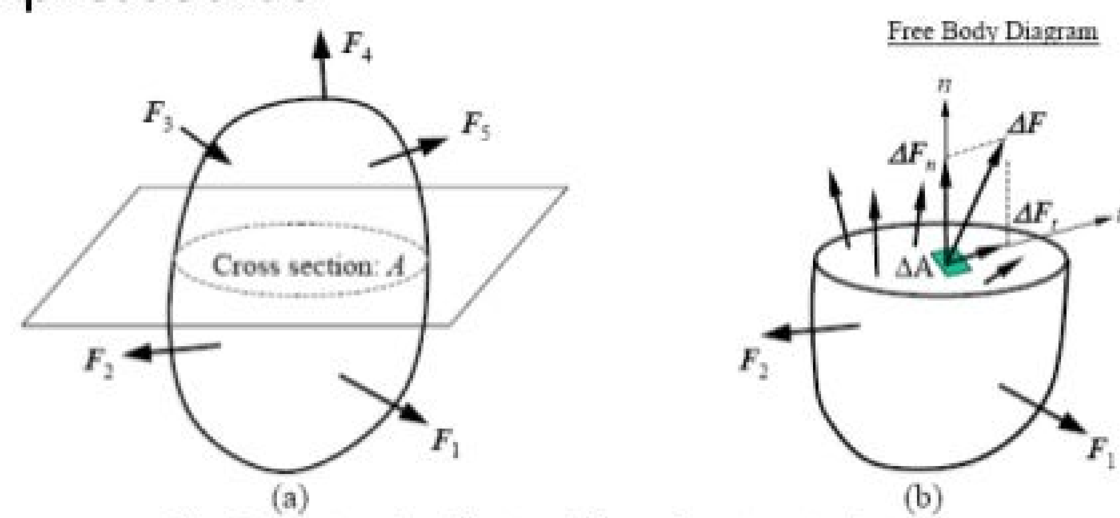
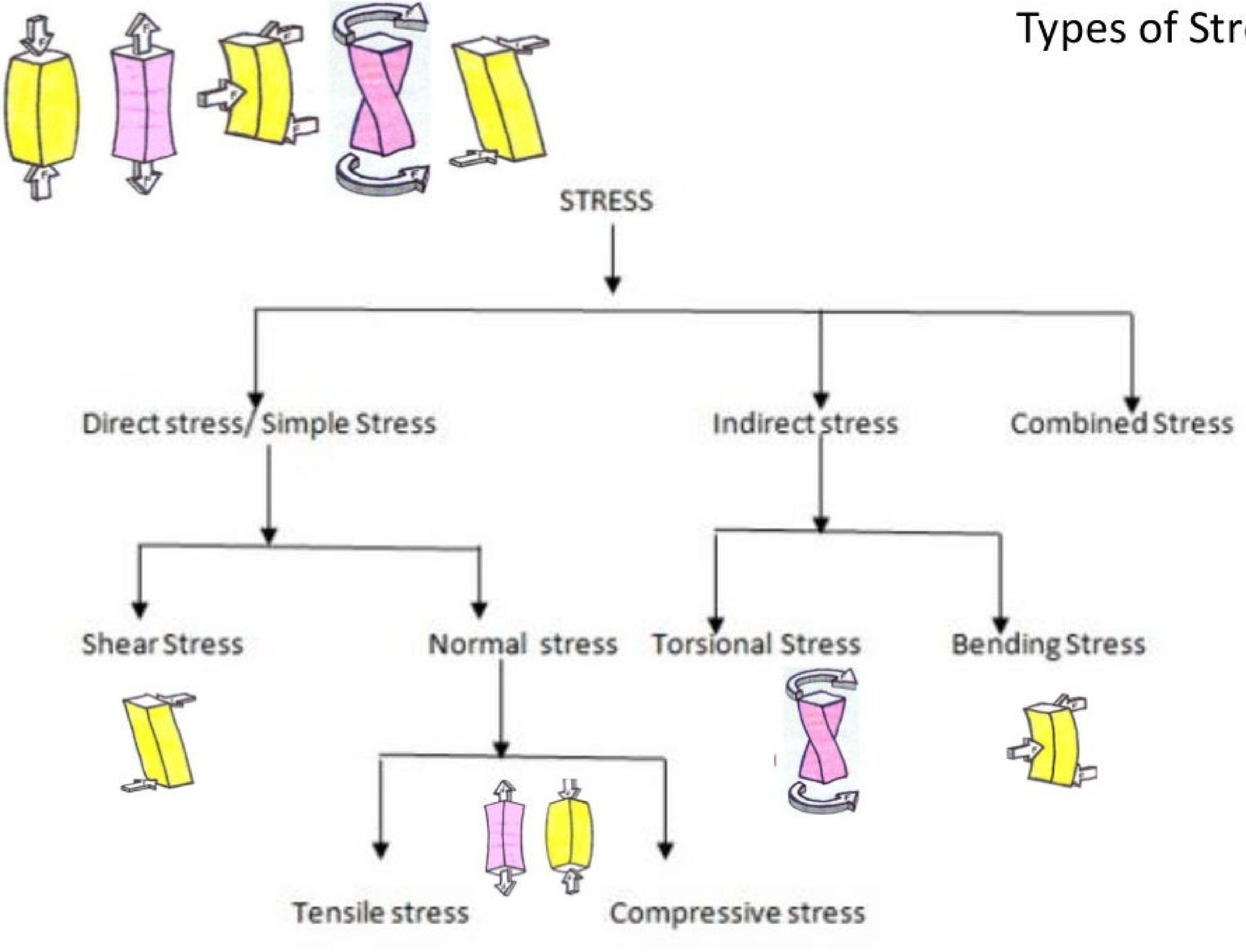
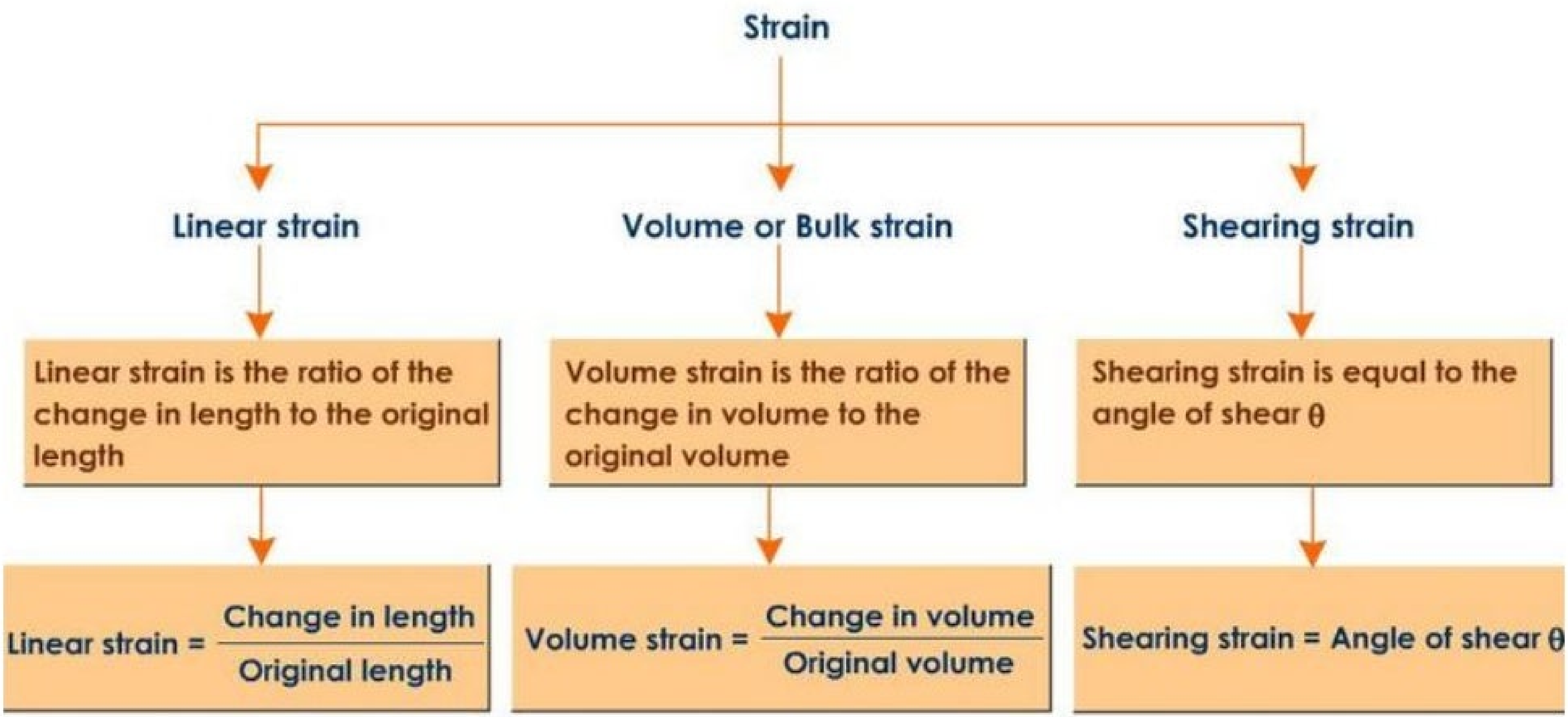


Fig. 2.1 External and internal forces in a structural member

Types of Stresses



Types of Strains

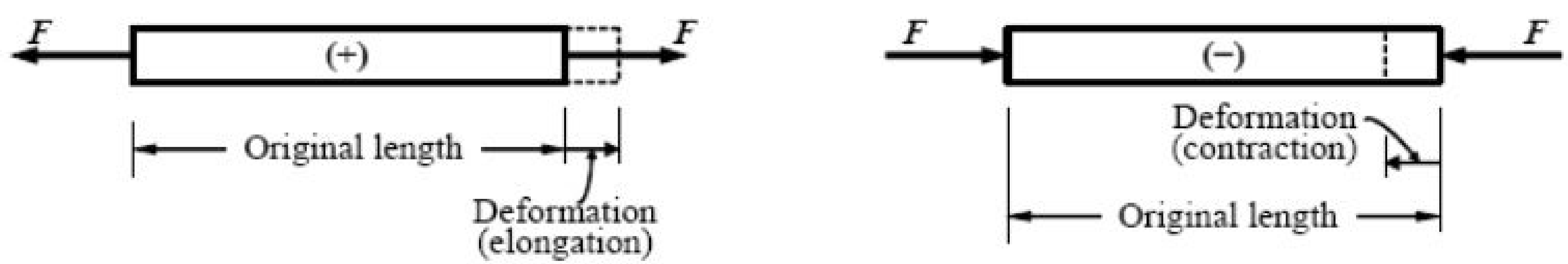


Terminologies

Normal Stress: The intensity of the force or force per unit area acting normally to section *A is called* ***Normal Stress, σ (sigma), and it is expressed as:***

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$

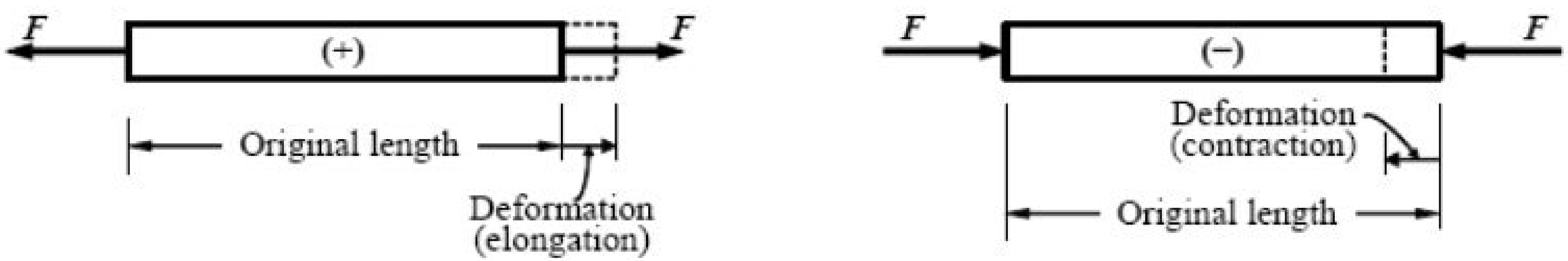
If this stress "pulls" on the area it is referred as ***Tensile Stress and defined as Positive. If it*** "pushes" on the area it is called ***Compressive Stress and defined as Negative.***



Shear Stress: The intensity or force per unit area acting tangentially to A is called ***Shear Stress, τ (tau), and*** it is expressed as:

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A}$$

Deformation: Whenever a force is applied to a body, its shape and size will change. These changes are referred as *deformations. These deformations can be thought of being either* ***positive*** **(elongation) or *negative (contraction) in sign as shown in Fig.***



Strain: The elongation (*+ve) or contraction (−ve) of a body per unit length is termed* ***Strain.***

THERMAL STRAIN

When the temperature of a body is changed, its *overall size will also change. In other words,* temperature change may cause the dimension or shape change in the material. More specially, if the temperature increases, generally a material expands. Whereas if the temperature decreases, the material will contract.

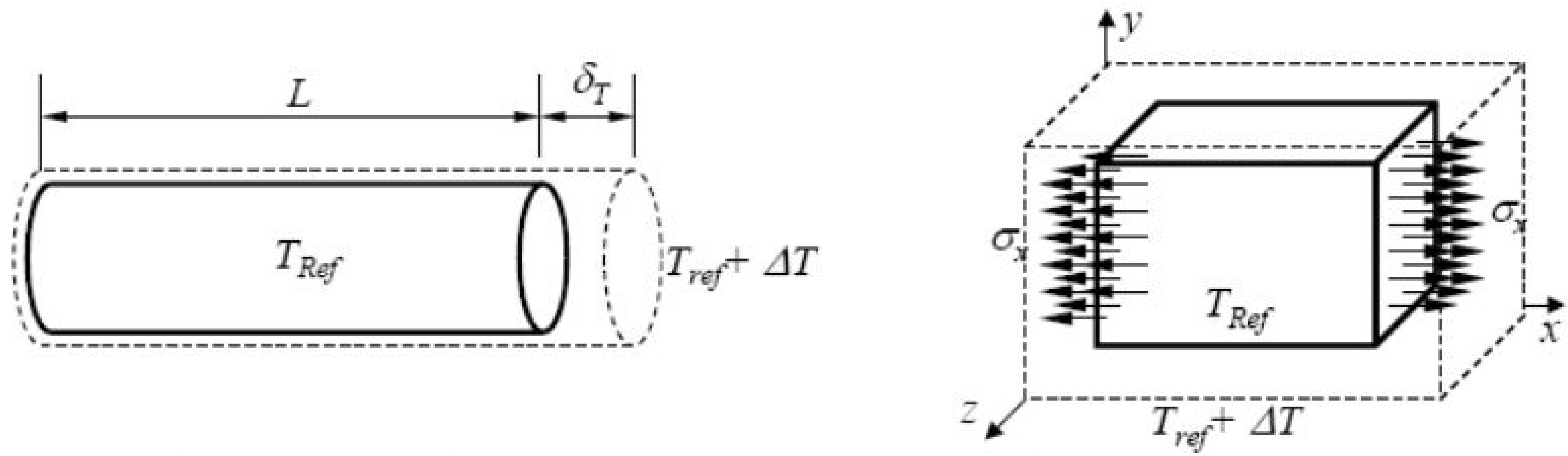
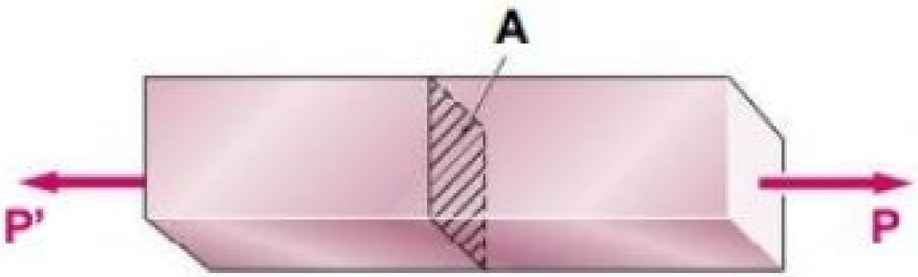


Fig. 2.6 Thermal and mechanical deformation

Thermal Strain

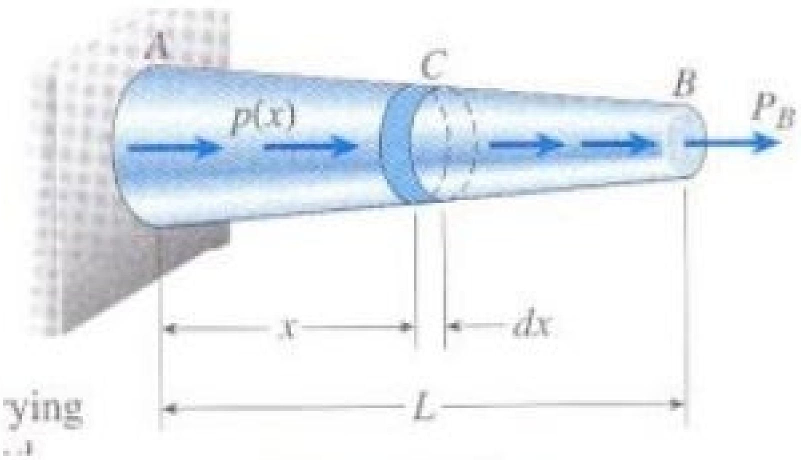
$$\varepsilon_{Thermal} = \frac{\delta_T}{L} = \alpha \cdot \Delta T$$

- Homogenous: material is the same throughout the bar
- Cross-section: section perpendicular to longitudinal axis of bar



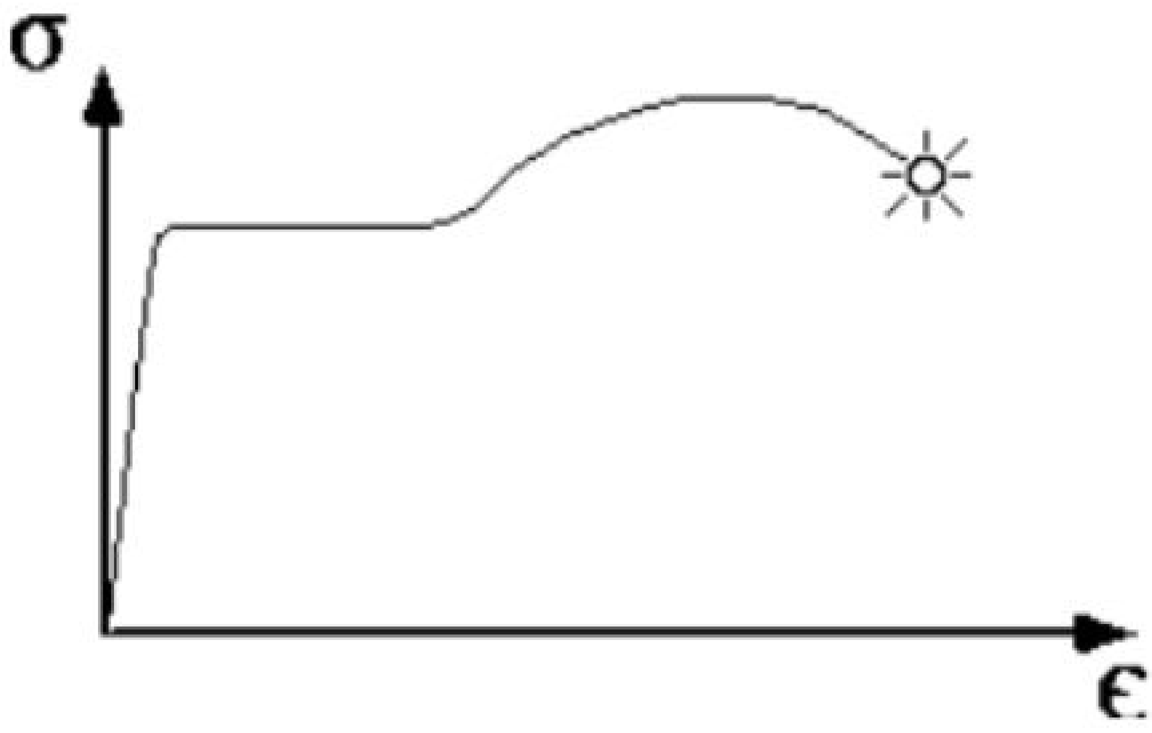
- Prismatic: cross-section does not change along axis of bar

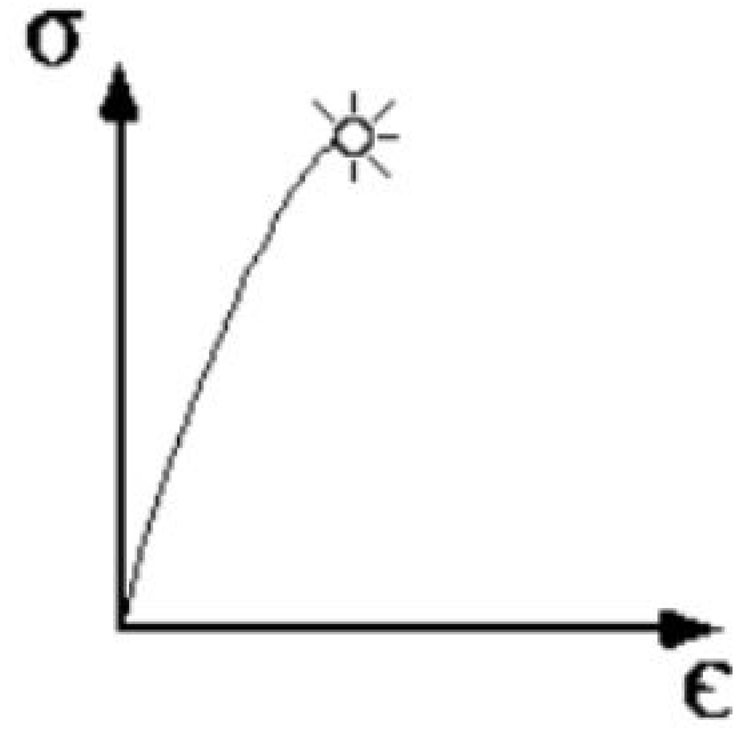
Prismatic



Non-Prismatic

Ductile Vs Brittle





Notice that for the ductile material, show on the left, larger strains occur before ultimate failure. In reality this means that (a) the material has a chance to change its shape in order to redistribute loads, and (b) if redistributing the loads does not prevent failure, there is often adequate visual warning (sagging beams, etc.) before failure occurs. For these reasons, ductile failure is preferable to brittle failure.

Stress-Strain Relationship Hook's law

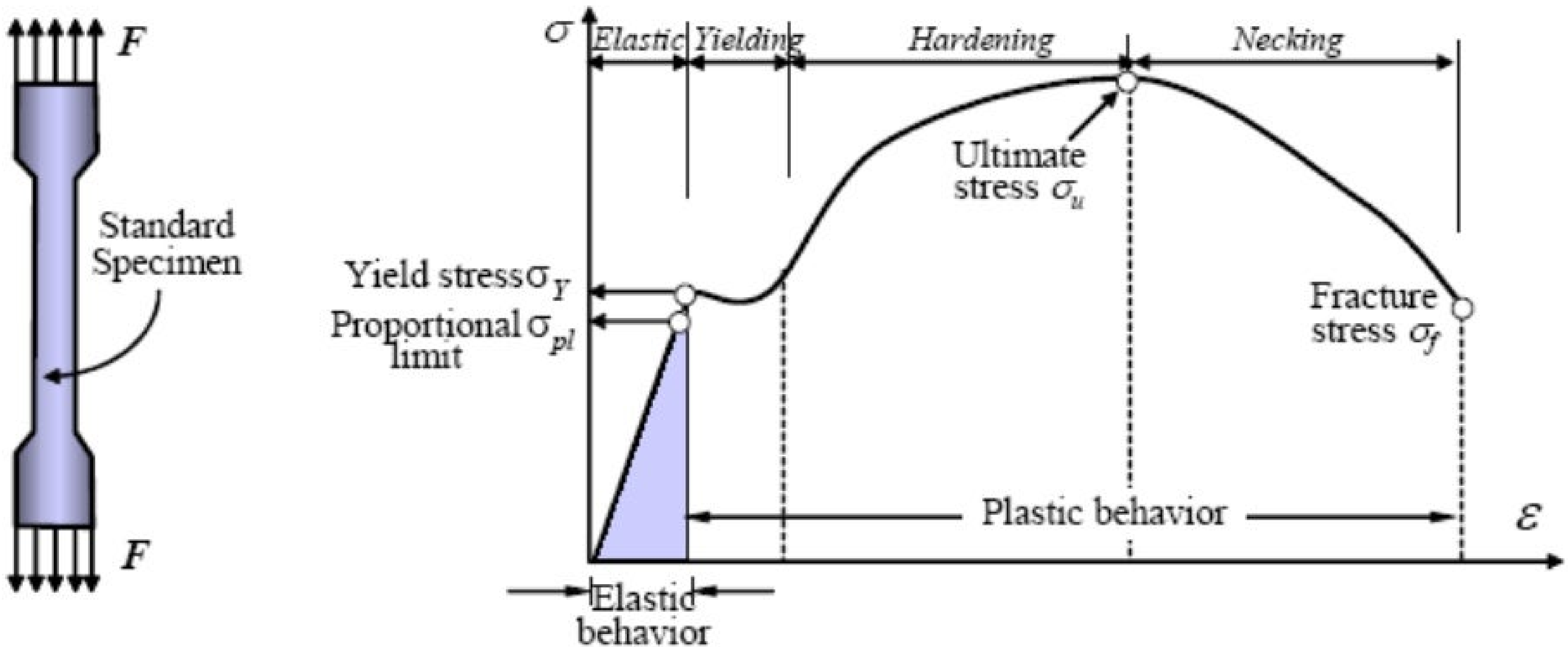


Fig. 2.4 Material test and *Stress-Strain Diagram*

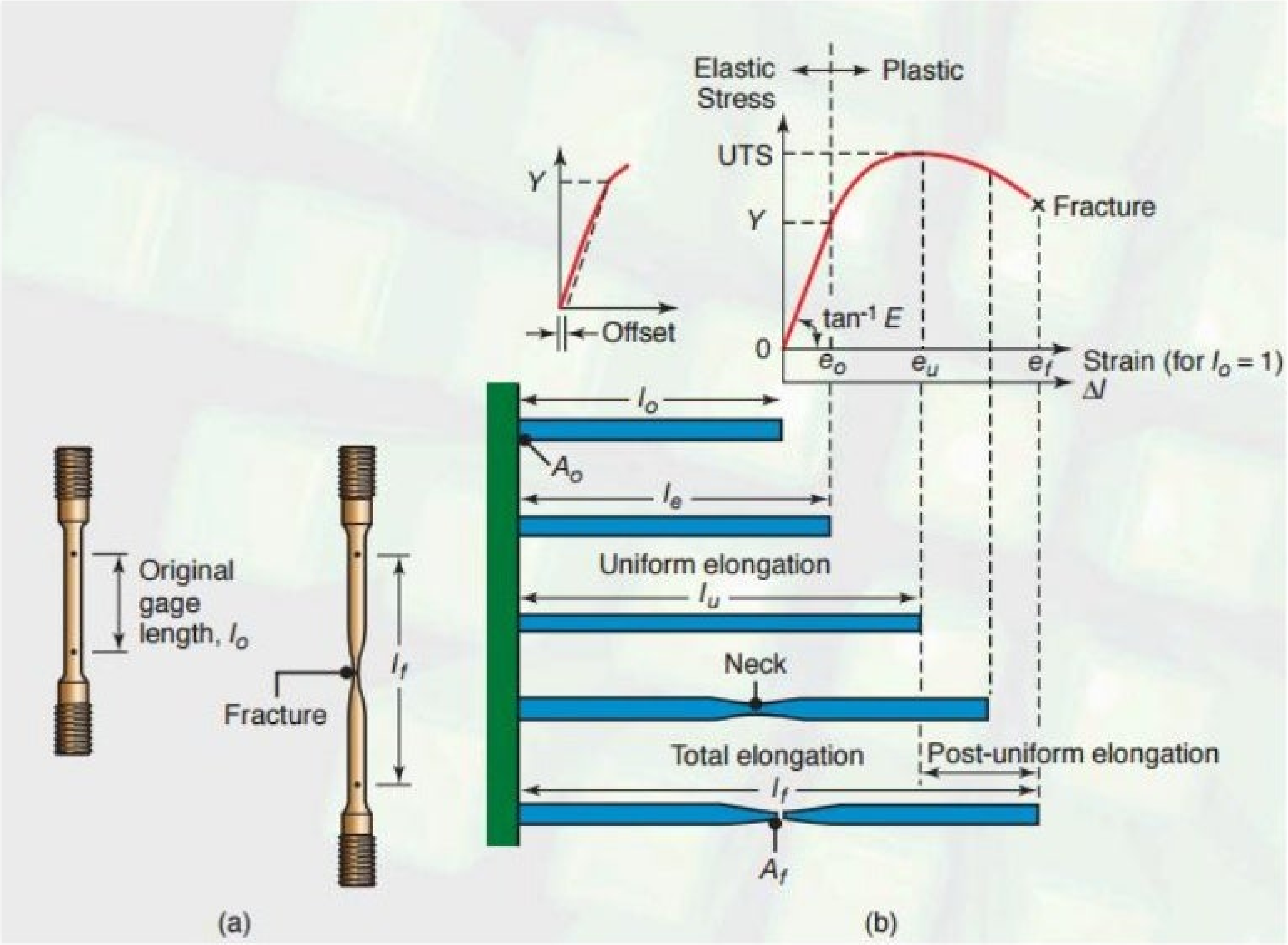
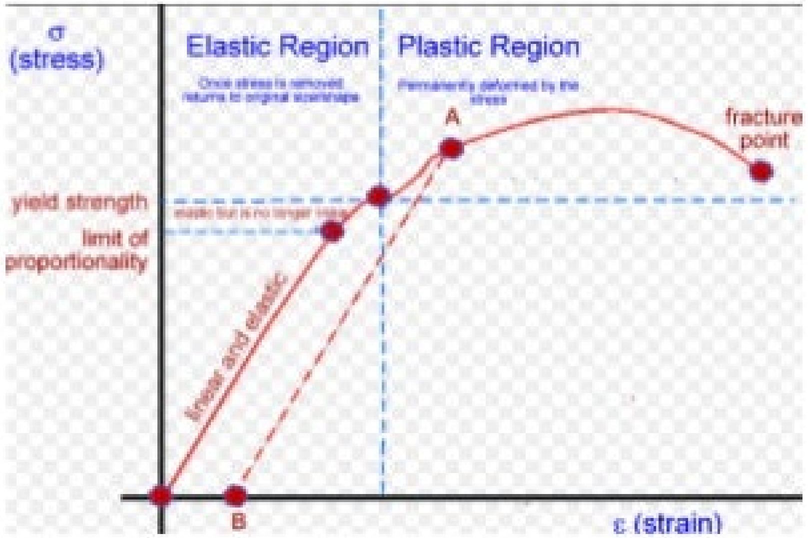


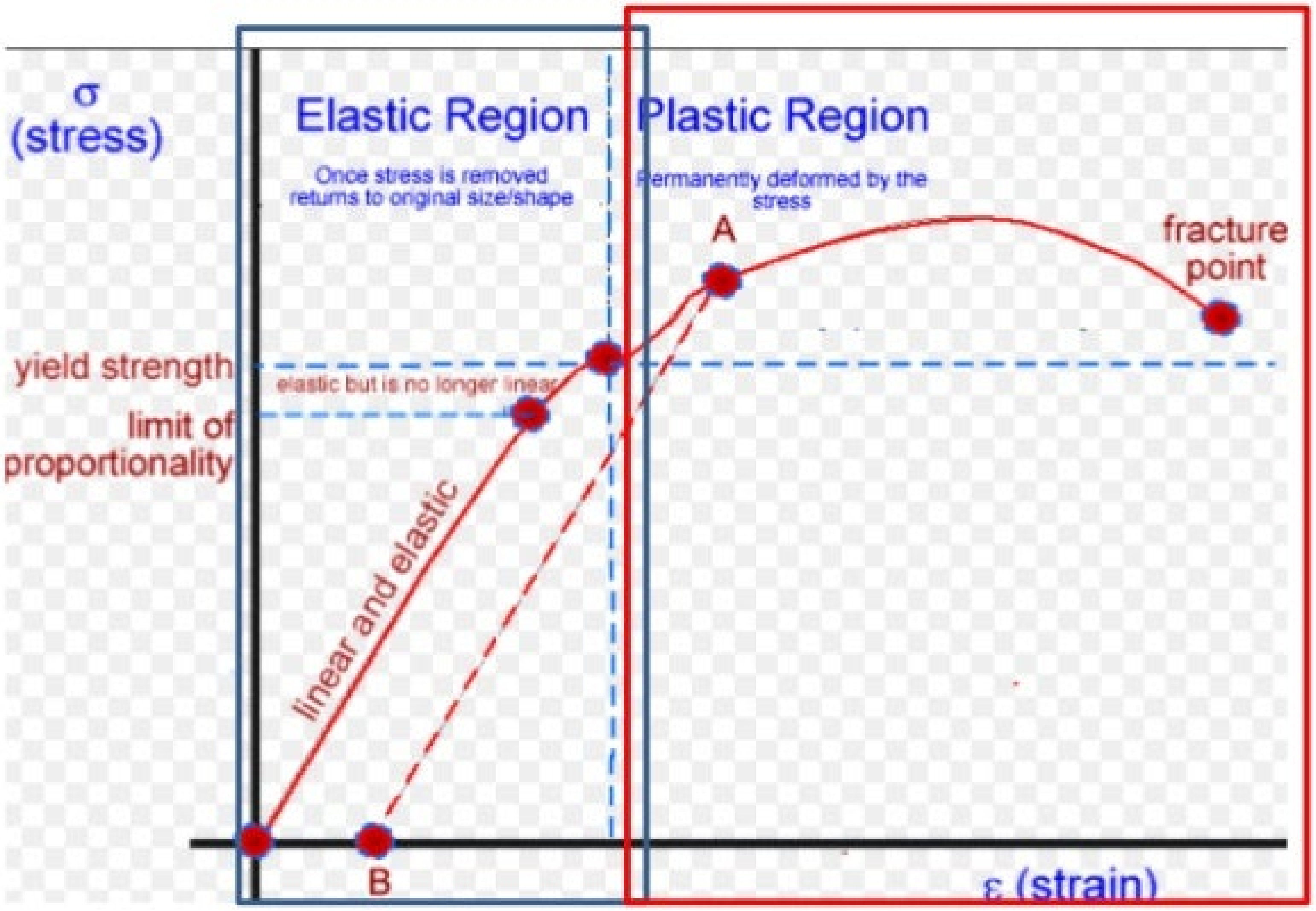
FIGURE: (a) Original and final shape of a standard tensile-test specimen. (b) Outline of a tensile-test sequence showing different stages in the elongation of the specimen.

Types of Analyses

There are two types of analyses (Linear/Nonlinear),

- **Linear static analysis (without inertial forces)**
- Linear dynamic analysis (with inertial forces)
- Nonlinear static analysis (without inertial forces)
- Nonlinear dynamic analysis (with inertial forces)





Types of Loads according to Member Behavior

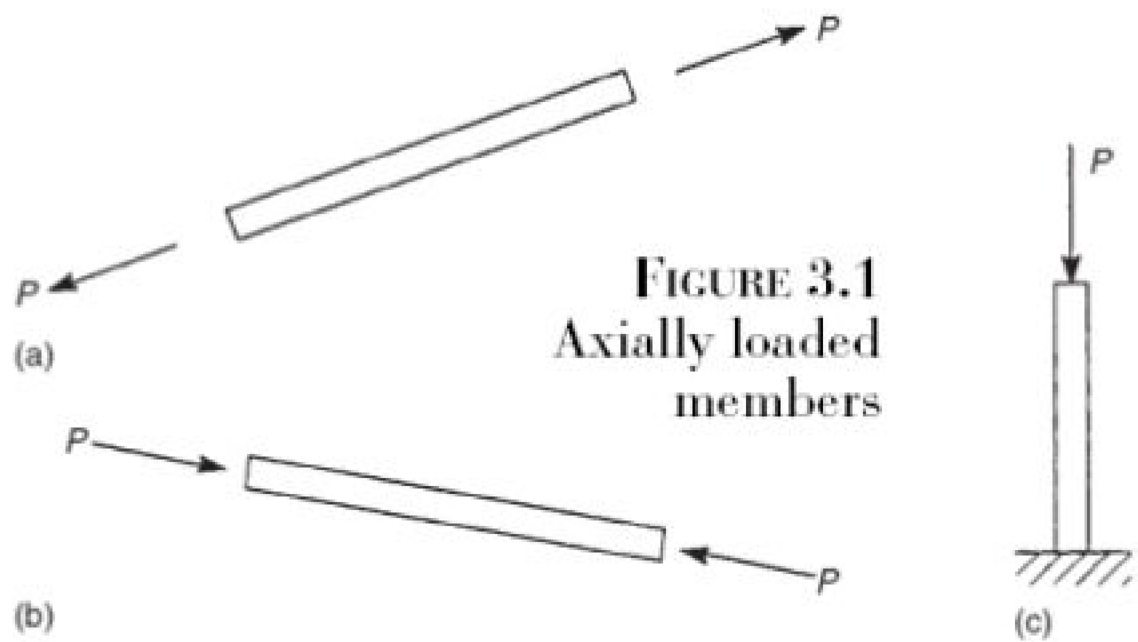


FIGURE 3.1 Axially loaded members

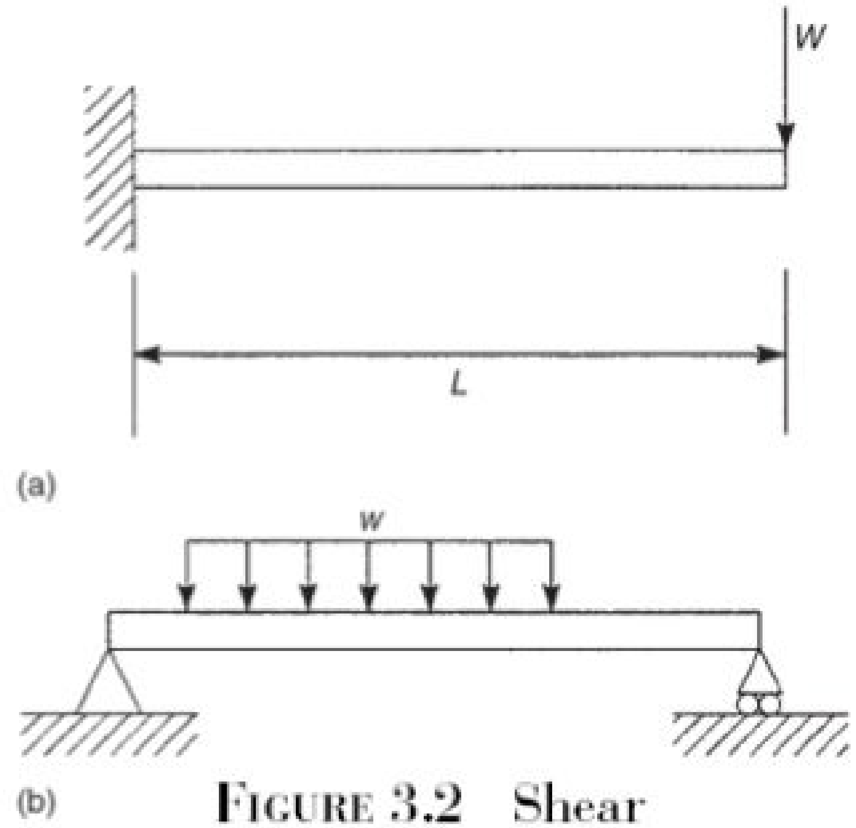


FIGURE 3.2 Shear loads applied to beams

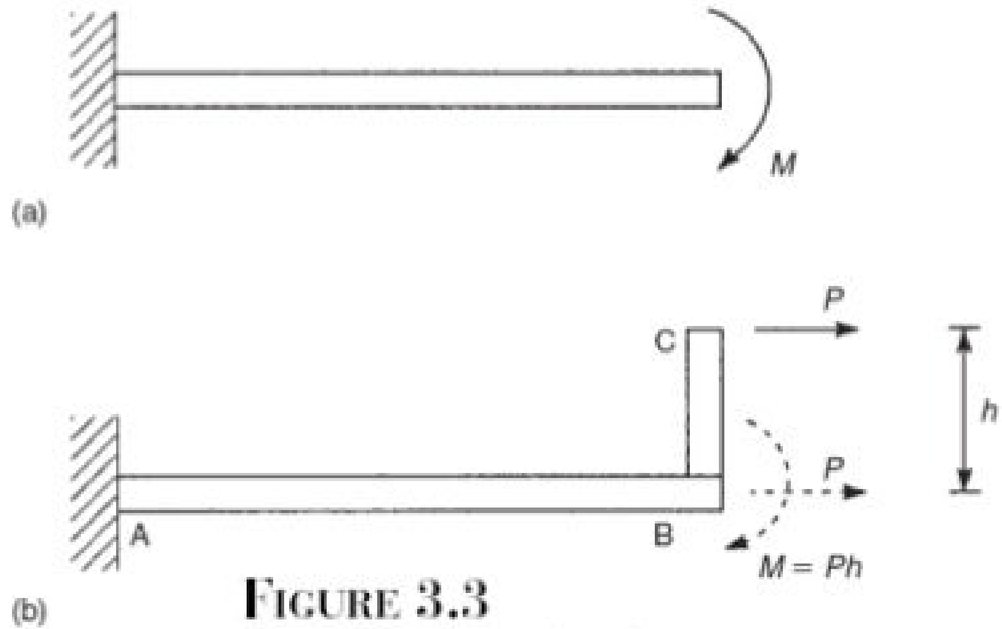


FIGURE 3.3 Moments applied to beams

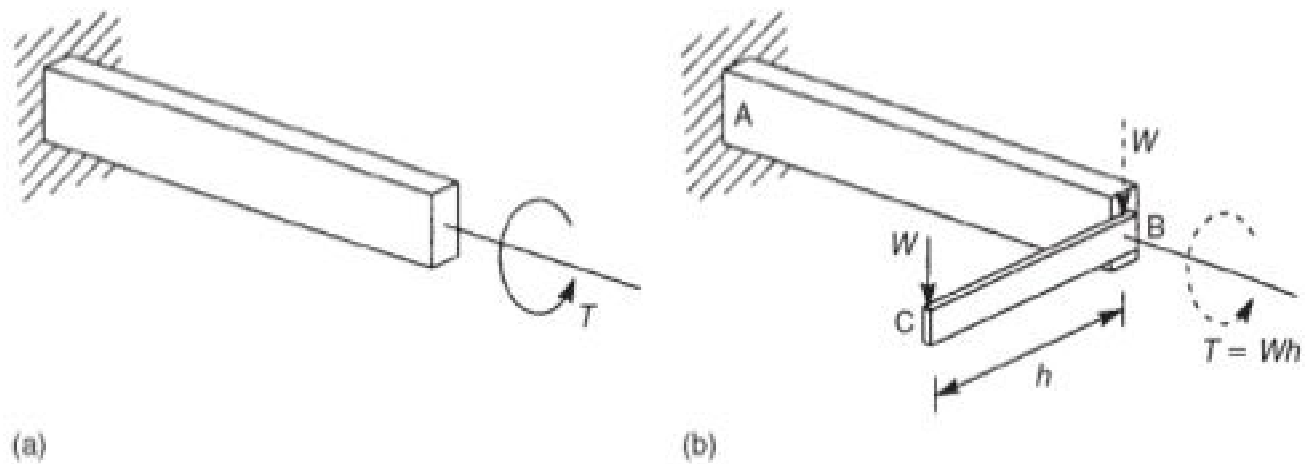
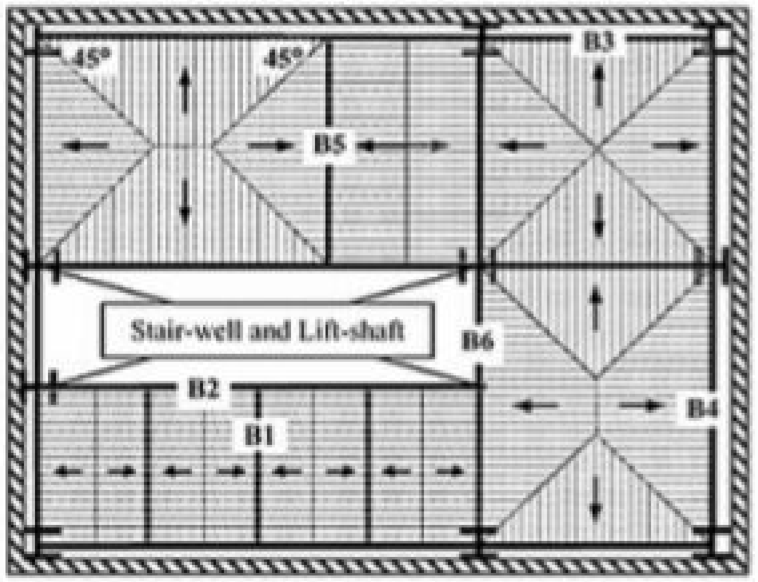
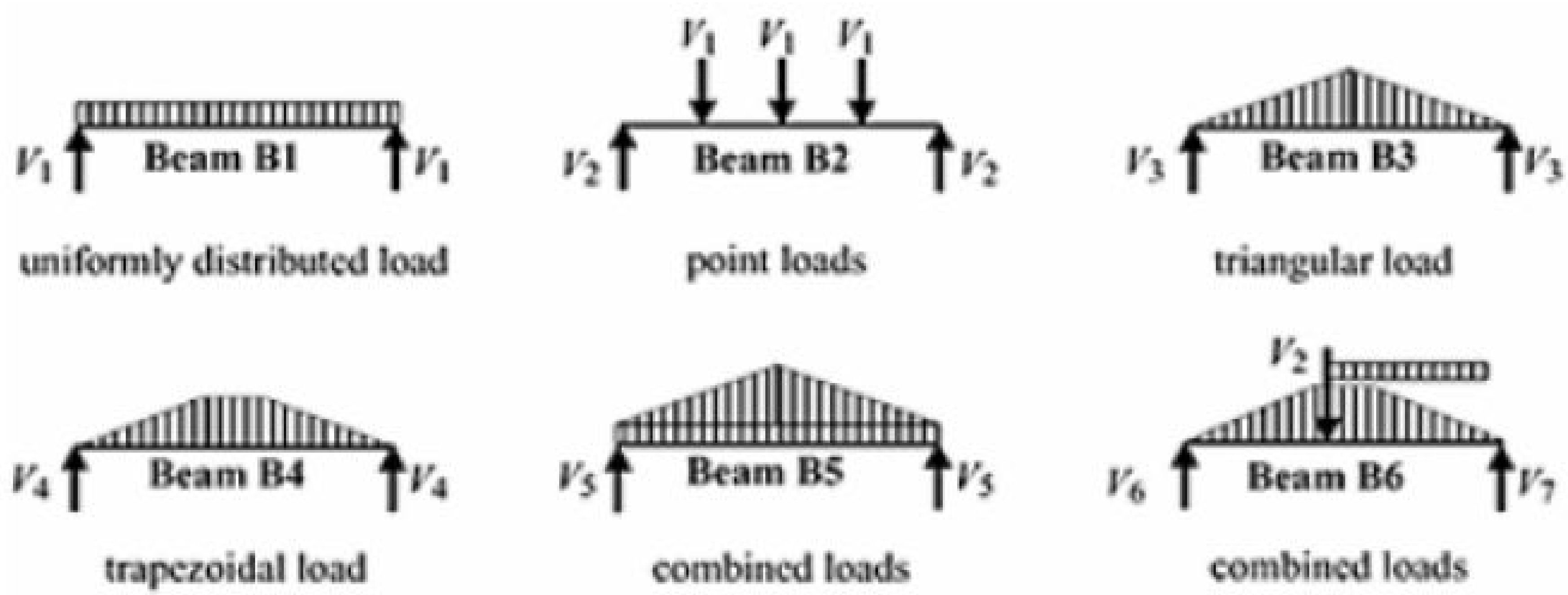


FIGURE 3.4 Torques applied to a beam

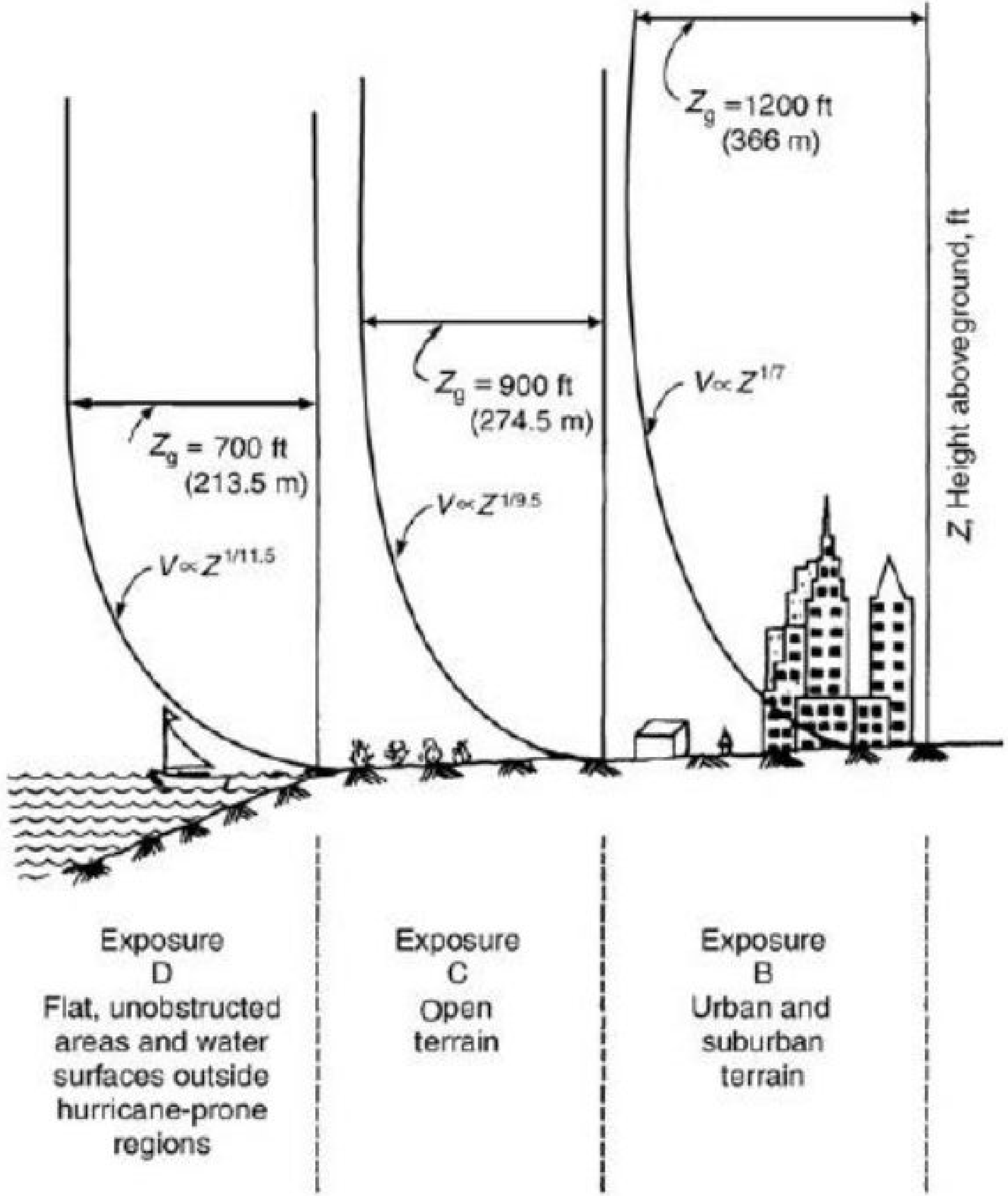
Gravity Loads (Dead/Live Loads)





Lateral Loads

(1) Wind Load
(2) Fluid/Water Load
(3) Earthquake/Seismic Load



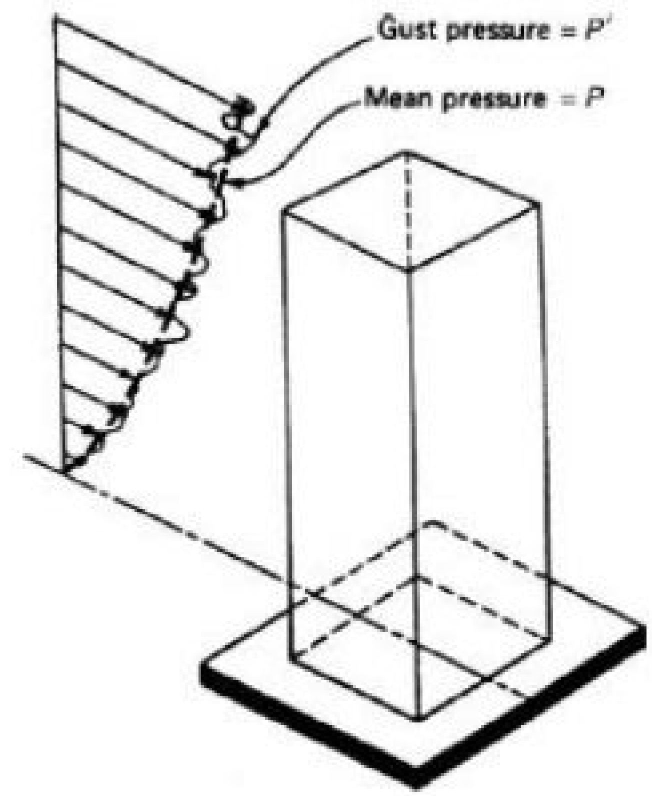
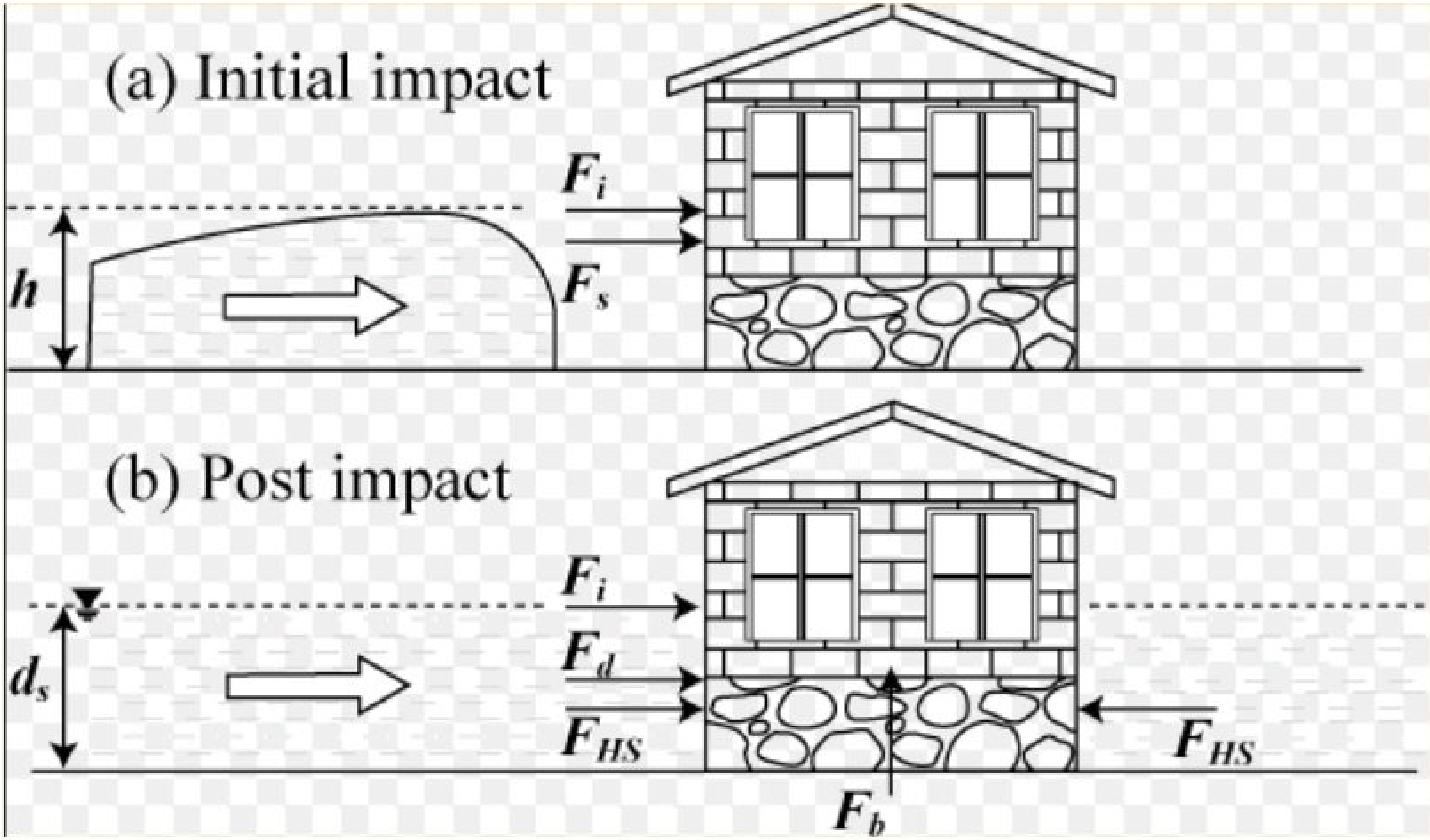
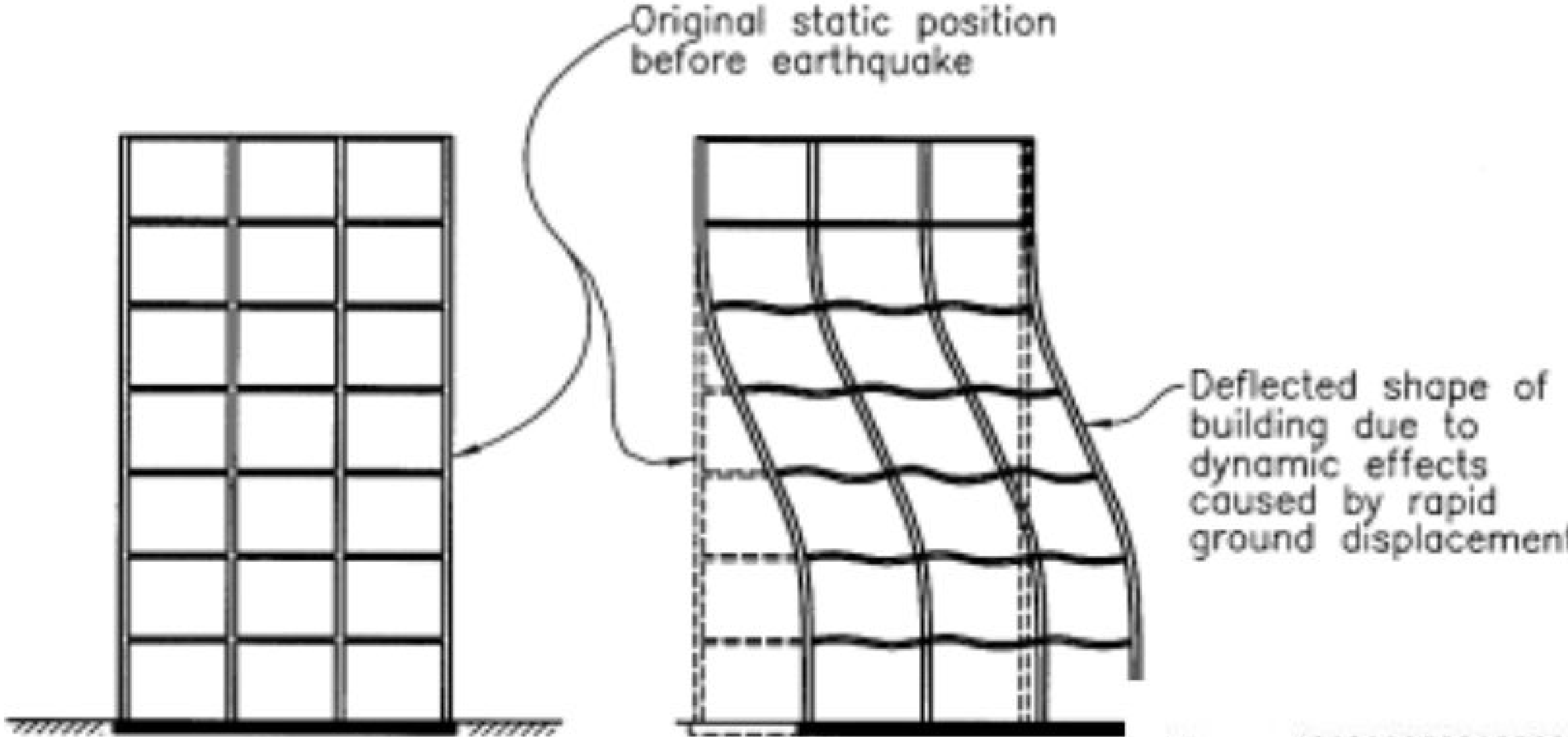


Figure 1.1. Influence of exposure terrain on variation of wind velocity with height.


(a) Initial impact
F_i
F_s
h
(b) Post impact
F_i
F_d
F_{HS}
d_s
F_{HS}
F_b



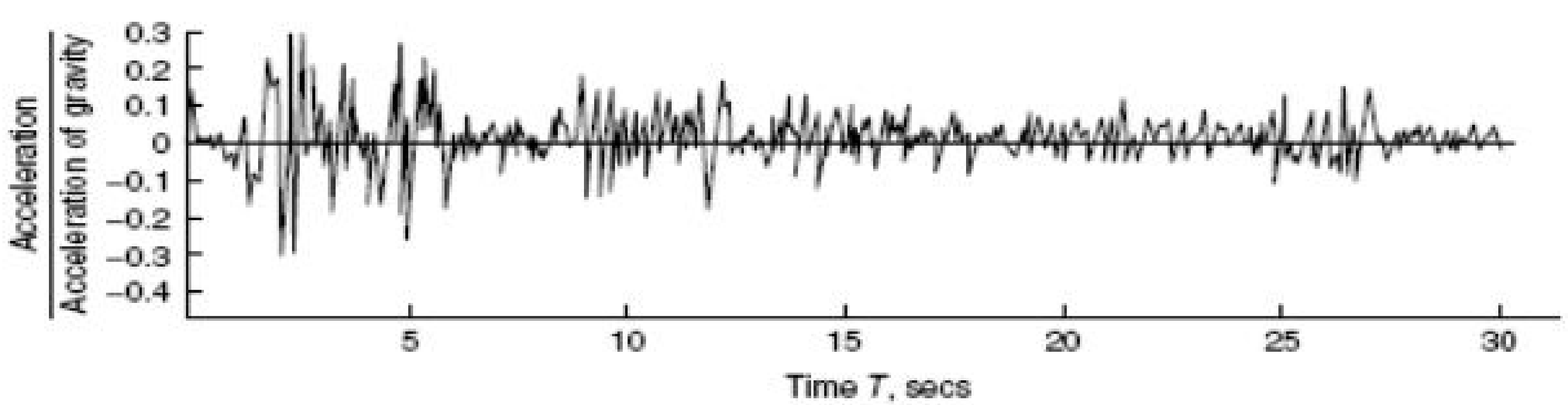
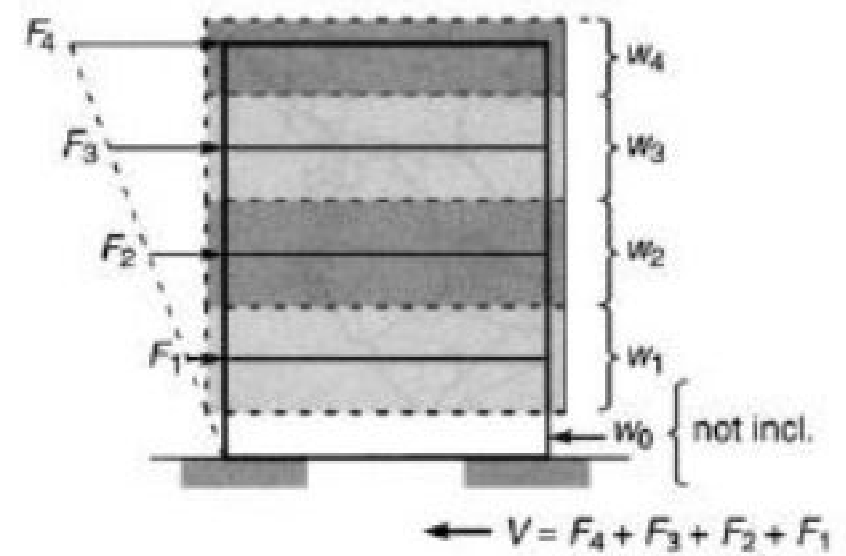
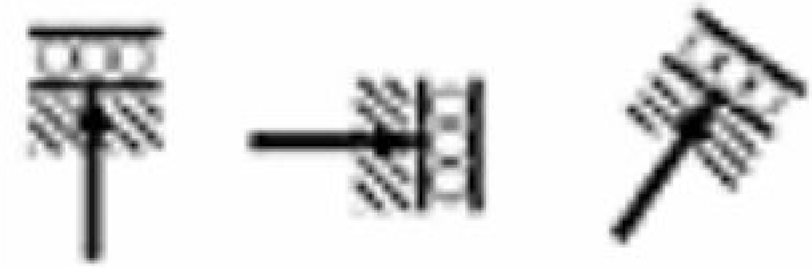


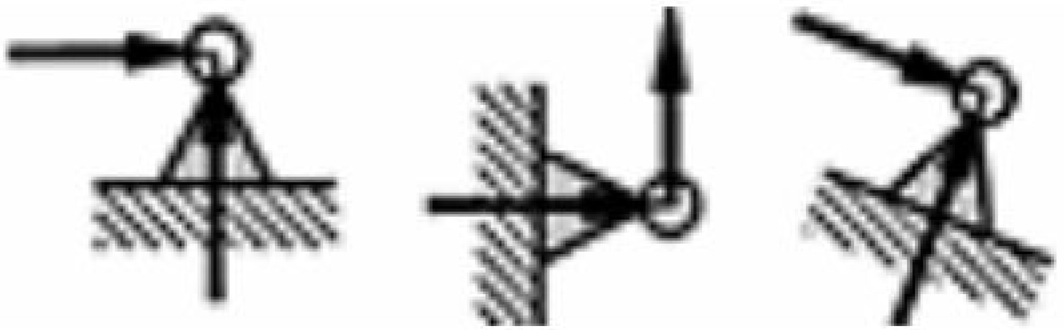
Figure 2.1. Behavior of a building during earthquak



roller supports: providing one restraint perpendicular to the roller.



pinned supports: providing two mutually perpendicular restraints



fixed supports: providing two mutually perpendicular restraints and one moment restraint.

Application of different Supports

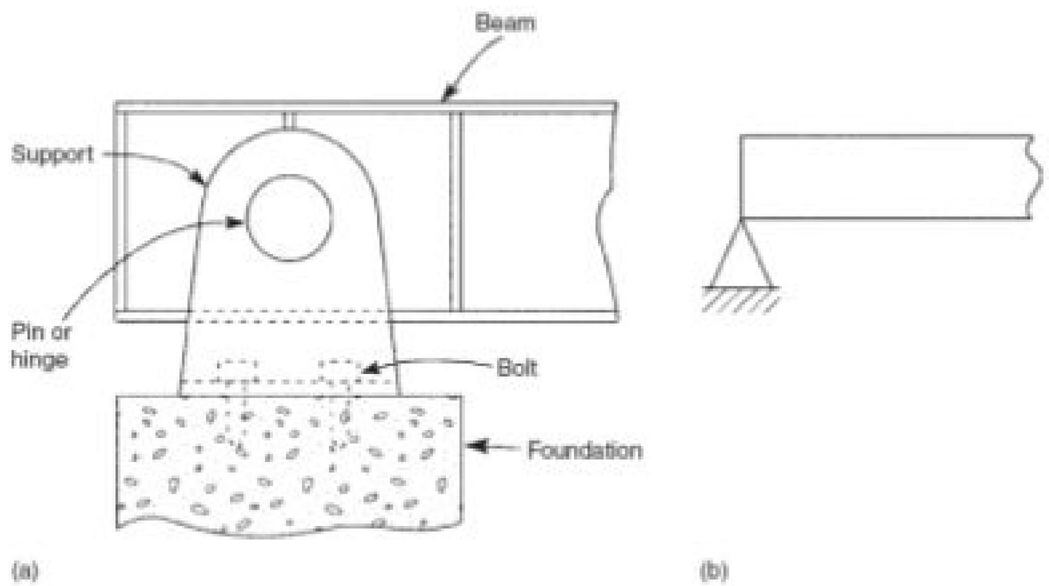


Figure 1.11 Idealization of a pinned support

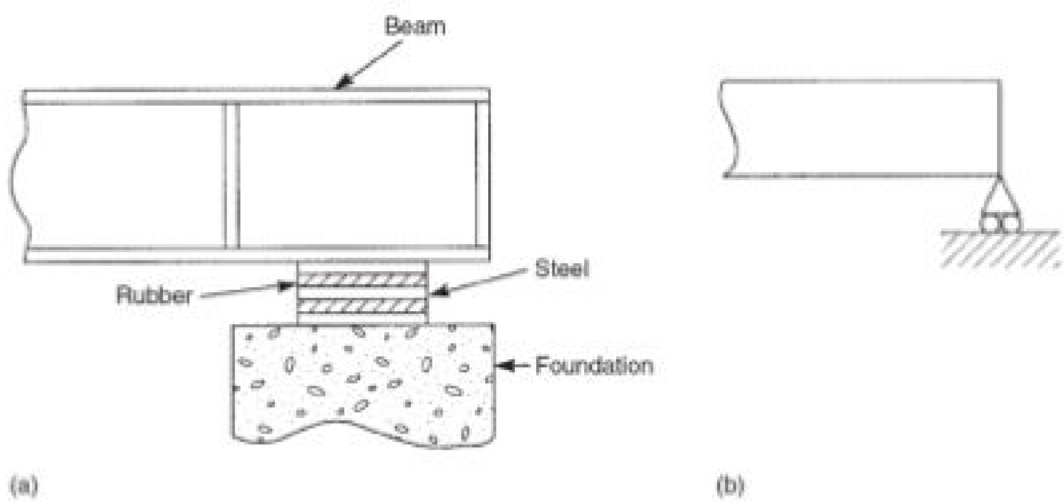


Figure 1.12 Idealization of a sliding or roller support

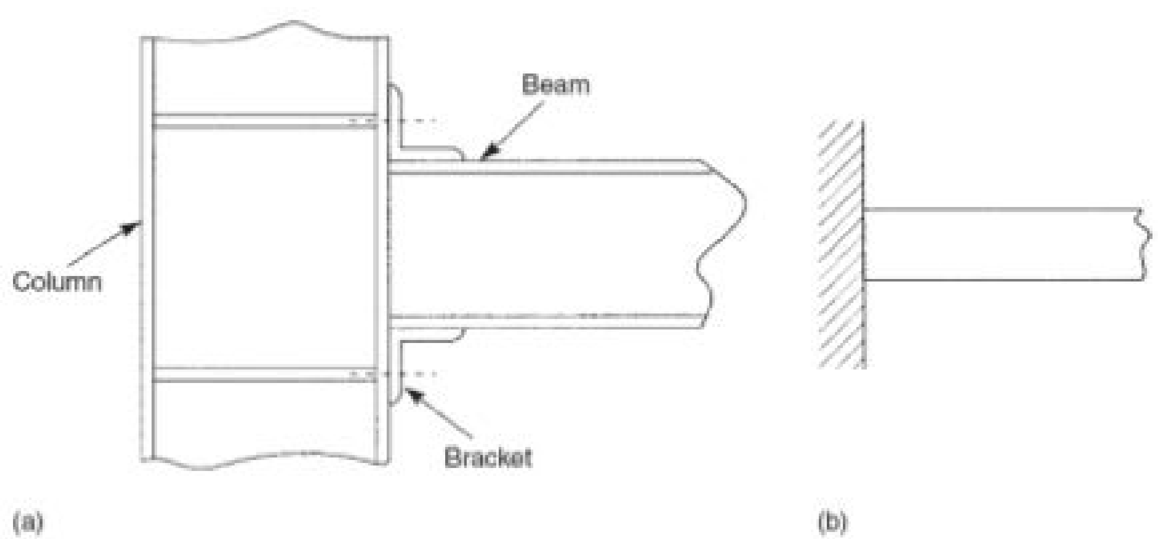


Figure 1.13 Idealization of a built-in support

Columns are usually subjected to two bending moments about two perpendicular axes (X and Y) as well as an axial force in the vertical Z direction (see Figure)

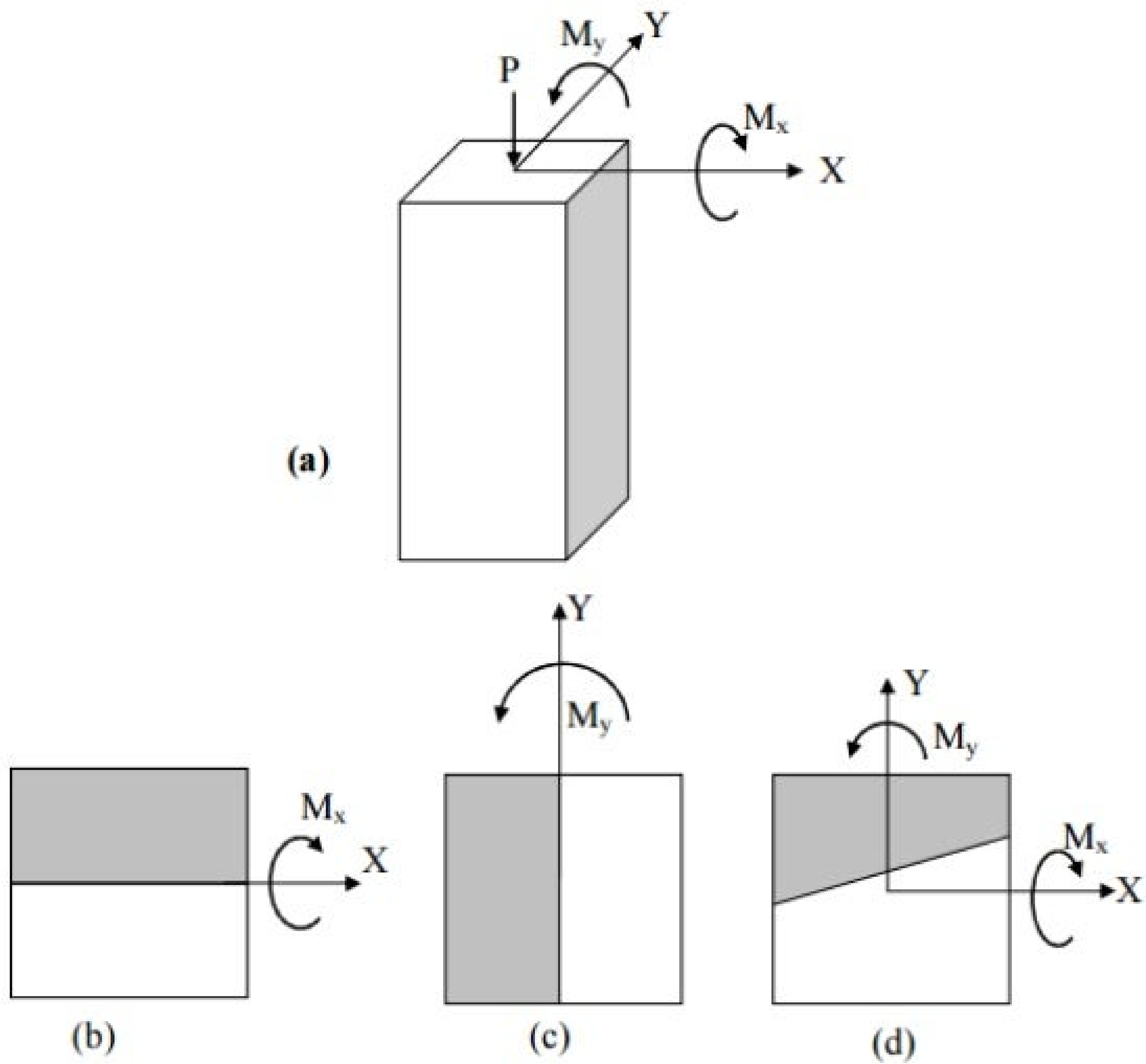


Figure 1: Biaxial bending of columns

(a): 3D view (b): Bending about *X*-axis

(c): Bending about *Y*-axis (d): Inclined neutral axis in biaxial bending

State of Stresses

(a) plane stress in sheet stretching; there are no stresses acting on the surfaces of the sheet (studied in MOS-I).

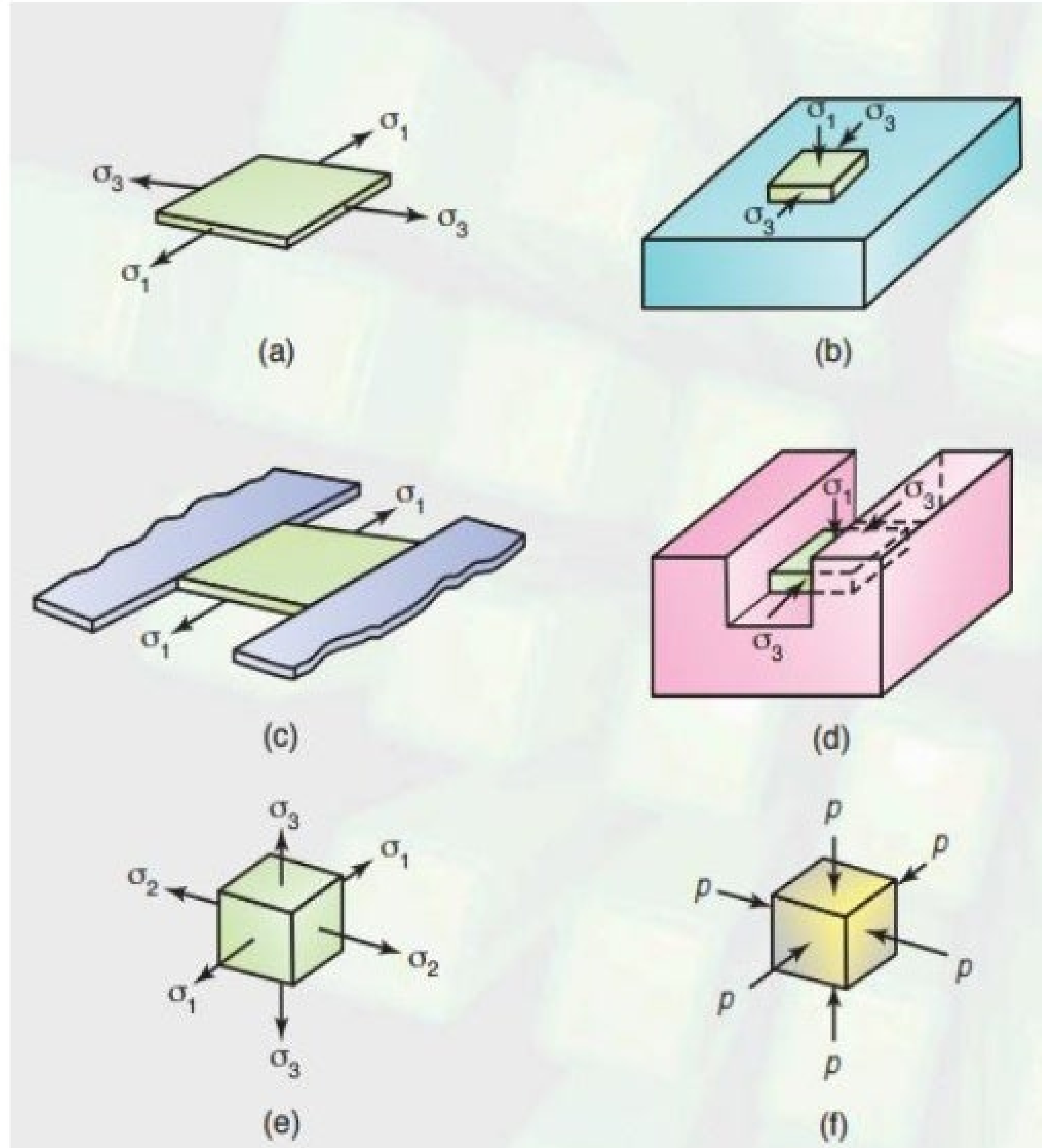
(b) plane stress in compression; there are no stresses acting on the sides of the specimen being compressed.

(c) plane strain in tension; the width of the sheet remains constant while being stretched.

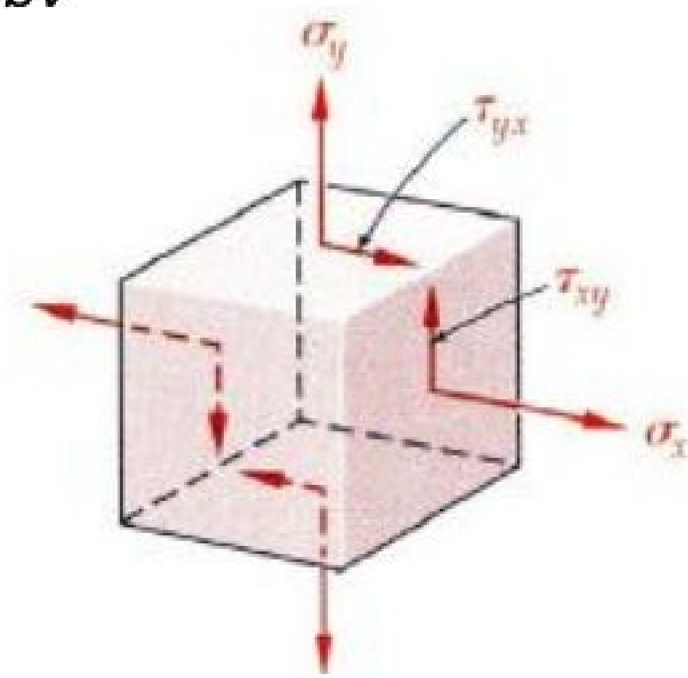
(d) plane strain in compression; the width of the specimen remains constant due to the restraint by the groove.

(e) Triaxial tensile stresses acting on an element.

(f) Hydrostatic compression of an element.

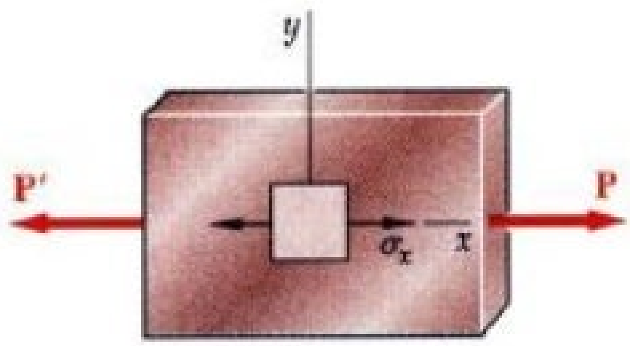


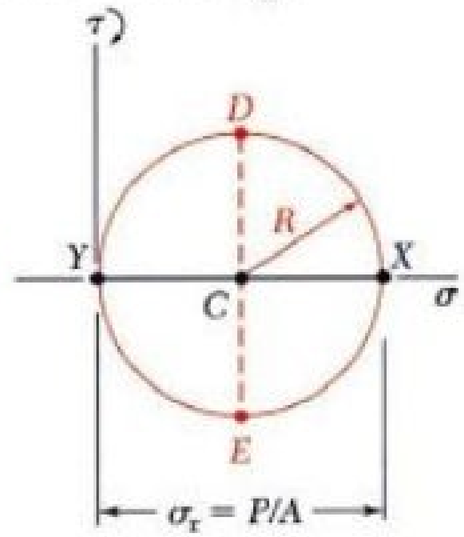
***Plane Stress*-state** *of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined bv*



$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

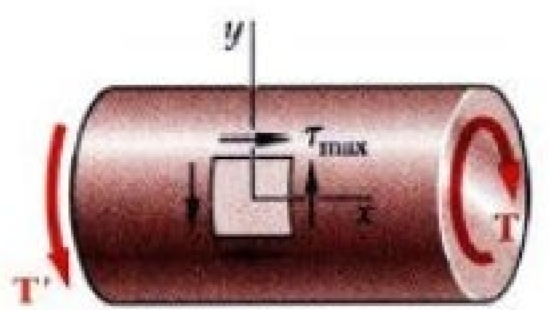
- Mohr's circle for centric axial loading:

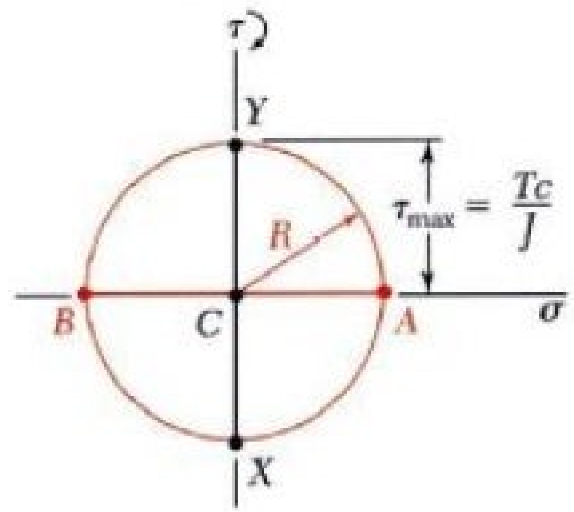




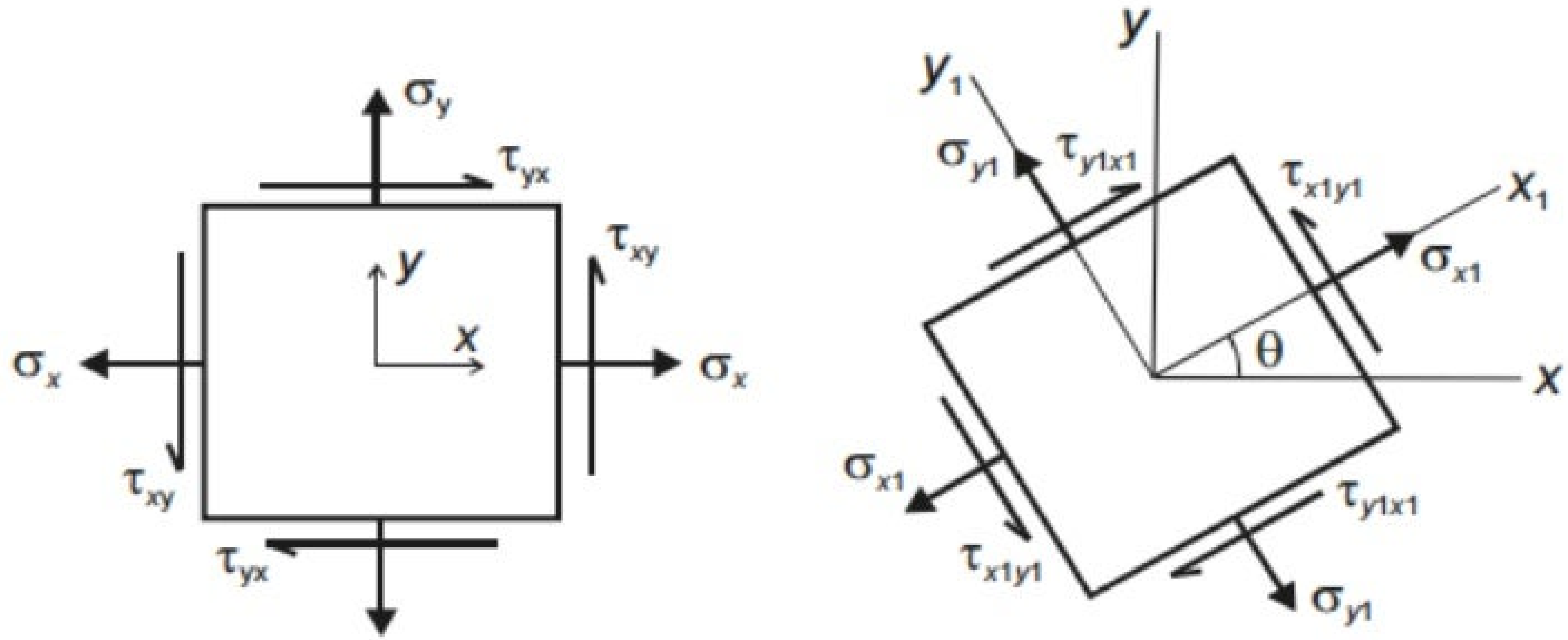
$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \tau_{xy} = 0$$

- Mohr's circle for torsional loading:





$$\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \frac{Tc}{J}$$

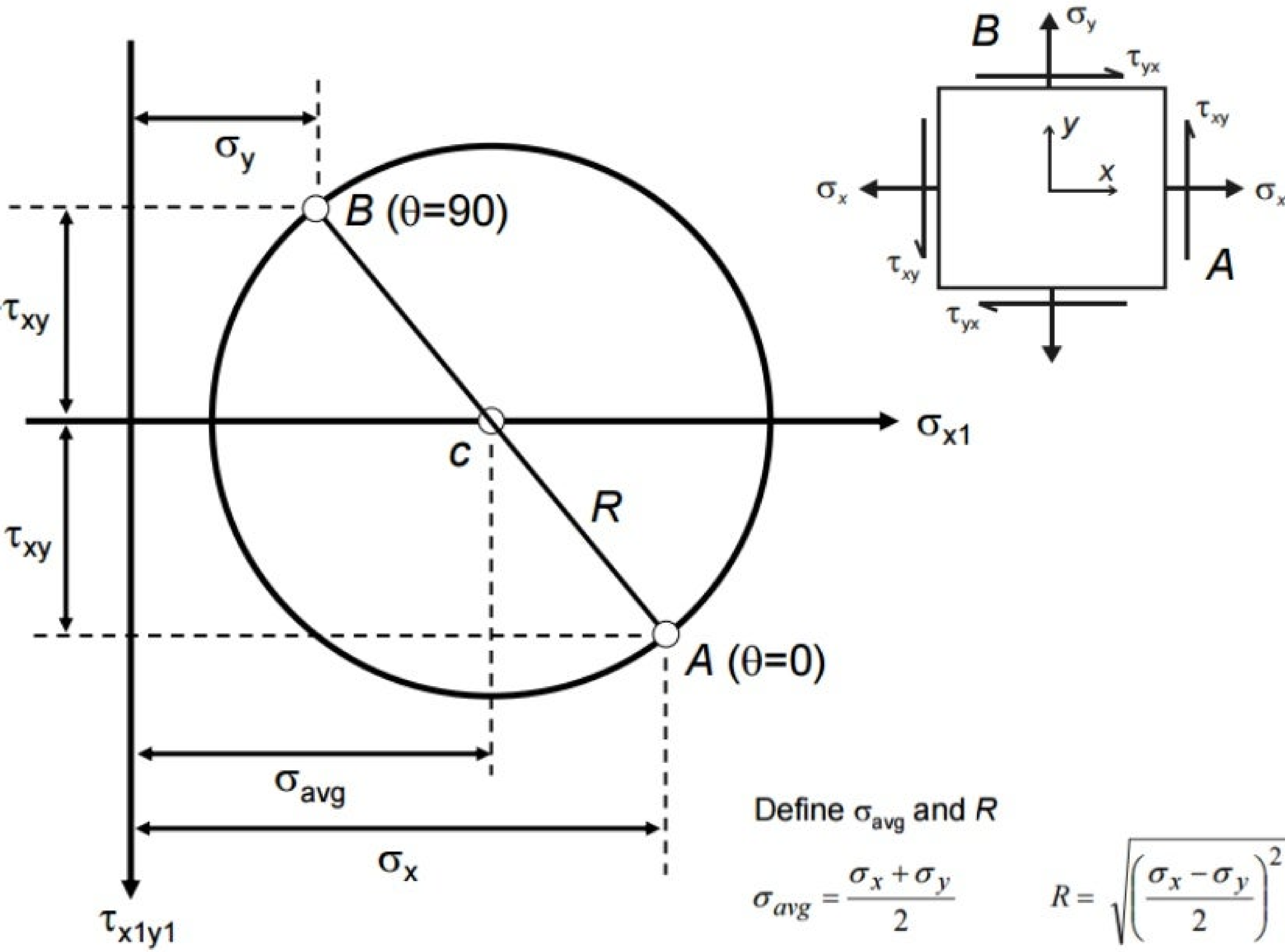


$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

If we vary θ from 0° to 360°, we will get all possible values of σ_{x1} and τ_{x1y1} for a given stress state. It would be useful to represent σ_{x1} and τ_{x1y1} as functions of θ in graphical form.

Mohr's Circle for Plane Stress

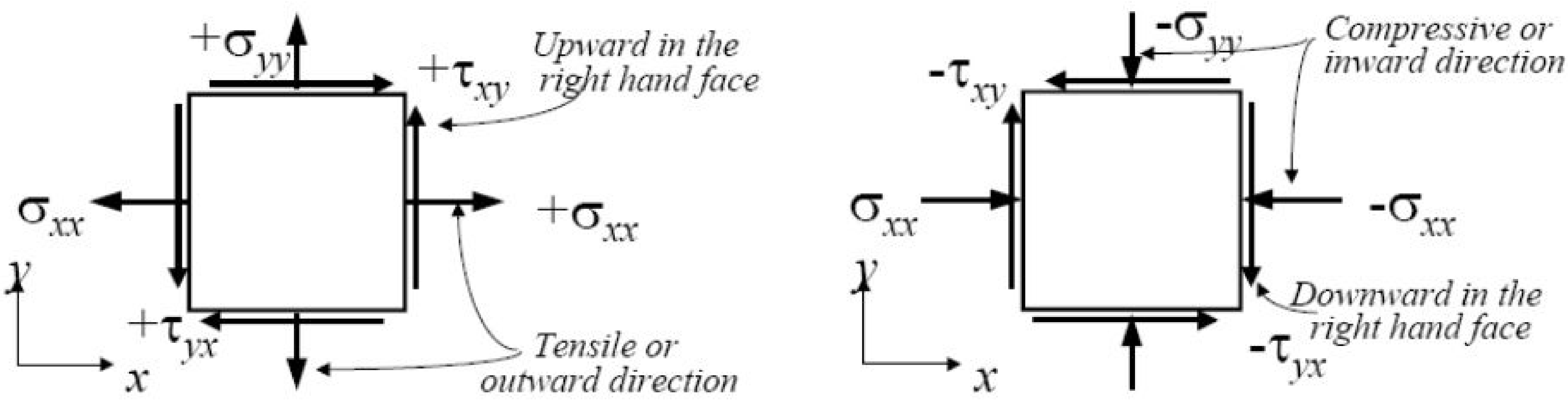


Define σ_{avg} and R

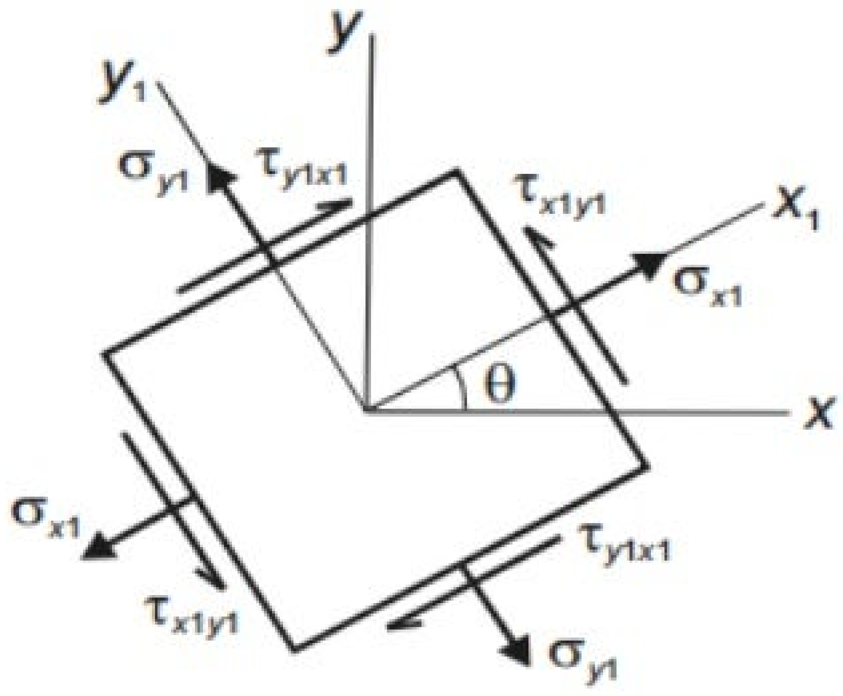
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}$$

Sign Convention

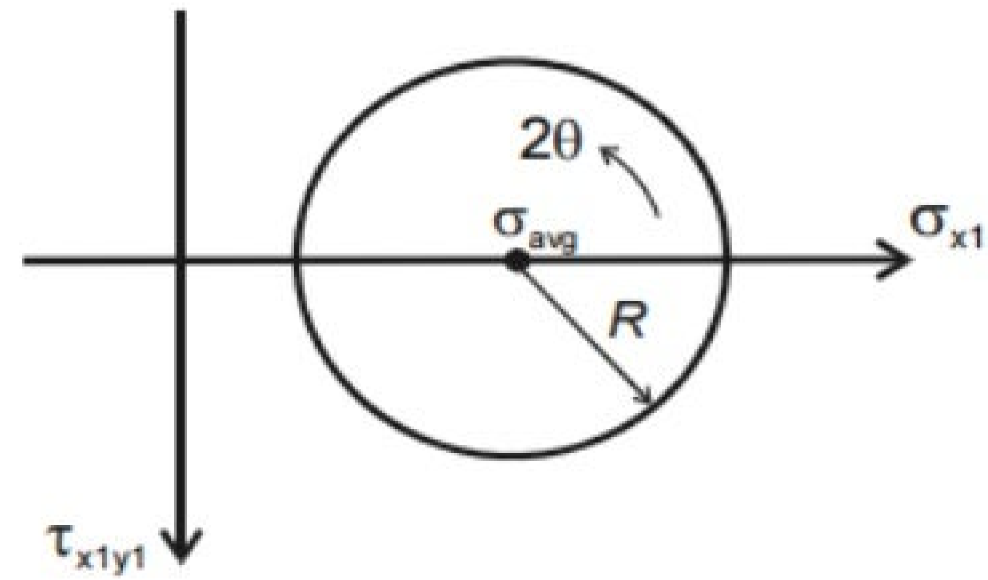
Before the transformation equations are derived, it is necessary for us to review the sign convention for the normal and shear stress components. As shown in Fig. below, the sign convention can be remembered by simply noting that positive normal stress acts outwards from all faces and positive shear stress acts upward on the right-hand face of the element.



Sign convention of stress components

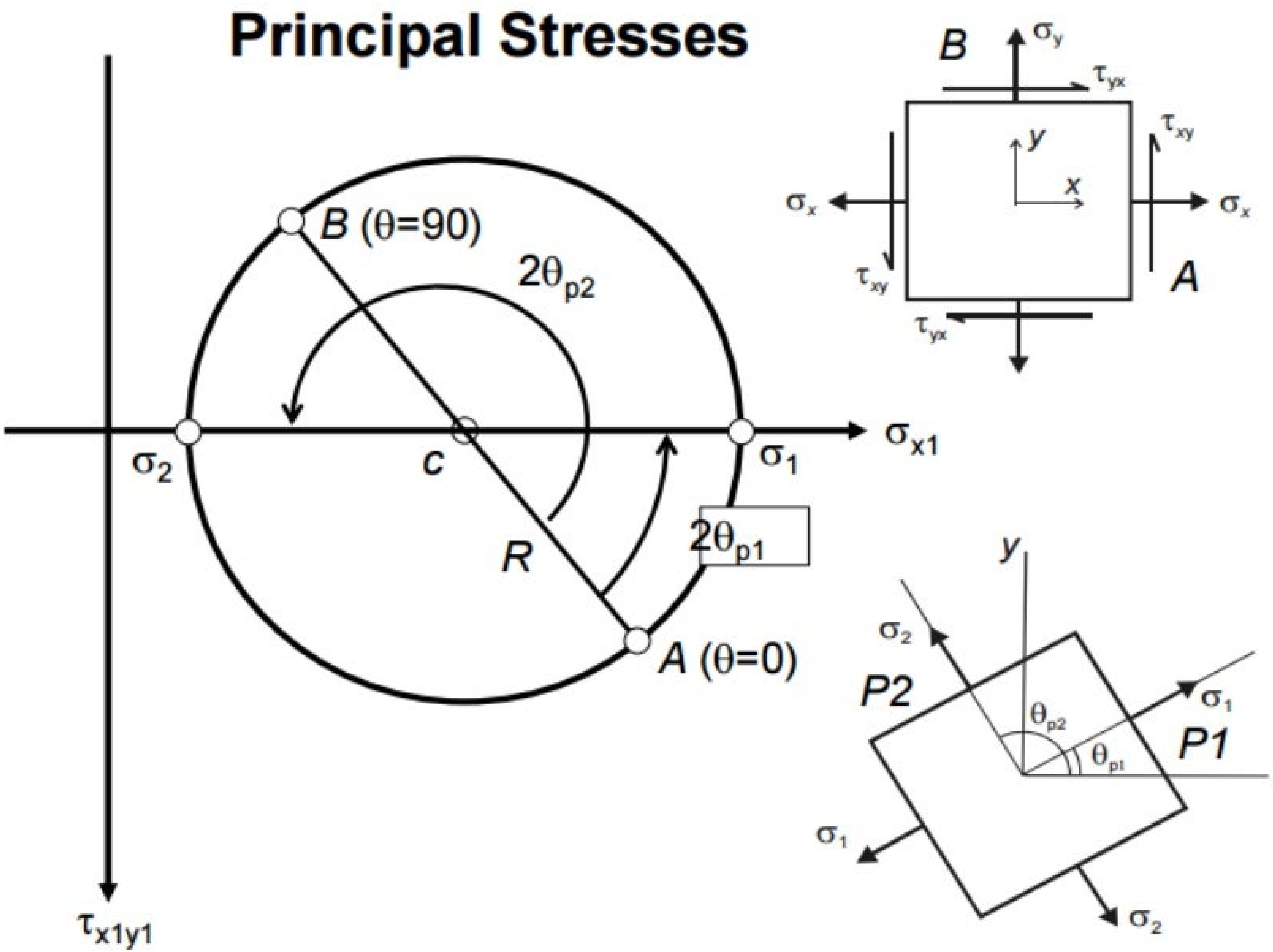


$$\left(\sigma_{x1} - \sigma_{avg}\right)^2 + \tau_{x1y1}{}^2 = R^2$$

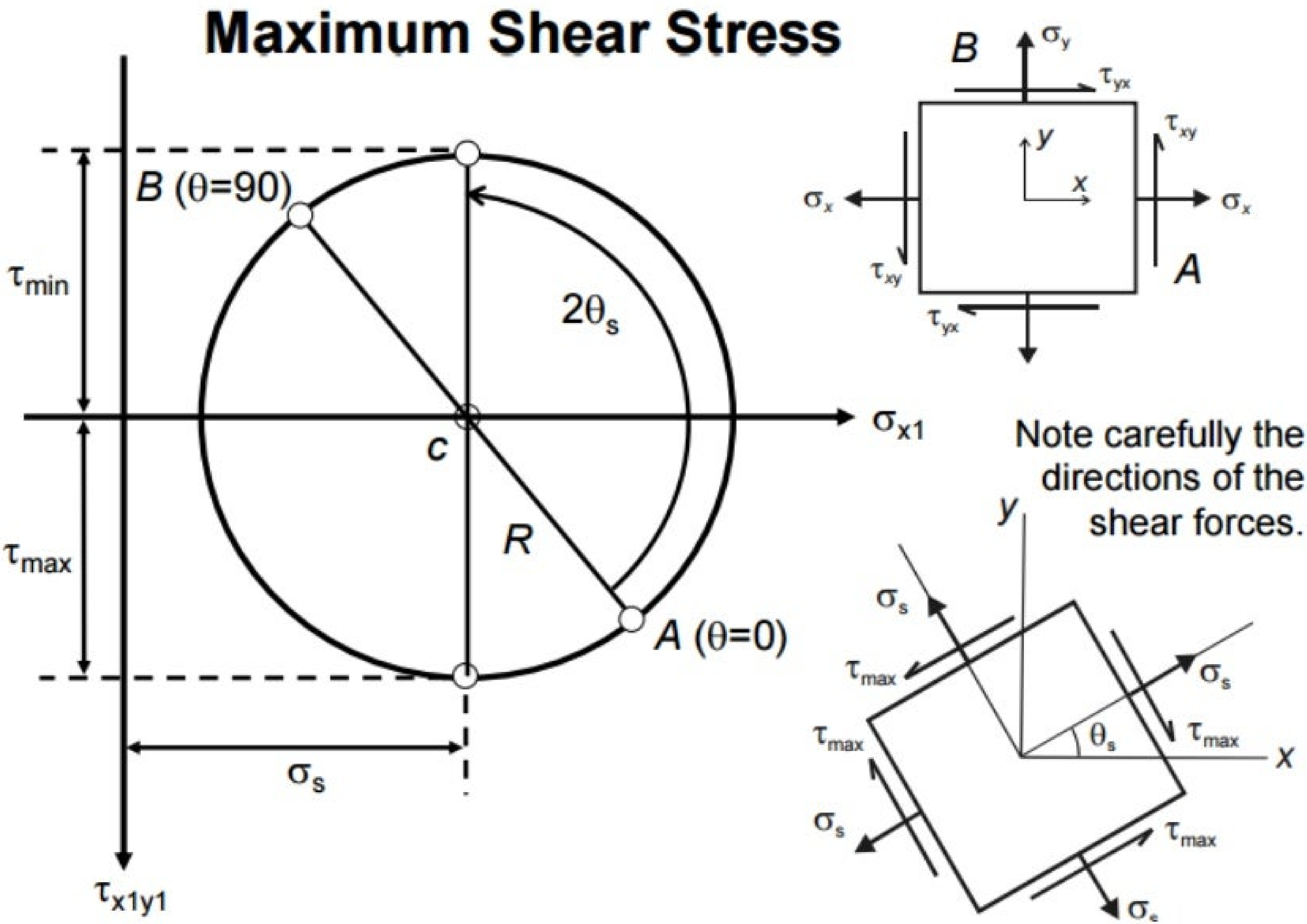


Notice that shear stress is plotted as positive downward. The reason for doing this is that 2θ is then positive counterclockwise, which agrees with the direction of 2θ used in the derivation of the tranformation equations and the direction of θ on the stress element.

Notice that although 2θ appears in Mohr's circle, θ appears on the stress element.


Principal Stresses
B (θ=90)
2θp2
σ2
c
σ1
σx1
2θp1
R
A (θ=0)
τx1y1
B
σy
τyx
τxy
y
x
σx
σx
τxy
A
τyx
y
σ2
P2
σ1
θp2
θp1
P1
x
σ1
σ2

Maximum Shear Stress



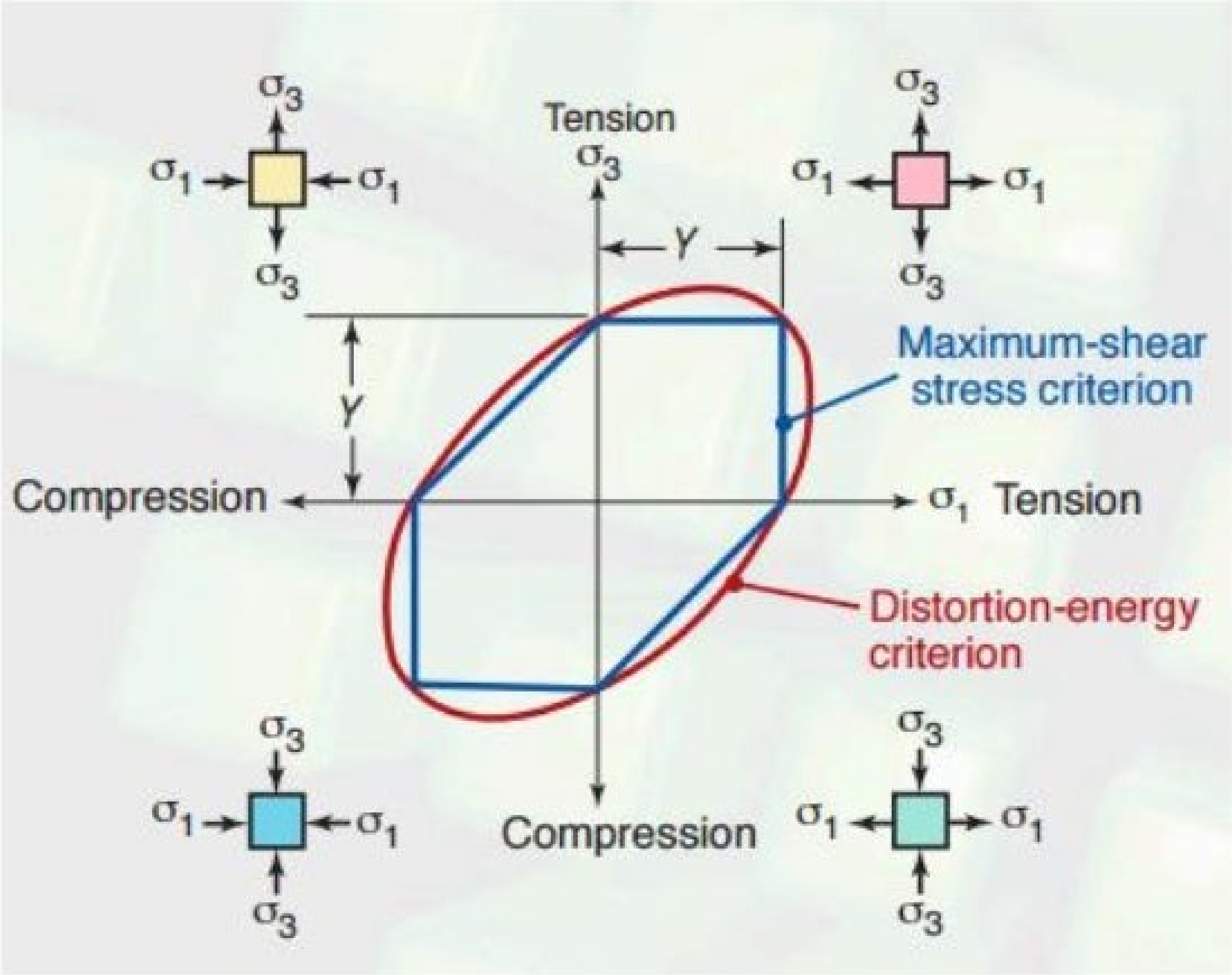
Application of Principle Stresses/Failure Theories/Yield Criterion

Maximum-shear-stress criterion:

$\sigma_{max} - \sigma_{min} = Y$

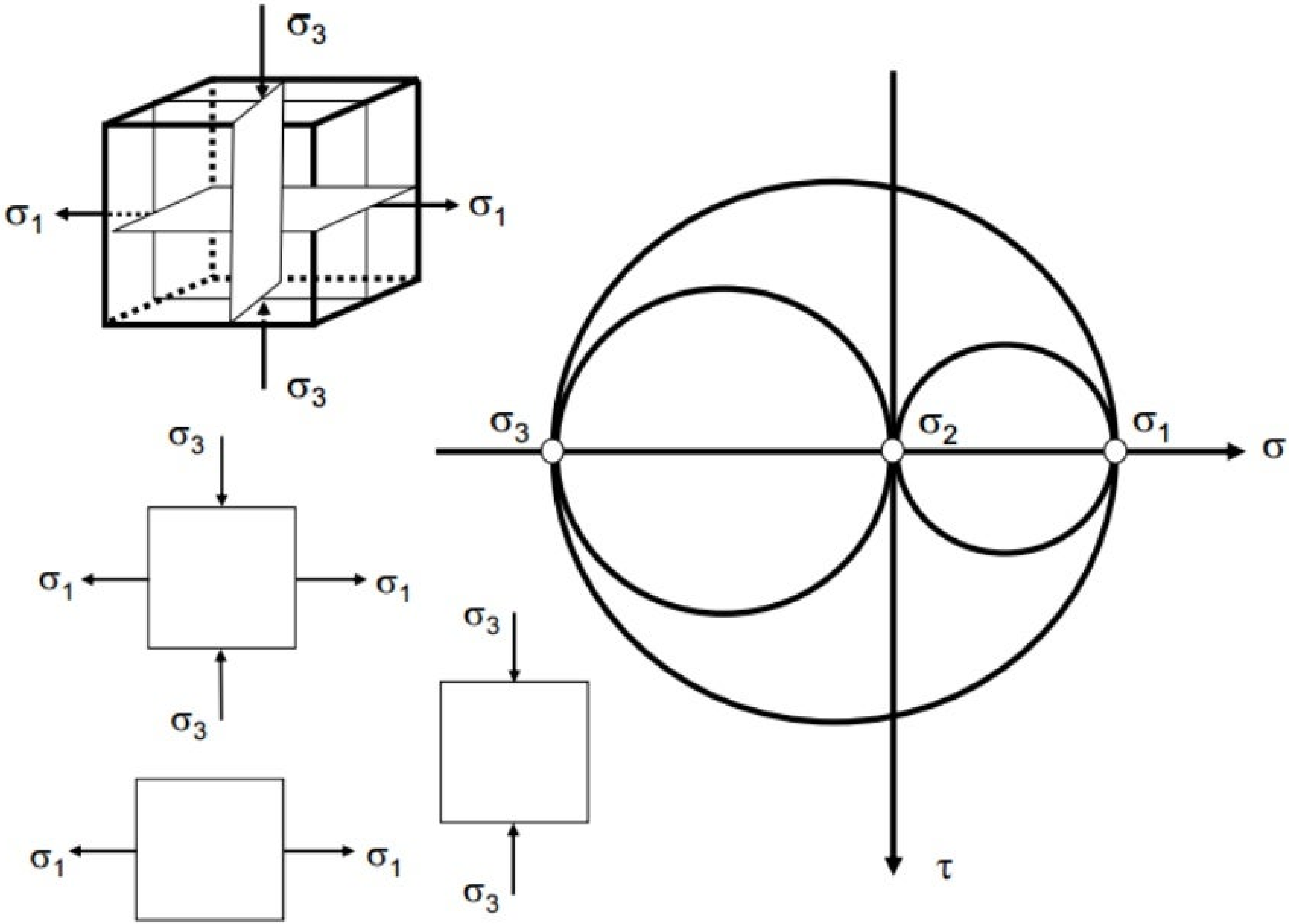
Distortion-energy criterion:

$(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2 = 2Y^2$

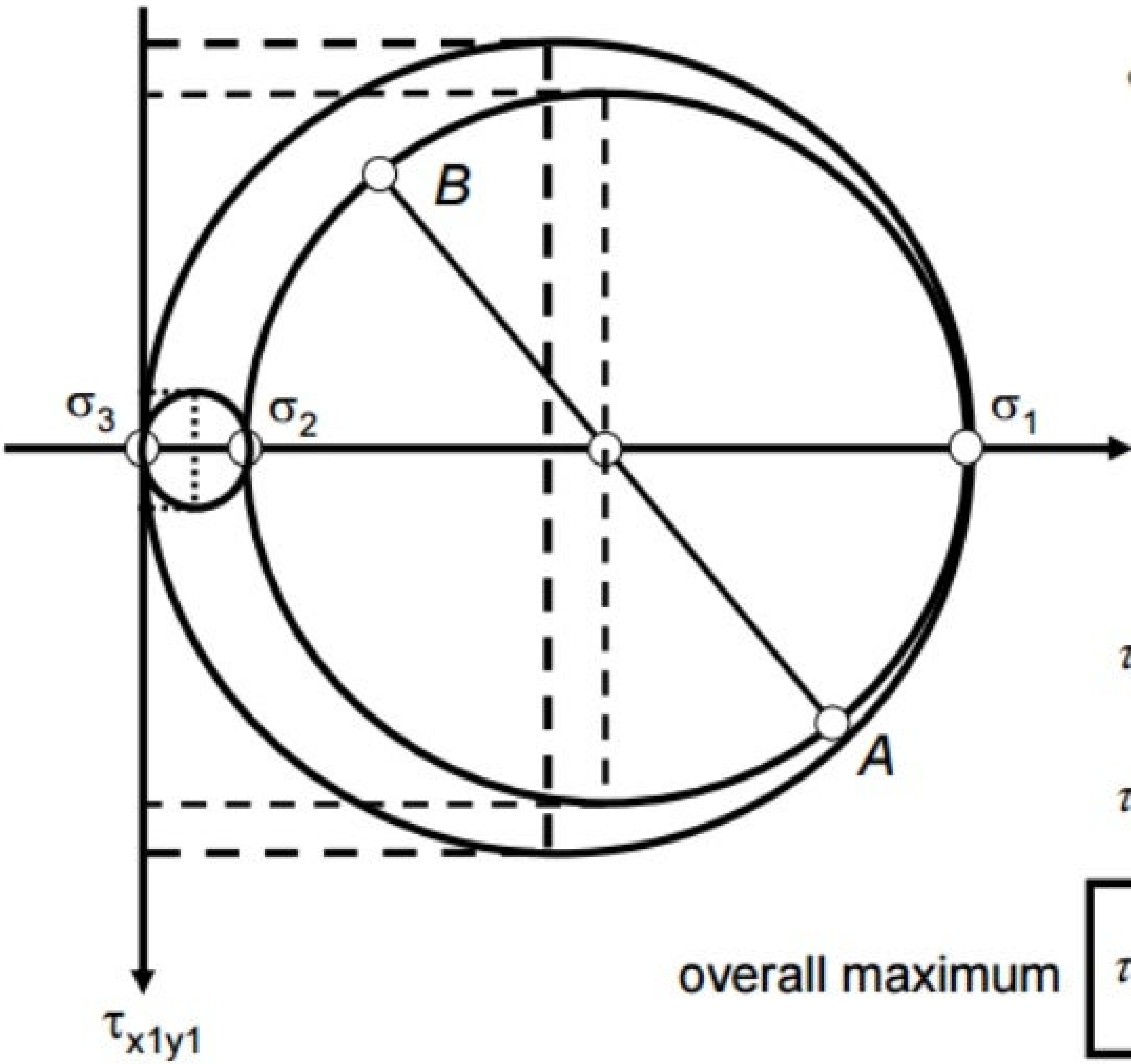


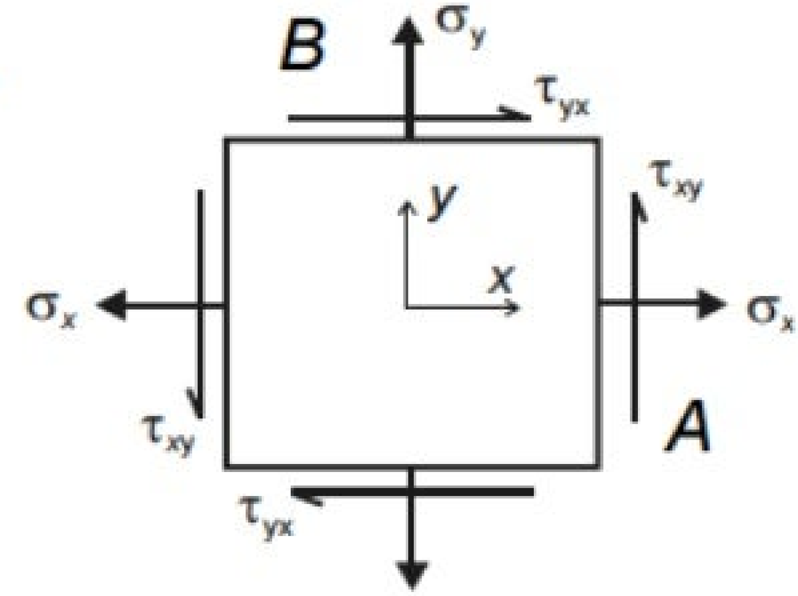
Plane-stress diagrams for maximum -shear-stress and distortion-energy criteria. Note that $\sigma 2 = 0$.

3D Stress Transformation



The stress element shown is in plane stress.
What is the maximum shear stress?





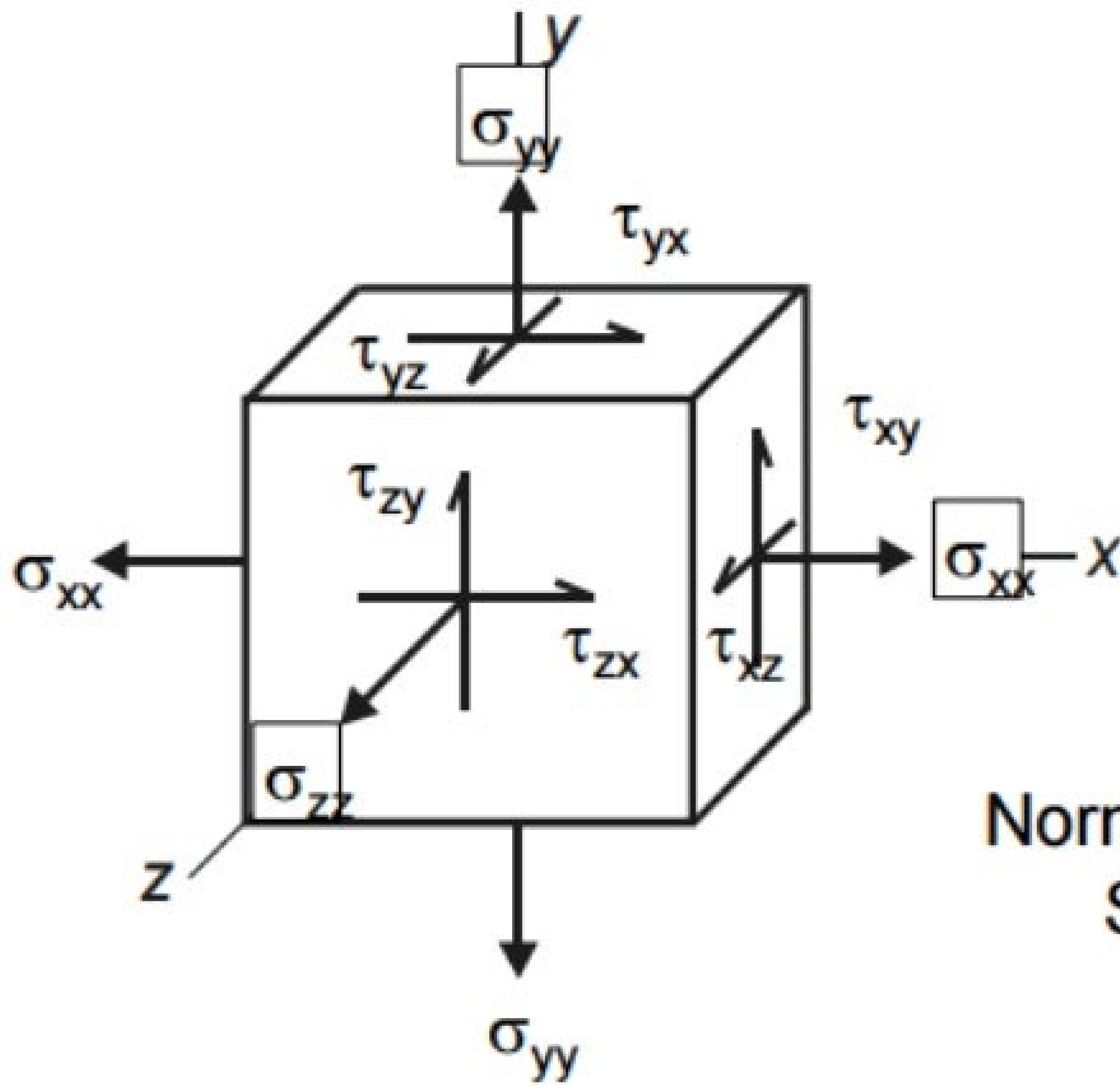
$$\tau_{max(1,2)} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{max(2,3)} = \frac{\sigma_2 - \sigma_3}{2} = \frac{\sigma_2}{2}$$

overall maximum

$$\tau_{max(1,3)} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_1}{2}$$

Introduction to the Stress Tensor



$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Normal stresses on the diagonal
Shear stresses off diagaonal
$\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

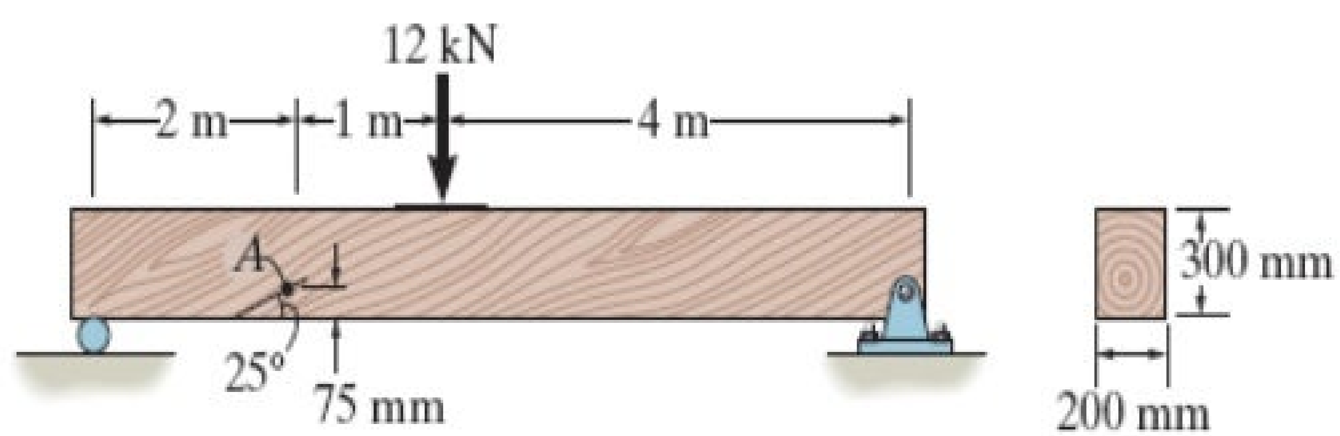
The normal and shear stresses on a stress element in 3D can be assembled into a 3x3 matrix known as the **stress tensor**.

From our analyses so far, we know that for a given stress system, it is possible to find a set of three principal stresses. We also know that if the principal stresses are acting, the shear stresses must be zero. In terms of the stress tensor,

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \Longrightarrow \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Example.#.01

•**9–23.** The wood beam is subjected to a load of 12 kN. If a grain of wood in the beam at point *A* makes an angle of 25° with the horizontal as shown, determine the normal and shear stress that act perpendicular and parallel to the grain due to the loading.



$$I = \frac{1}{12}(0.2)(0.3)^3 = 0.45(10^{-3})\ \text{m}^4$$

$$Q_A = \bar{y}A' = 0.1125(0.2)(0.075) = 1.6875(10^{-3})\ \text{n}$$

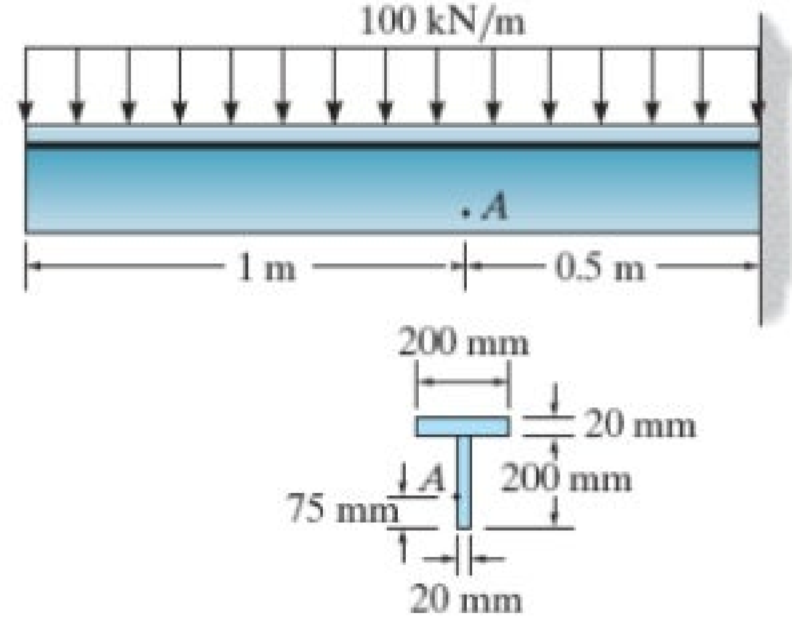
$$\sigma_A = \frac{M y_A}{I} = \frac{13.714(10^3)(0.075)}{0.45(10^{-3})} = 2.2857\ \text{MPa (T)}$$

$$\tau_A = \frac{VQ_A}{It} = \frac{6.875(10^3)(1.6875)(10^{-3})}{0.45(10^{-3})(0.2)} = 0.1286\ \text{MPa}$$

$\sigma_x = 2.2857$ MPa $\quad \sigma_y = 0 \quad \tau_{xy} = -0.1286$ MPa $\quad \theta = 115°$

Example.#.02

9–22. The T-beam is subjected to the distributed loading that is applied along its centerline. Determine the principal stress at point A and show the results on an element located at this point.



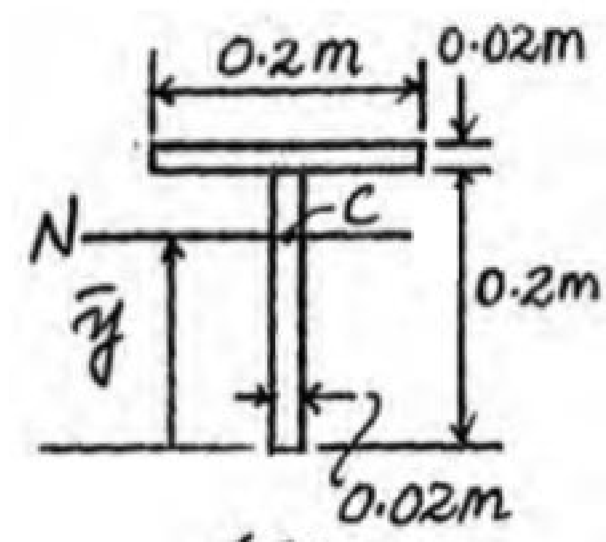
The location of the centroid c of the T cross-section, Fig. a, is

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A} = \frac{0.1(0.2)(0.02) + 0.21(0.02)(0.2)}{0.2(0.02) + 0.02(0.2)} = 0.155 \text{ m}$$

$$I = \frac{1}{12}(0.02)(0.2^3) + 0.02(0.2)(0.155 - 0.1)^2$$

$$+ \frac{1}{12}(0.2)(0.02^3) + 0.2(0.02)(0.21 - 0.155)^2$$

$$= 37.6667(10^{-6}) \text{ m}^4$$



(a)

Referring to Fig. b,

$$Q_A = \bar{y}' A' = 0.1175(0.075)(0.02) = 0.17625(10^{-3}) \text{ m}^3$$

Using the method of sections and considering the FBD of the left cut segment of the beam, Fig. *c*,

$+\uparrow \Sigma F_y = 0;$ $V - 100(1) = 0$ $V = 100\text{ kN}$

$\circlearrowleft + \Sigma M_C = 0;$ $100(1)(0.5) - M = 0$ $M = 50\text{ kN}\cdot\text{m}$

The normal stress developed is contributed by bending stress only. For point A, $y = 0.155 - 0.075 = 0.08$ m. Thus

$$\sigma = \frac{My}{I} = \frac{50(10^3)(0.08)}{37.6667(10^{-6})} = 106\text{ MPa}$$

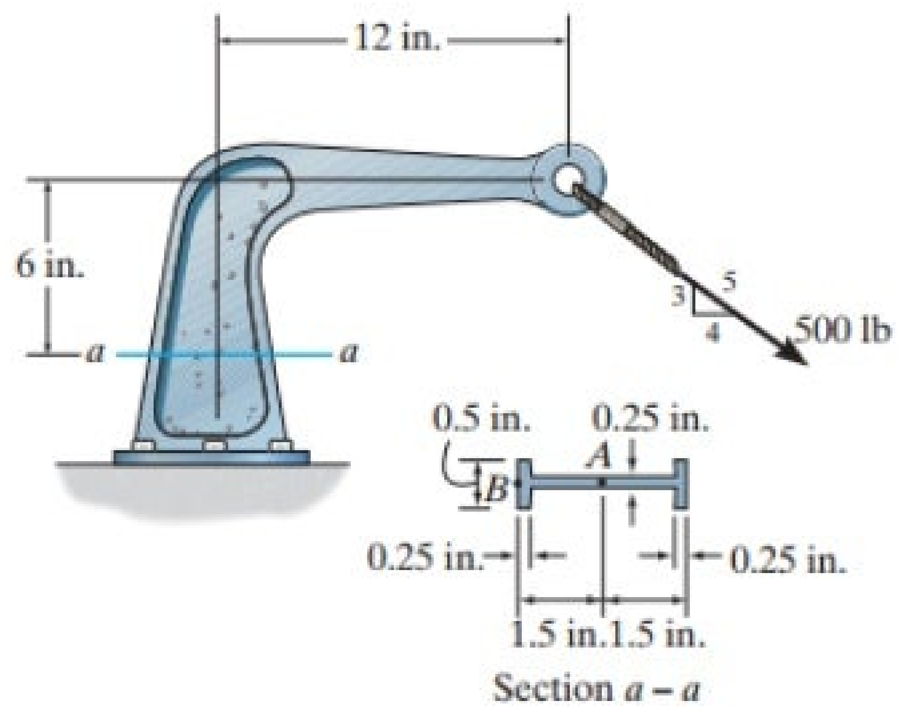
The shear stress is contributed by the transverse shear stress only. Thus,

$$\tau = \frac{VQ_A}{It} = \frac{100(10^3)[0.17625(10^{-3})]}{37.6667(10^{-6})(0.02)} = 23.40(10^6)\text{Pa} = 23.40\text{ MPa}$$

The state of stress of point *A* can be represented by the element shown in Fig. *c*.

Here, $\sigma_x = -106.19$ MPa, $\sigma_y = 0$ and $\tau_{xy} = 23.40$ MPa.

9–95. Determine the principal stress and absolute maximum shear stress developed at point B on the cross section of the bracket at section a–a.



Internal Loadings: Considering the equilibrium of the free - body diagram of the bracket's upper cut segment, Fig. a,

$$+\uparrow \Sigma F_y = 0; \qquad N - 500\left(\frac{3}{5}\right) = 0 \qquad N = 300 \text{ lb}$$

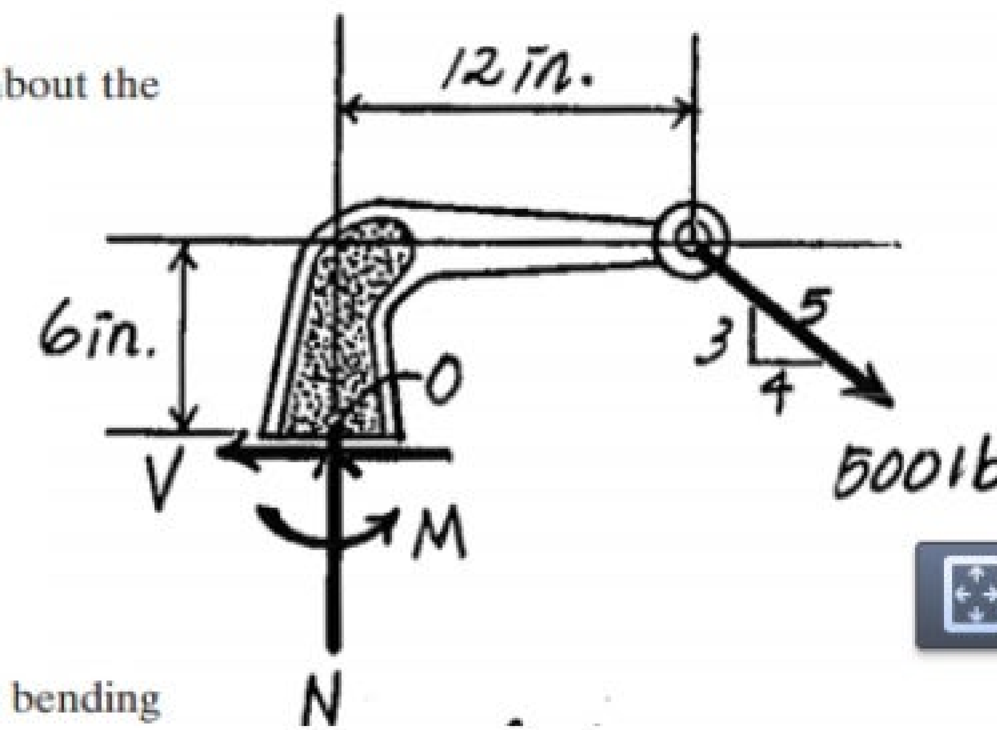
$$\overset{+}{\leftarrow} \Sigma F_x = 0; \qquad V - 500\left(\frac{4}{5}\right) = 0 \qquad V = 400 \text{ lb}$$

$$\Sigma M_O = 0;\ M - 500\left(\frac{3}{5}\right)(12) - 500\left(\frac{4}{5}\right)(6) = 0 \qquad M = 6000 \text{ lb} \cdot \text{in}$$

Section Properties: The cross - sectional area and the moment of inertia about the centroidal axis of the bracket's cross section are

$$A = 0.5(3) - 0.25(2.5) = 0.875 \text{ in}^2$$

$$I = \frac{1}{12}(0.5)(3^3) - \frac{1}{12}(0.25)(2.5^3) = 0.79948 \text{ in}^4$$



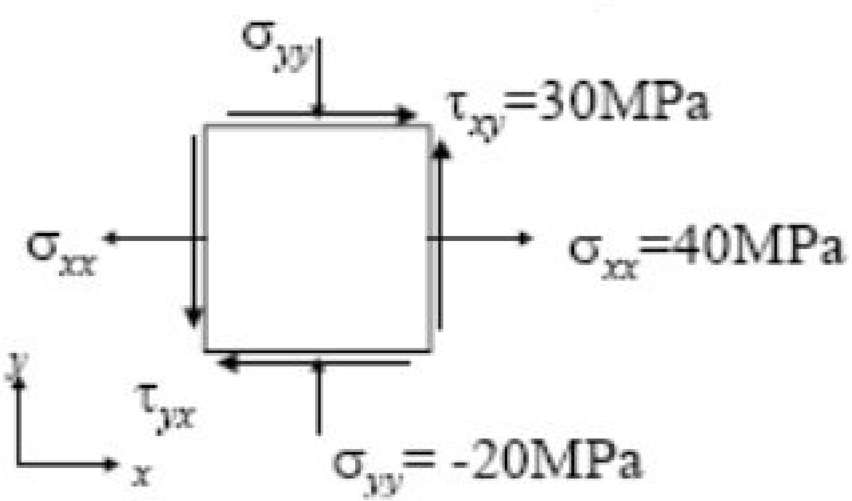
Referring to Fig. b,

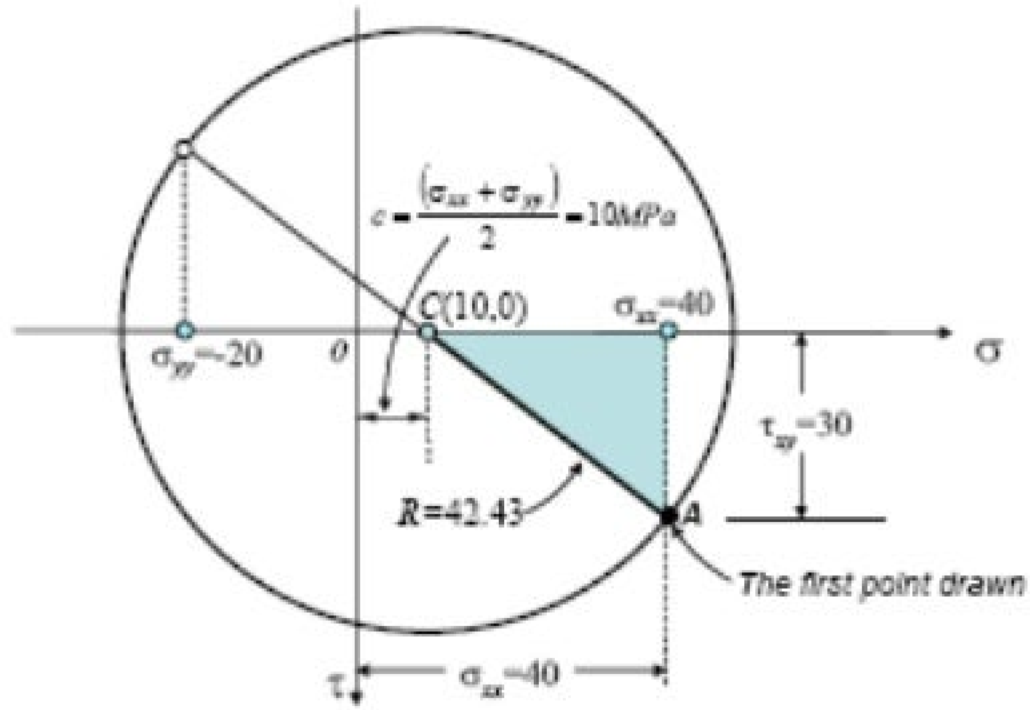
$$Q_B = 0$$

Normal and Shear Stress: The normal stress is a combination of axial and bending

Example 10.2 Find σ_1, σ_2, τ_{max} and their orientations for the following stress system: $\sigma_{xx} = 40\ MPa$, $\sigma_{yy} = -20\ MPa$, $\tau_{xy} = 30\ MPa$.

Step 1: Draw the state of stresses

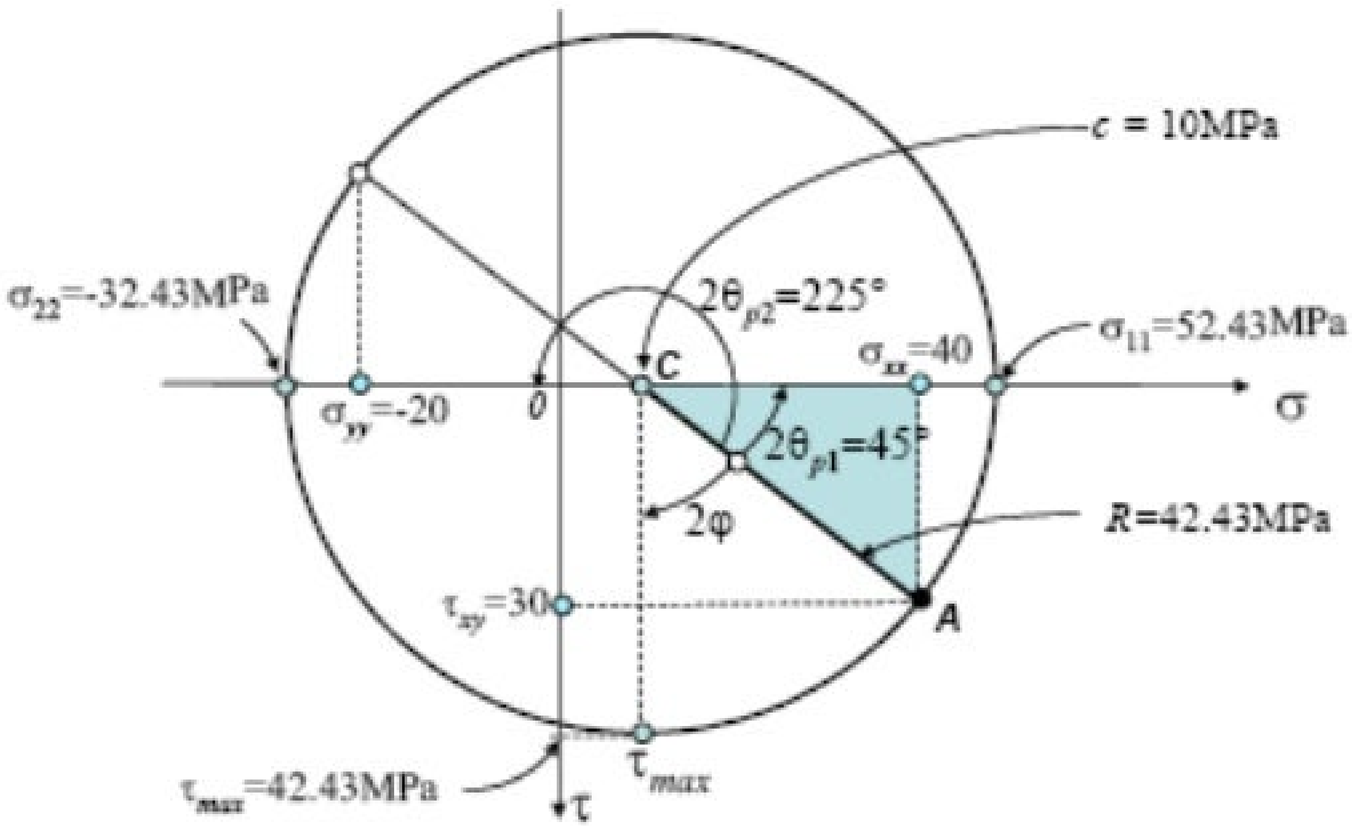




Step 2: Draw the **Mohr Circle** It is necessary to first establish σ and τ axis. Since σ_{xx}, σ_{yy}, and τ_{xy} are known, the center of the circle can then be plotted at C(10,0). To obtain the radius, one can either plot point A (40,30) as shown above or compute the value as follows.

Example.#.04

Radius of Mohr Circle: $R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{40-(-20)}{2}\right)^2 + (30)^2} = 42.43MPa$



Center of Mohr Circle:

$$c = \frac{(\sigma_{xx} + \sigma_{yy})}{2} = \frac{40+(-20)}{2} = 10MPa$$

<u>Step 3: Determine the orientation of the principle stress:</u>

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})} = \frac{2\times 30}{40-(-20)} = 1.0$$

$$\therefore \theta_{p1} = \frac{\arctan(1.0)}{2} = 22.5^\circ \text{ and}$$

$$\therefore \theta_{p2} = 90^\circ + \theta_{p1} = 112.5^\circ$$

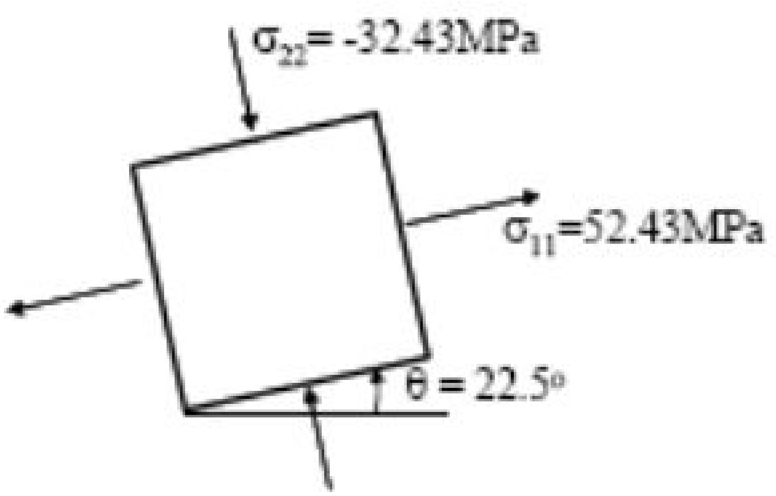
Step 4: Compute the principle stresses and the maximum shear stress

$$\begin{matrix}\sigma_{11}\\ \sigma_{22}\end{matrix} = \frac{(\sigma_{xx}+\sigma_{yy})}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = c \pm R = 10 \pm 42.43 = \begin{matrix}52.43MPa\\ -32.43MPa\end{matrix}$$

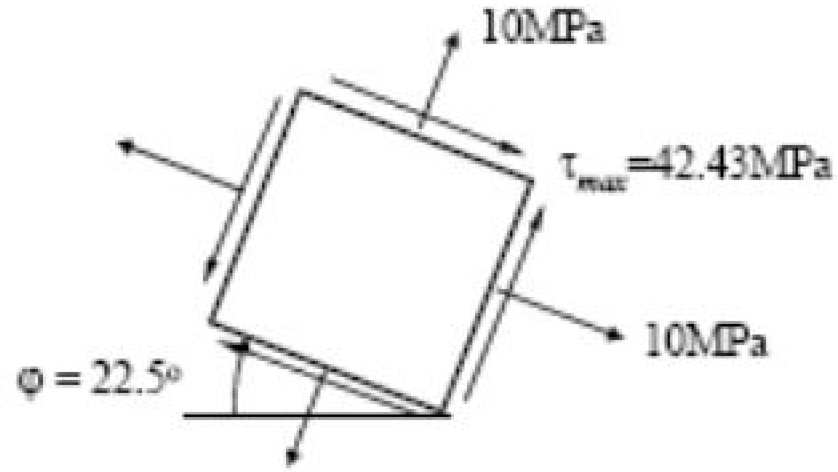
$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{40-(-20)}{2}\right)^2 + (30)^2} = 42.43MPa$$

Step 5: Draw infinitesimal elements indicating magnitude and orientations of both the Principal Stresses (the left hand of the figure) and the Maximum Shear Stress (the right hand of the figure). From the Mohr circle, σ_{11} rotates from the bold line by $2\theta_{p1}$ anticlockwise. So we rotate the oriented element showing the principal stress by $\theta=\theta_{p1}$ in the same direction as given on the left below. Similarly, in the Mohr circle, τ_{max} rotates by 2φ clockwise; hence the oriented element showing the maximum shear stress should be rotated by φ clockwise as given on the right figure below

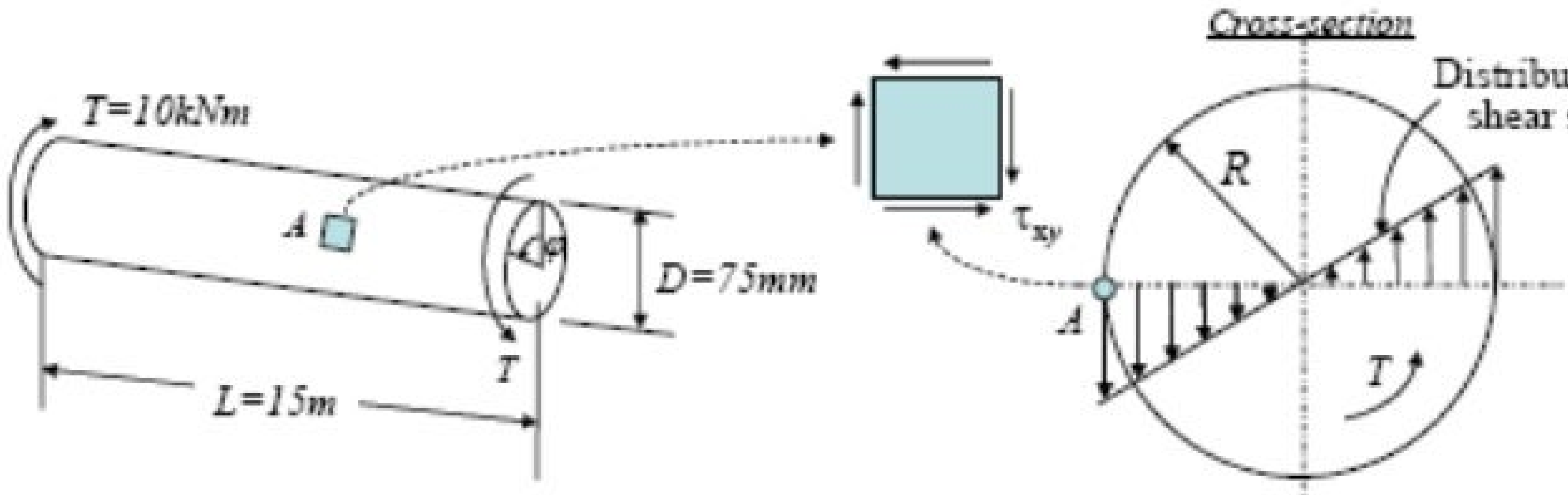
Orientation of Principal Stresses



Orientation of Maximum Shear Stresses



Example 10.3 When the torque of T=10kN/m is applied to a torsional shaft as shown, it produces a state of pure shear stress in the material. Determine the detailed stress status at element A for: a) the maximum shear stress and the principal stresses; b) the orientation of principal stresses.



Step 1: Compute the shear stress in the cross section

$$J = \frac{\pi D^4}{32} = \frac{\pi(0.075)^4}{32} = 3.106\times10^{-6}\,m^4$$

$$\tau = \frac{TR}{J} = \frac{T(D/2)}{J} = \frac{10\times10^3\times0.0375}{3.106\times10^{-6}} = 120.7MPa$$

$\tau_{xy} = -\tau = -120.7MPa$;   No normal stress at A, so we have $\sigma_{xx} = \sigma_{yy} = 0$

Step 2: Compute the maximum shear stress

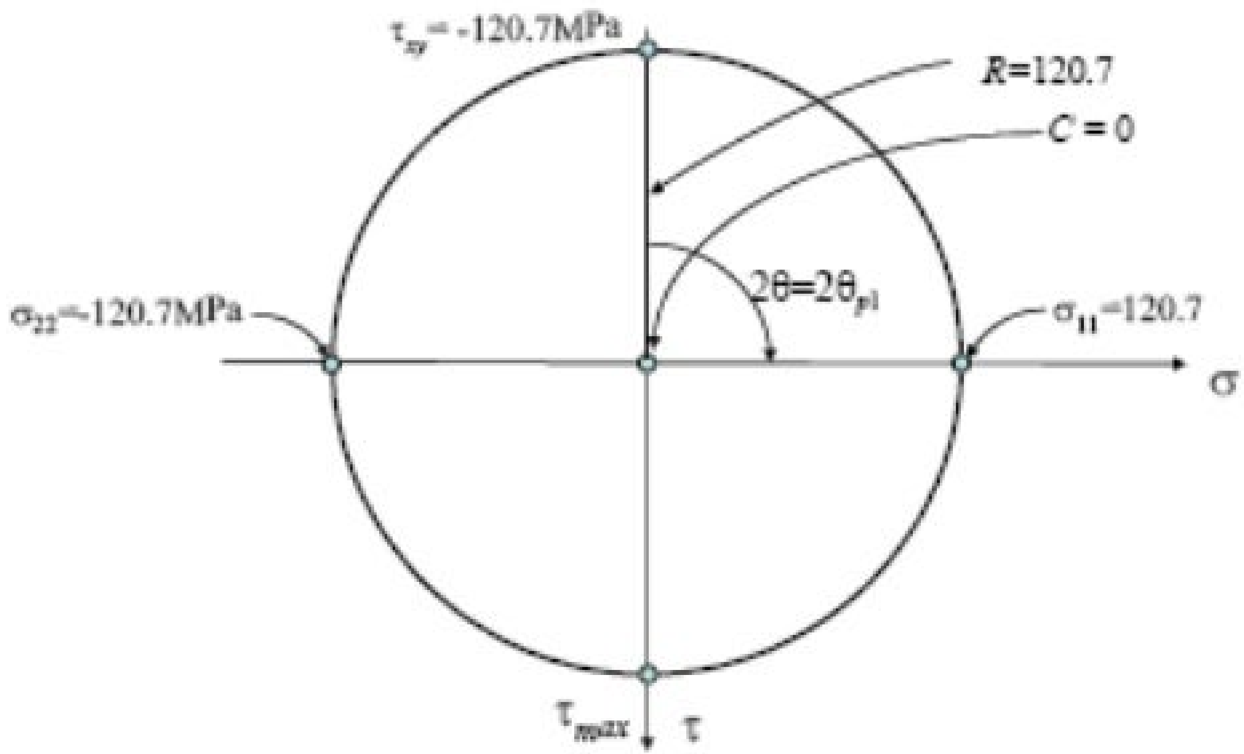
$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0-0}{2}\right)^2 + (-120.7)^2} = 120.7MPa$$

As expected, the maximum shear stress corresponds to a pure shear status. Through experiment, it has been found that ductile will fail due to shear stress.

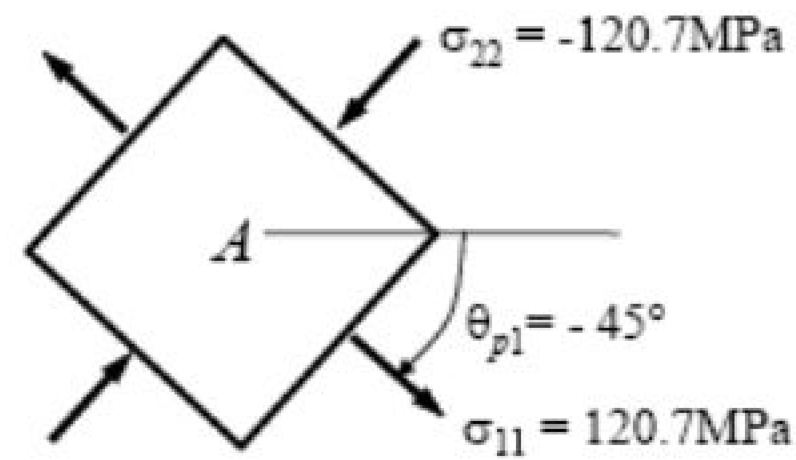
Step 3: Compute the principal stresses and their orientation

$$\begin{matrix}\sigma_{11}\\ \sigma_{22}\end{matrix} = \frac{(\sigma_{xx}+\sigma_{yy})}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \frac{(0+0)}{2} \pm \sqrt{\left(\frac{0-0}{2}\right)^2 + (-120.7)^2} = \begin{matrix}120.7MPa\\ -120.7MPa\end{matrix}$$

Orientation of principle stress: $tan\,2\theta = \frac{2\times(-120.7)}{0-(0)} = -\infty \;\therefore \theta_{p1} = \frac{arctan(-\infty)}{2} = -45°$



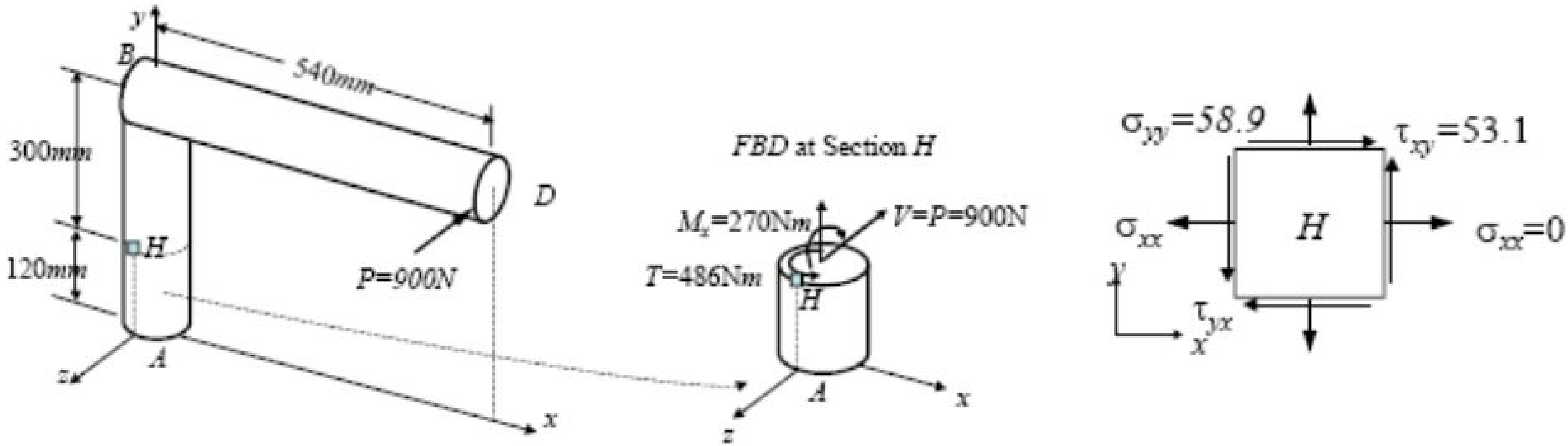
Orientation of Principal Stresses



Example.#.05

Thus the first principal stress $\sigma_1 = \tau_{xy}$ acts at $\theta_{p1}=-45°$ and the second principal stress $\sigma_2=-|\tau_{xy}|$ acts at $\theta_{p2}=90°+\theta_{p1}=45°$. Brittle material fails due to normal stress. That is why when a brittle material such as cast iron and chalk, is subjected to torsion (because usually its allowable tensile stress is much smaller than its allowable compressive stress) it will fail in tension at a 45°inclination.

Example 10.4 A single horizontal force *P*=900N is applied to end D of lever ABD. Knowing that portion AB of lever has a diameter of 36mm. Determine a) the normal and shear stress status at element at point *H*; b) the principal planes and principal stresses at *H*.



Step 1: Internal loads at the section *H*

Shear force:	$V = P = 900\text{N}$
Torque:	$T = 900 \times 0.540 = 486\text{N}m$
Bending moment:	$M_x = 900 \times 0.300 = 270\text{N}m$

Step 2: Compute normal bending stress at *H*

$$\sigma = -\frac{MR}{I} = -\frac{(-270) \times 0.018}{\pi \times (0.018)^4 / 4} = 58.9MPa\,, \quad \therefore \sigma_{yy} = \sigma = 58.9MPa$$

Step 3: Compute combined shear stress at H **Example.#.06**

Shear stress consists of torsional shear component T and transverse shear component due to V. However, transverse shear τ_V is zero at H from Table 7.1 and Example 7.2. We have

$$\tau = \tau_T + \tau_V = \frac{TR}{J} + \frac{VQ}{It} = \frac{486 \times 0.018}{\pi \times (0.018)^4 / 2} + 0 = 53.1MPa \qquad \therefore \tau_{xy} = \tau = 53.1MPa$$

Step 4: Determine stress status as shown in the right hand side of the top figure.

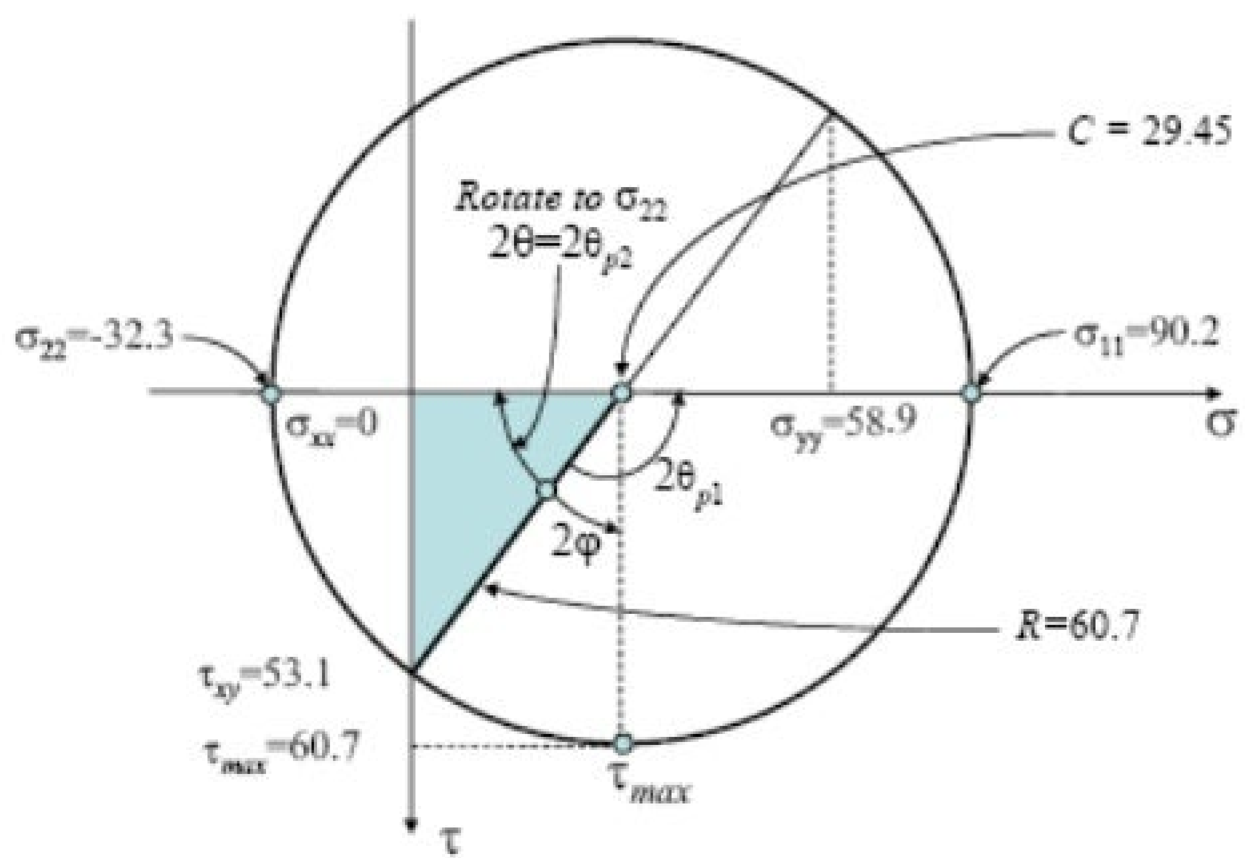
Step 5: Compute principal stress and their orientation

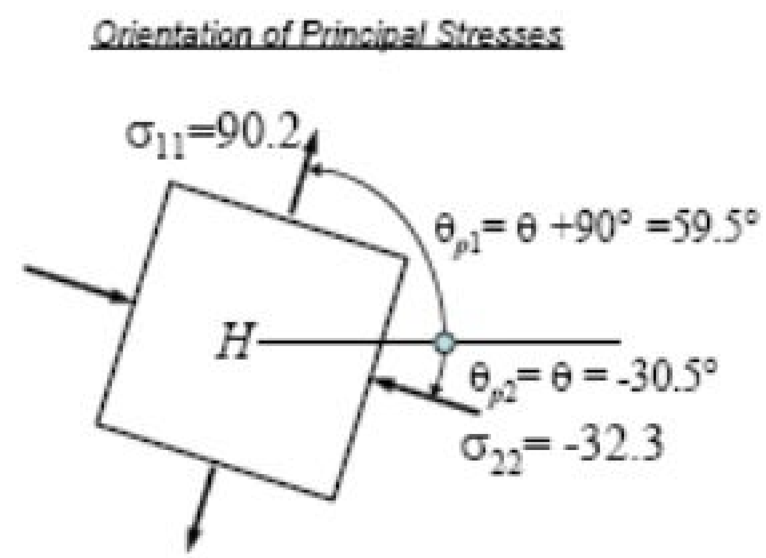
$$\begin{matrix}\sigma_{11} \\ \sigma_{22}\end{matrix} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \frac{(0 + 58.9)}{2} \pm \sqrt{\left(\frac{0 - 58.9}{2}\right)^2 + (53.1)^2} = \begin{matrix}90.2MPa \\ -32.3MPa\end{matrix}$$

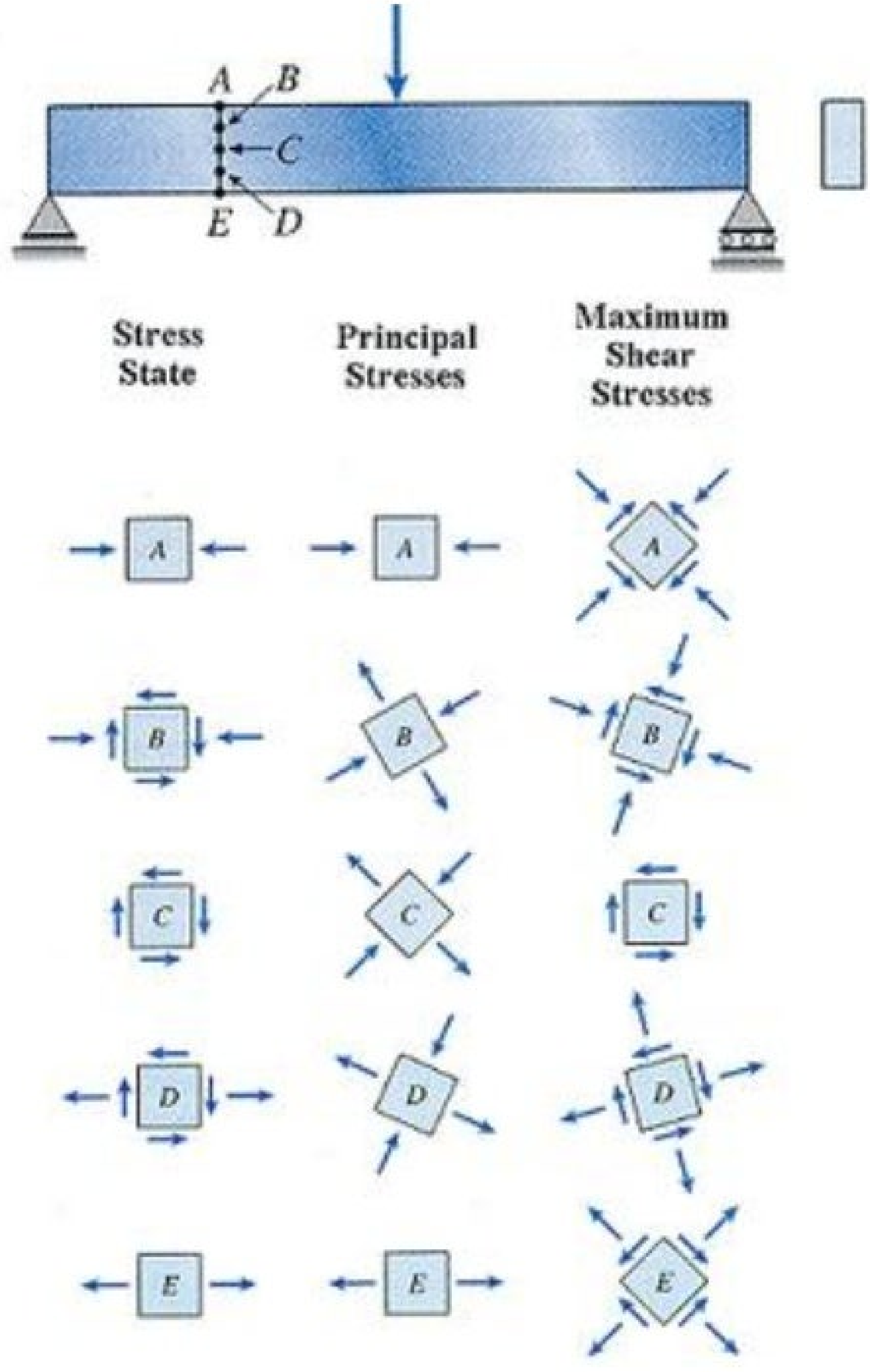
Orientation of principle stress: $tan\, 2\theta = \frac{2 \times 53.1}{0 - 58.9} = -1.8$

$$\therefore \theta_{p2} = \theta = \frac{arctan(-1.8)}{2} = -30.5^\circ \qquad \therefore \theta_{p1} = 90 + (-30.5^\circ) = 59.5^\circ$$

Step 6: Draw Mohr Circle



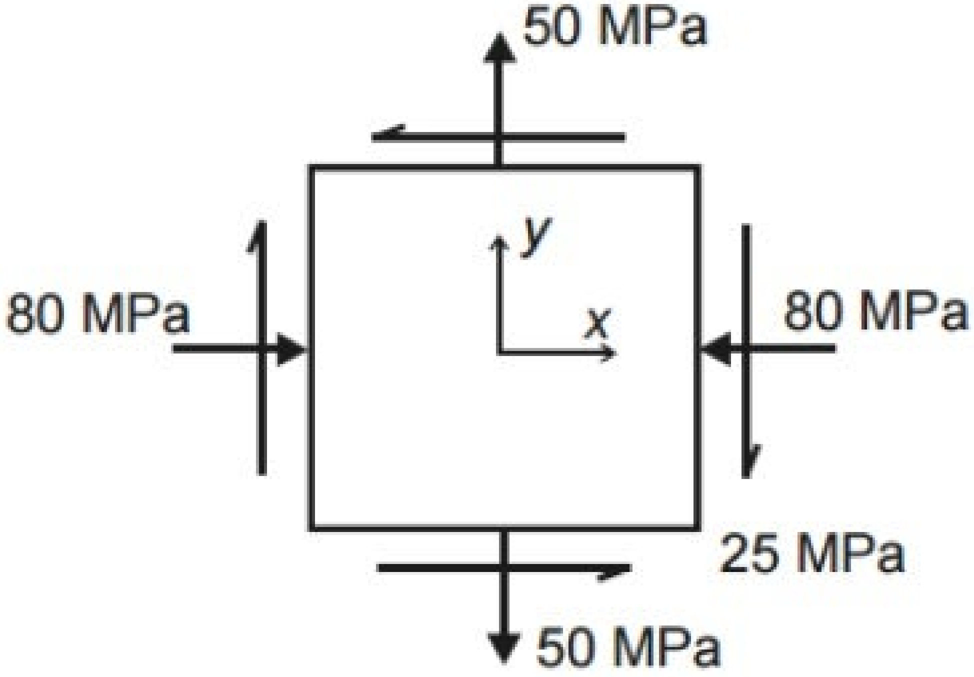




A
B
C
D
E
Stress State
Principal Stresses
Maximum Shear Stresses
A
A
A
B
B
B
C
C
C
D
D
D
E
E
E

Principal stresses and Stress Transformation

Example: The state of plane stress at a point is represented by the stress element below. Find the principal stresses.



$$M = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix}$$

We must find the eigenvalues of this matrix.

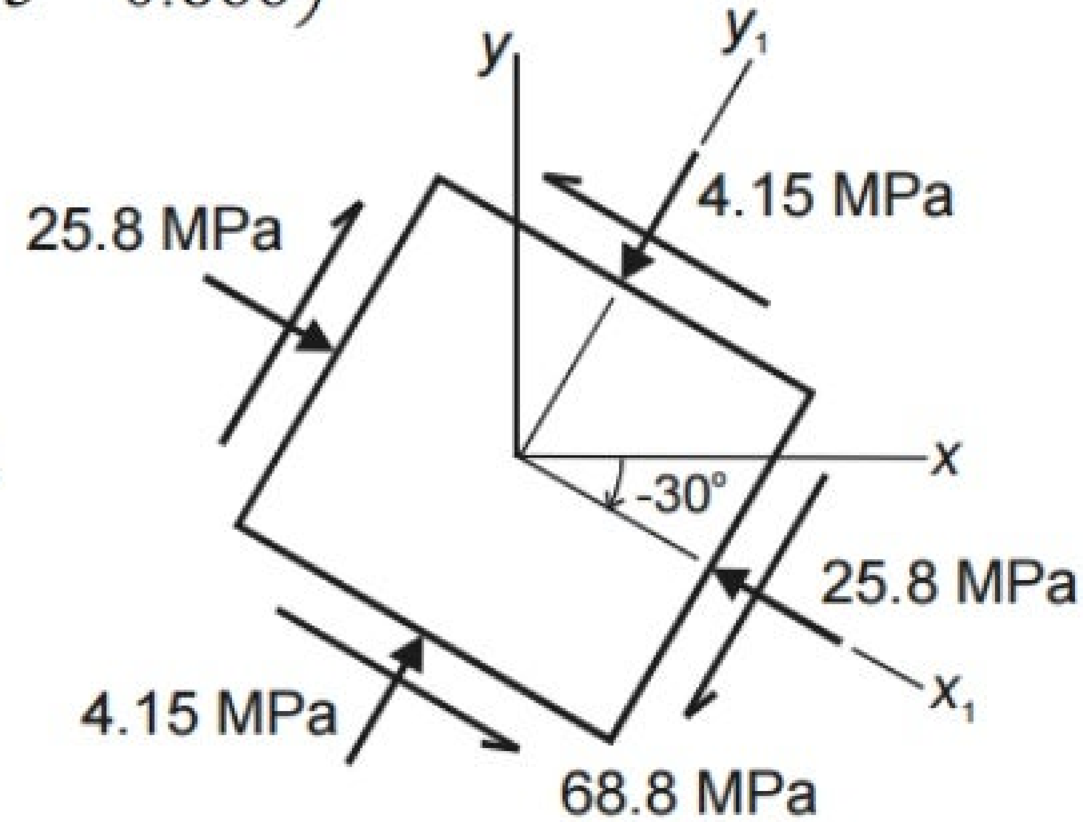
we can use the rotation matrix approach to find the stresses on an inclined element with θ = -30°.

$$R = \begin{pmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{pmatrix} = \begin{pmatrix} 0.866 & 0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

$$M' = R^T M \ R$$

$$M' = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix} \begin{pmatrix} -80 & -25 \\ -25 & 50 \end{pmatrix} \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix}$$

$$M' = \begin{pmatrix} -25.8 & -68.8 \\ -68.8 & -4.15 \end{pmatrix} = \begin{pmatrix} \sigma_{x1} & \tau_{xy} \\ \tau_{yx} & \sigma_{y1} \end{pmatrix}$$



Again, the transformation equations, Mohr's circle, and the stress tensor approach all give the same result.

3D Stress Components

The most general state of stress at a point may be represented by 6 components

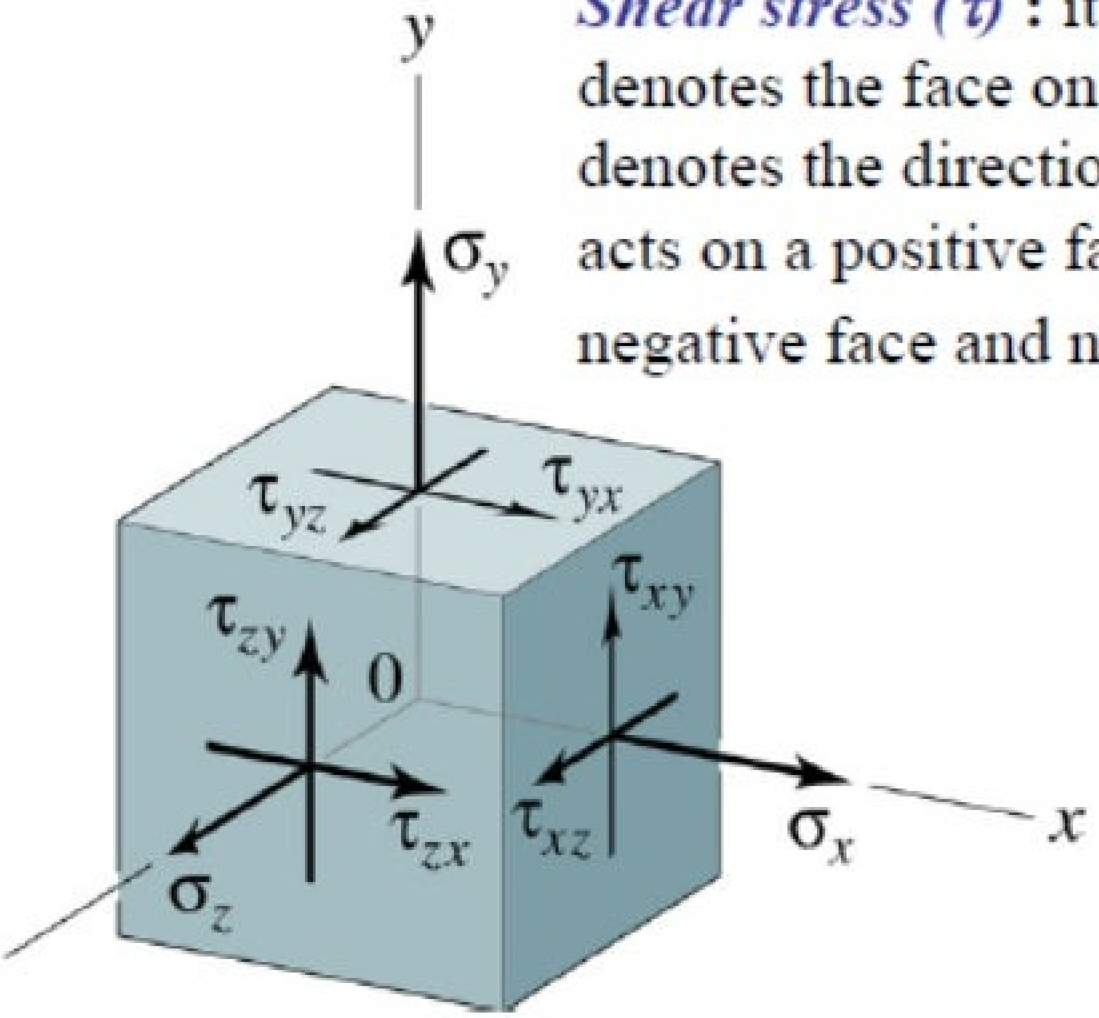
From equilibrium principles:

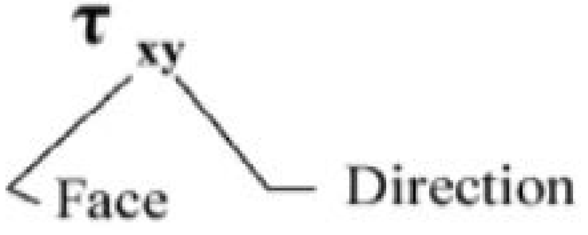
$\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$

Normal Stresses	σ_x	σ_y	σ_z
Shear Stresses	τ_{xy}	τ_{yz}	τ_{xz}

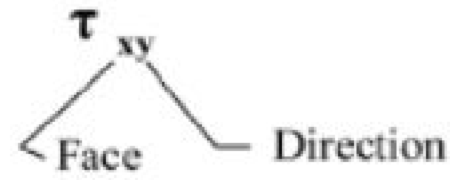
Normal stress (σ) : the subscript identifies the face on which the stress acts. Tension is positive and compression is negative.

Shear stress (τ) : it has two subscripts. The first subscript denotes the face on which the stress acts. The second subscript denotes the direction on that face. A shear stress is positive if it acts on a positive face and positive direction or if it acts in a negative face and negative direction.





For static equilibrium $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$ resulting in six independent scalar quantities. These six scalars can be arranged in a 3x3 matrix, giving us a ***stress tensor***.

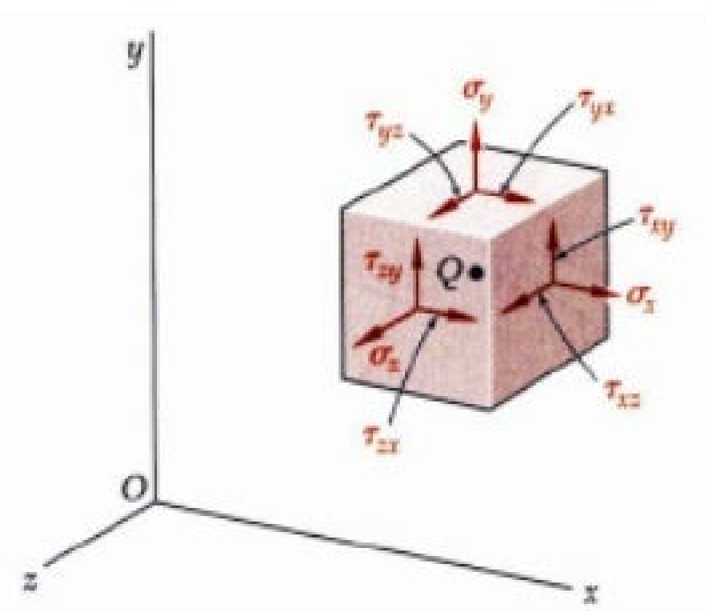


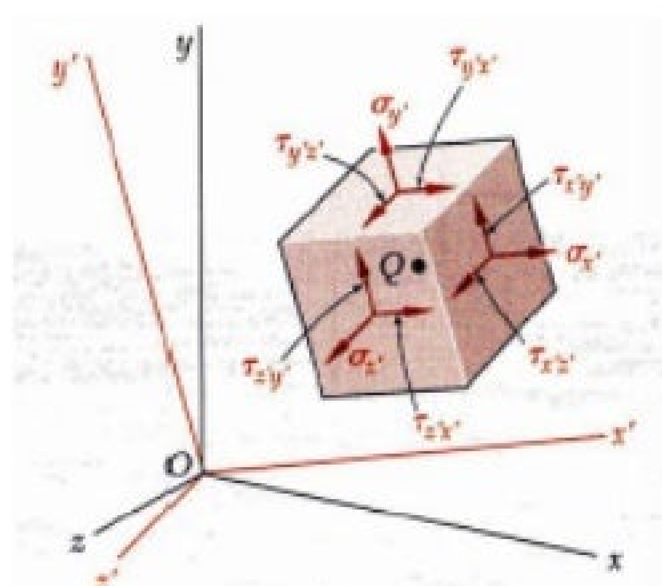
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

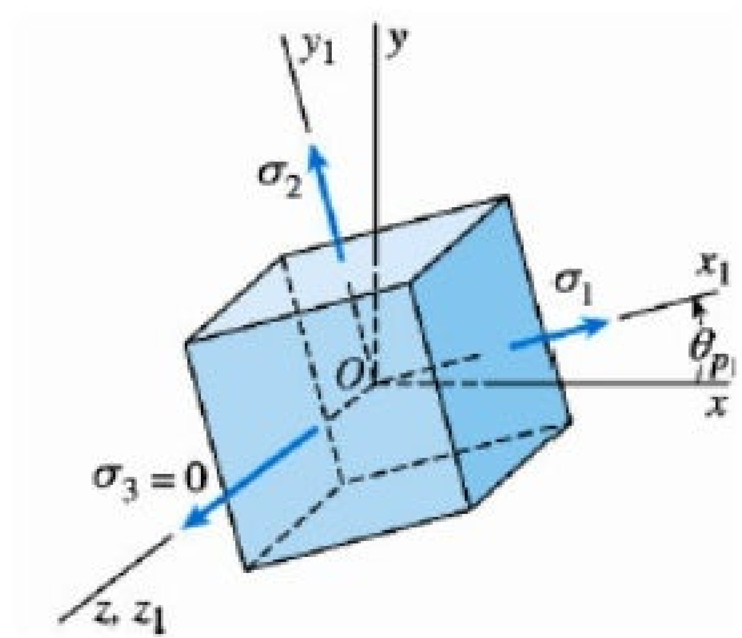
The sign convention for the stress elements is that a positive force on a positive face or a negative force on a negative face is positive. All others are negative.

The stress state is a second order tensor since it is a quantity associated with two directions (two subscripts direction of the surface normal and direction of the stress).

Same state of stress is represented by a different set of components if axes are rotated. There is a special set of components (when axes are rotated) where all the shear components are zero (principal stresses).







A property of a symmetric tensor is that there exists an orthogonal set of axes *1*, *2* and *3* ***(called principal axes)*** with respect to which the tensor elements are all zero except for those in the diagonal.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \xrightarrow{\text{Eigen values}} \boldsymbol{\sigma}' = \sigma'_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

In matrix notation the transformation is known as the ***Eigen-values***.

The principal stresses are the ***"new-axes"*** coordinate system. The angles between the ***"old-axes"*** and the ***"new-axes"*** are known as the ***Eigen-vectors***.

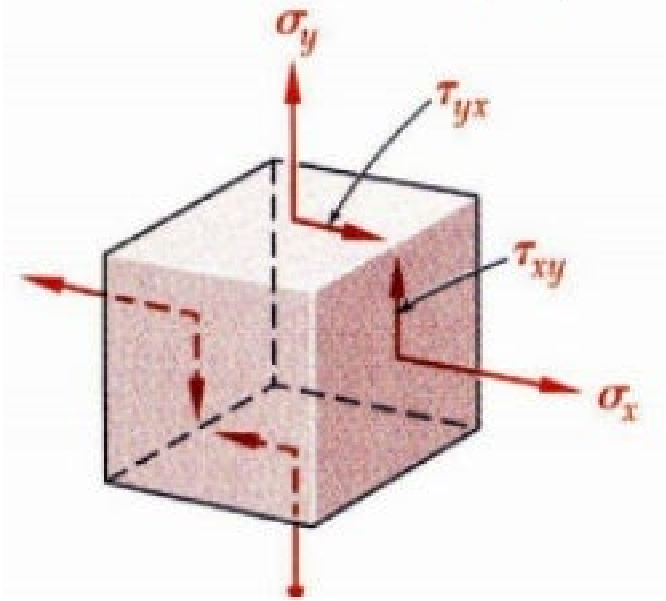
"old" axes

	x1	x2	x3
x'1	a11	a12	a13
"new" x'2	a21	a22	a23
x'3	a31	a32	a33

principal stress	Cosine of angle between X and the principal stress	Cosine of angle between Y and the principal stress	Cosine of angle between Z and the principal stress
σ_1	k1	l1	m1
σ_2	k2	l2	m2
σ_3	k3	l3	m3

Plane Stress

State of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by



$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

Sign Conventions for Shear Stress and Strain

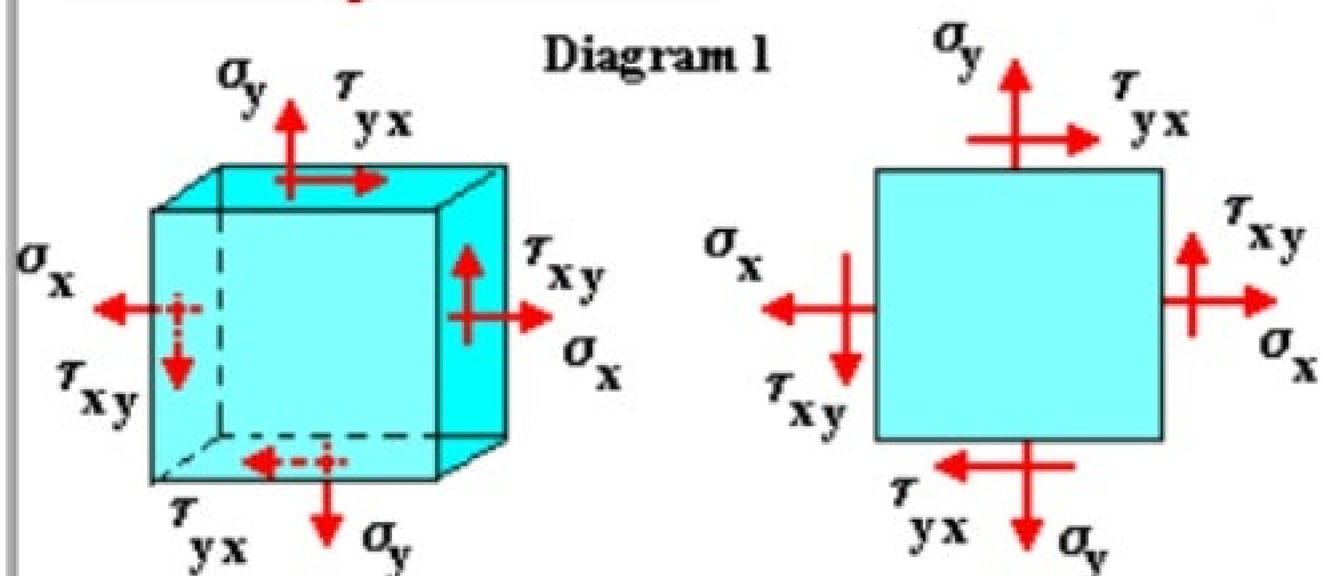


Diagram 1

The Shear Stress will be considered positive when a pair of shear stress acting on opposite sides of the element produce a counterclockwise (ccw) torque (couple).

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{vmatrix} = 0$$

How to calculate principal stresses in 3D elements

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

Expanding the determinant:

- $(\sigma_x - \sigma)(\sigma_y - \sigma)(\sigma_z - \sigma) + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x - \sigma)\tau_{yz}^2 - (\sigma_y - \sigma)\tau_{zx}^2 - (\sigma_z - \sigma)\tau_{xy}^2 = 0$

- $(\sigma_x\sigma_y - \sigma\sigma_y - \sigma_x\sigma + \sigma^2)(\sigma_z - \sigma) + 2\tau_{xy}\tau_{yz}\tau_{zx}$
  $- \sigma_x\tau_{yz}^2 + \sigma\tau_{yz}^2 - \sigma_y\tau_{zx}^2 + \sigma\tau_{zx}^2 - \sigma_z\tau_{xy}^2 + \sigma\tau_{xy}^2 = 0$

- $\sigma_x\sigma_y\sigma_z - \sigma\sigma_y\sigma_z - \sigma_x\sigma\sigma_z + \sigma^2\sigma_z - \sigma_x\sigma_y\sigma + \sigma^2\sigma_y + \sigma_x\sigma^2 - \sigma^3 + 2\tau_{xy}\tau_{yz}\tau_{zx}$
  $- \sigma_x\tau_{yz}^2 + \sigma\tau_{yz}^2 - \sigma_y\tau_{zx}^2 + \sigma\tau_{zx}^2 - \sigma_z\tau_{xy}^2 + \sigma\tau_{xy}^2 = 0$

- $\sigma_x\sigma_y\sigma_z - \sigma_y\sigma_z\sigma - \sigma_z\sigma_x\sigma + \sigma_z\sigma^2 - \sigma_x\sigma_y\sigma + \sigma_y\sigma^2 + \sigma_x\sigma^2 - \sigma^3 + 2\tau_{xy}\tau_{yz}\tau_{zx}$
  $- \sigma_x\tau_{yz}^2 + \tau_{yz}^2\sigma - \sigma_y\tau_{zx}^2 + \tau_{zx}^2\sigma - \sigma_z\tau_{xy}^2 + \tau_{xy}^2\sigma = 0$

How to calculate principal stresses in 3D elements

Rearranging terms we can write:

$$-\sigma^3 + (\sigma_x + \sigma_y + \sigma_z)\sigma^2 - (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma +$$
$$(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0$$

Multiplying both sides by -1 we get:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma -$$
$$(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0$$

The above equation is known as the characteristic equation of the stress tensor and it can be written as:

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0}$$

How to calculate principal stresses in 3D elements

where:

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$
$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix}$$
$$= \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

Notice that:

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = \det([\tilde{\sigma}])$$

I_1, I_2, I_3 → Principal scalar invariants of the stress tensor
I_1 → First scalar invariant of the stress tensor
I_2 → Second scalar invariant of the stress tensor
I_3 → Third scalar invariant of the stress tensor

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

How to calculate principal stresses in 3D elements

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \quad \text{or} \quad I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

Since the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

is cubic in σ, it has three possible roots. It can be shown that always the three roots are real.

We will denote by $\sigma_1, \sigma_2, \sigma_3$ the roots of the cubic equation and we will always use the following convention:

$$\boxed{\sigma_1 \geq \sigma_2 \geq \sigma_3}$$

$\sigma_1, \sigma_2, \sigma_3$ → Principal stresses

σ_1 → Maximum principal stress (maximum algebraic value)

σ_3 → Minimum principal stress (minimum algebraic value)

For Direction Cosines

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Before proceeding with this problem, we must define the invariants of stress.

Stress Invariants – Step 1

When the stress tensor is expressed with reference to sets of axes oriented in different directions, the components of the tensor change. However, certain functions of the components do not change. These are known as stress invariants, expressed as I_1, I_2, I_3, where:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

The expression for the first invariant, I_1, indicates that for a given stress state, whatever its orientation, the values of the three normal stresses will add up to the same value I_1.

Stress Invariants - Step 2

When the principal stresses have to be calculated from the components of the stress tensor, a cubic equation can be used for finding the three values σ_1, σ_2, σ_3:

$$\sigma^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

or

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

Because the values of the principal stresses must be independent of the choice of axes, the coefficients I_1, I_2, I_3 must be invariant with respect to the orientation of the axes. It can also be noted from the first invariant that:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3$$

For Direction Cosines

Stress Invariants – Step 3

Each principal stress is related to a principal stress axis, whose direction cosines can be obtained, for example for σ_1, through a set of simultaneous, homogeneous equations in λ_{x1}, λ_{y1}, λ_{z1}, based on the dot product theorem of vector analysis:

$$\frac{\lambda_{x1}}{A} = \frac{\lambda_{y1}}{B} = \frac{\lambda_{z1}}{C} = K$$

Where:

$$A = \begin{vmatrix} \sigma_{yy} - \sigma_1 & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} - \sigma_1 \end{vmatrix} \qquad B = -\begin{vmatrix} \sigma_{xy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zz} - \sigma_1 \end{vmatrix}$$

$$C = \begin{vmatrix} \sigma_{xy} & \sigma_{yy} - \sigma_1 \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix}$$

Substituting for λ_{x1}, λ_{y1}, λ_{z1} in the dot product relation for any unit vector gives:

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

$$\lambda_{x1} = A/(A^2 + B^2 + C^2)^{1/2}$$

$$\lambda_{y1} = B/(A^2 + B^2 + C^2)^{1/2}$$

$$\lambda_{z1} = C/(A^2 + B^2 + C^2)^{1/2}$$

Stress Invariants – Step 4

Proceeding in a similar way, the vectors of direction cosines for the intermediate and minor principal stresses axes, i.e. (λ_{x2}, λ_{y2} , λ_{z2}) and (λ_{x3}, λ_{y3} , λ_{z3}) are obtained by repeating the calculations but substituting σ_2 and σ_3.

$$\frac{\lambda_{x2}}{D} = \frac{\lambda_{y2}}{E} = \frac{\lambda_{z2}}{F} = K$$

Where:

$$D = -\begin{vmatrix} \sigma_{xy} & \sigma_{zx} \\ \sigma_{yz} & \sigma_{zz} - \sigma_2 \end{vmatrix} \qquad E = \begin{vmatrix} \sigma_{xx} - \sigma_2 & \sigma_{zx} \\ \sigma_{zx} & \sigma_{zz} - \sigma_2 \end{vmatrix}$$

$$F = -\begin{vmatrix} \sigma_{xx} - \sigma_2 & \sigma_{xy} \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix}$$

$$\frac{\lambda_{x3}}{G} = \frac{\lambda_{y3}}{H} = \frac{\lambda_{z3}}{I} = K$$

Where:

$$G = \begin{vmatrix} \sigma_{xy} & \sigma_{zx} \\ \sigma_{yy} - \sigma_3 & \sigma_{yz} \end{vmatrix} \qquad H = -\begin{vmatrix} \sigma_{xx} - \sigma_3 & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yz} \end{vmatrix}$$

$$I = \begin{vmatrix} \sigma_{xx} - \sigma_3 & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \sigma_3 \end{vmatrix}$$

For Direction Cosines

Stress Invariants – Step 5

The procedure for calculating the principal stresses and the orientations of the principal stress axes is simply the determination of the eigenvalues of the stress matrix, and the eigenvector for each eigenvalue. Thus, some simple checks can be performed to assess the correctness of the solution:

Invariance of the sum of the normal stresses requires that:

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

The condition of orthogonality requires that each of the three dot products of the vectors of the direction cosines must vanish:

$$\lambda_{x1}\,\lambda_{x2} + \lambda_{y1}\,\lambda_{y2} + \lambda_{z1}\,\lambda_{z2} = 0$$

For Principal Stresses

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. **Step 1**: Solving the stress invariants we get:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 14.0 + 34.8 + 16.1 = \mathbf{64.9\ MPa}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$
$$= (14.0)(34.8) + (34.8)(16.1) + (16.1)(14.0) - (-0.6)^2 - (6.0)^2 - (-2.1)^2$$
$$= \mathbf{1232.1\ MPa}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$
$$= (14.0)(34.8)(16.1) + 2(-0.6)(6.0)(-2.1) - (14.0)(6.0)^2 - (34.8)(-2.1)^2 - (16.1)(-0.6)^2$$
$$= \mathbf{7195.8\ MPa}$$

For Principal Stresses

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = −0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = −2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 2: Substituting these values into the cubic equation we get:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$\sigma^3 - 64.9\sigma^2 + 1232.1\sigma - 7195.8 = 0$$

Step 3: Solving the cubic equation gives: $\sigma = \begin{bmatrix} 36.6 \\ 16.0 \\ 12.3 \end{bmatrix}$ MPa

Thus: $\sigma_1 = 36.6$ MPa $\sigma_2 = 16.0$ MPa $\sigma_3 = 12.3$ MPa

Principal Stresses and DCs

EXAMPLE 2.1 ***Principal Stresses and Principal Directions***

The state of stress at a point in a machine part is given by $\sigma_{xx} = -10$, $\sigma_{yy} = 30$, $\sigma_{xy} = 15$, and $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$; see Figure E2.1*a*. Determine the principal stresses and orientation of the principal axes at the point.

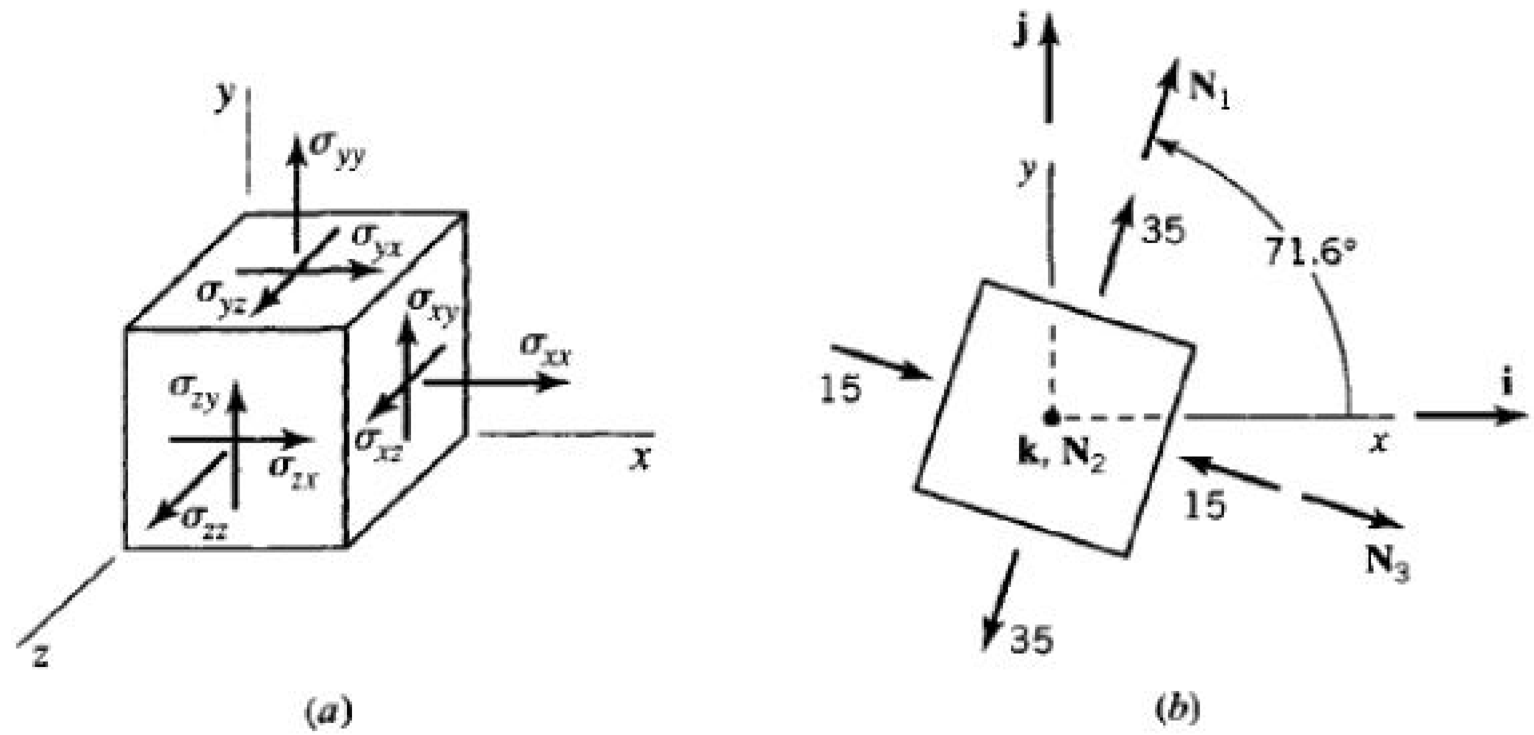


FIGURE E2.1

Solution

By Eq. 2.21 the three stress invariants are

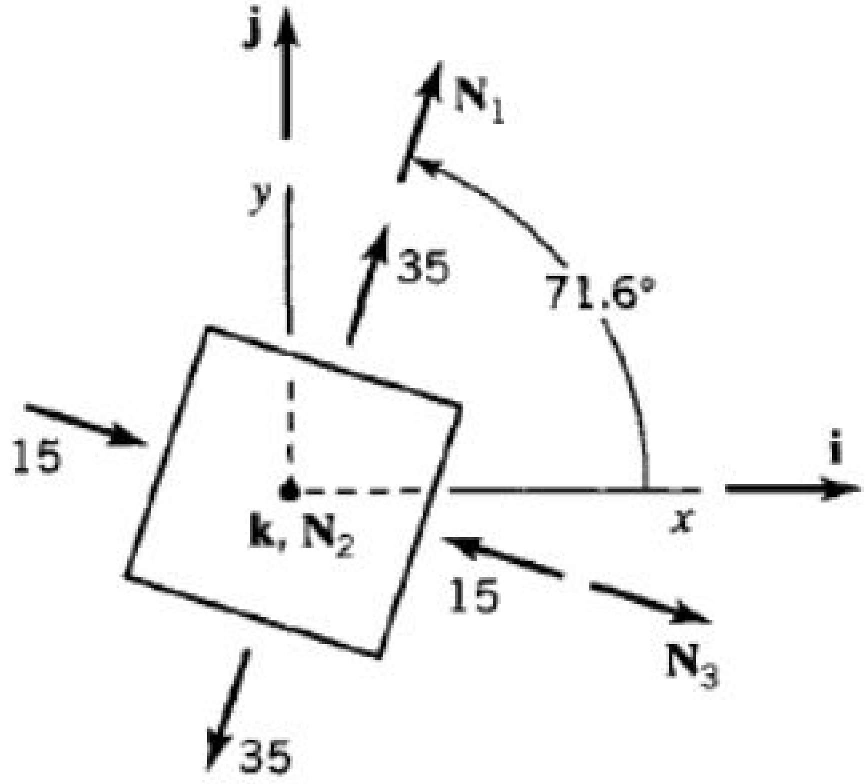
$$I_1 = 20, \quad I_2 = -525, \quad \text{and} \quad I_3 = 0$$

Substituting the invariants into Eq. 2.20 and solving for the three roots of this equation, we obtain the principal stresses

$$\sigma_1 = 35, \quad \sigma_2 = 0, \quad \text{and} \quad \sigma_3 = -15$$

Principal Stresses and DCs

$$l(\sigma_{xx} - \sigma) + m\sigma_{xy} + n\sigma_{xz} = 0$$
$$l\sigma_{xy} + m(\sigma_{yy} - \sigma) + n\sigma_{yz} = 0$$
$$l\sigma_{xz} + m\sigma_{yz} + n(\sigma_{zz} - \sigma) = 0$$



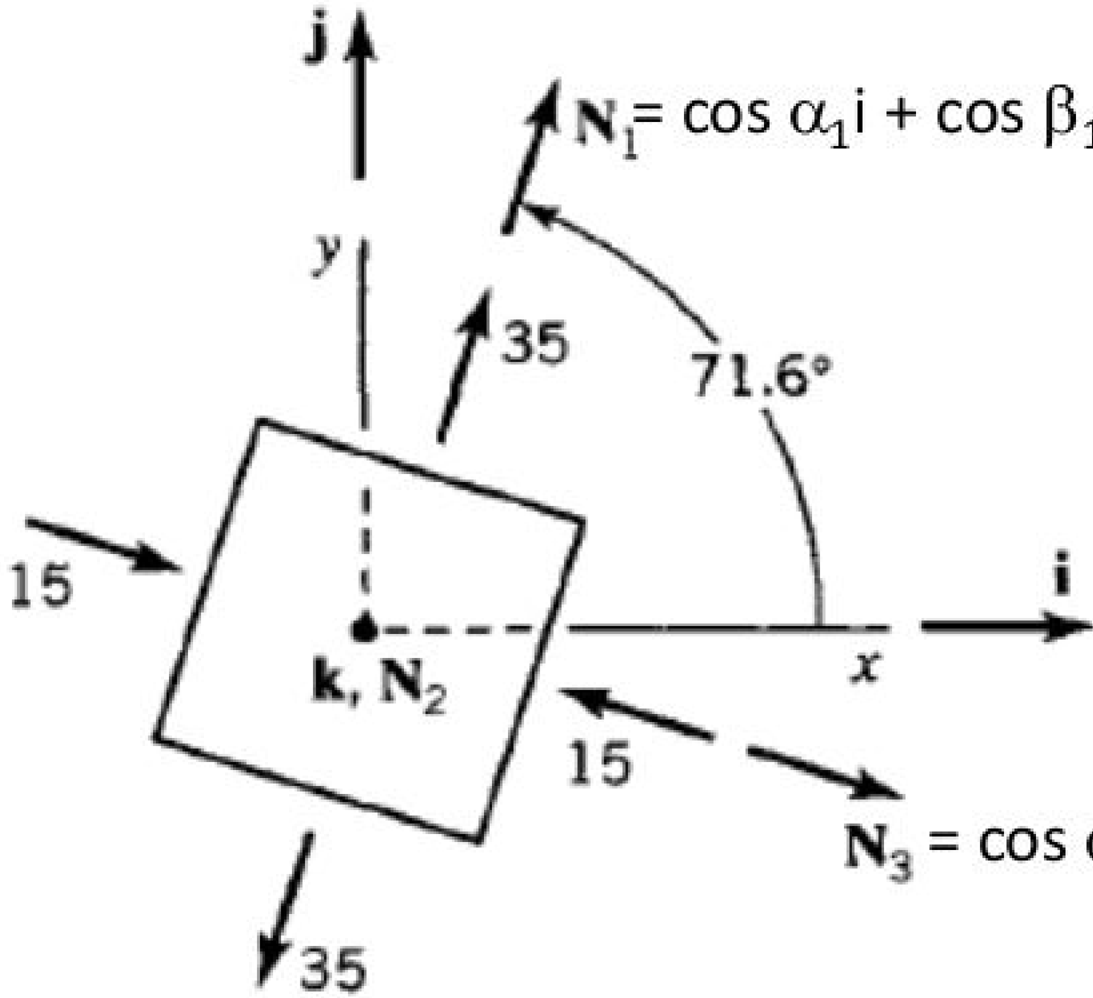
Note: In above equation l, m, and n are direction cosines (DCs).

For $\sigma 1$ these l, m and n becomes l1, m1 and n1 and DCs for direction vector N1 which is normal to the principal plane 1.

Similarly,

l2, m2 and n2 are DCs for direction vector N2 which is normal to the principal plane 2.
l3, m3 and n3 are DCs for direction vector N3 which is normal to the principal plane 3.

Relationship b/w DCs and Principal plane angles



$N_1 = \cos\alpha_1 i + \cos\beta_1 j + \cos\gamma_1 k = l_1 i + m_1 j + n_1 k$

Here $\alpha 1$ = 71.6 degree
While $\beta 1$ = 18.4 degree
$\gamma 1$ = 90.0

$N_3 = \cos\alpha_3 i + \cos\beta_3 j + \cos\gamma_3 k$

Here $\alpha 3$ = 18.4 degree
While $\beta 3$ = 71.6 degree
$\gamma 3$ = 90.0

Properties of DCs

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$$

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$
$$l_1 n_1 + l_2 n_2 + l_3 n_3 = 0$$
$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$$

$$l_1^2 + l_2^2 + l_3^2 = 1,$$
$$m_1^2 + m_2^2 + m_3^2 = 1,$$
$$n_1^2 + n_2^2 + n_3^2 = 1,$$

Also,

$$\boxed{l_1^2 + m_1^2 + n_1^2 = 1}$$

How to calculate of DCs

$$l(\sigma_{xx} - \sigma) + m\sigma_{xy} + n\sigma_{xz} = 0$$
$$l\sigma_{xy} + m(\sigma_{yy} - \sigma) + n\sigma_{yz} = 0$$
$$l\sigma_{xz} + m\sigma_{yz} + n(\sigma_{zz} - \sigma) = 0$$

In above equations replace value of s by s1 and l, m, and n by l1, m1 and n1.

$$-45l_1 + 15m_1 = 0$$
$$15l_1 - 5m_1 = 0$$
$$-35n_1 = 0$$
$$l_1^2 + m_1^2 + n_1^2 = 1$$

How to calculate of DCs

Only two of the first three of these equations are independent. Equation (c) gives

$$n_1 = 0$$

Simultaneous solution of Eqs. (b) and (d) yields the result

$$l_1^2 = 0.10$$

or

$$l_1 = \pm 0.3162$$

Substituting into Eq. (b) for l_1, we obtain

$$m_1 = \pm 0.9487$$

where the order of the + and – signs corresponds to those of l_1. Note also that Eq. (a) is satisfied with these values of l_1, m_1, and n_1. Thus, the first principal axis is directed along unit vector $\mathbf{N}_1$, where

$$\mathbf{N}_1 = 0.3162\mathbf{i} + 0.9487\mathbf{j} \; ; \; \theta_x = 71.6^\circ \quad \text{(e)}$$

or

$$\mathbf{N}_1 = -0.3162\mathbf{i} - 0.9487\mathbf{j} \quad \text{(f)}$$

How to calculate of DCs

The orientation of the second principal axis is found by substitution of $\sigma = \sigma_2 = 0$ into Eq. 2.18, which yields

$$l_2 = 0 \quad \text{and} \quad m_2 = 0$$

Proceeding as for σ_1, we then obtain

$$n_2 = \pm 1$$

from which

$$\mathbf{N}_2 = \pm \mathbf{k}$$

where **k** is a unit vector along the z axis.

The orientation of the third principal axis is found in a similar manner:

$$l_3 = \pm 0.9487$$

$$m_3 = \mp 0.3162$$

$$n_3 = 0$$

To establish a definite sign convention for the principal axes, we require them to form a right-handed triad. If $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit vectors that define the directions of the first two principal axes, then the unit vector $\mathbf{N}_3$ for the third principal axis is determined by the right-hand rule of vector multiplication. Thus, we have

$$\mathbf{N}_3 = \mathbf{N}_1 \times \mathbf{N}_2$$

How to calculate of DCs

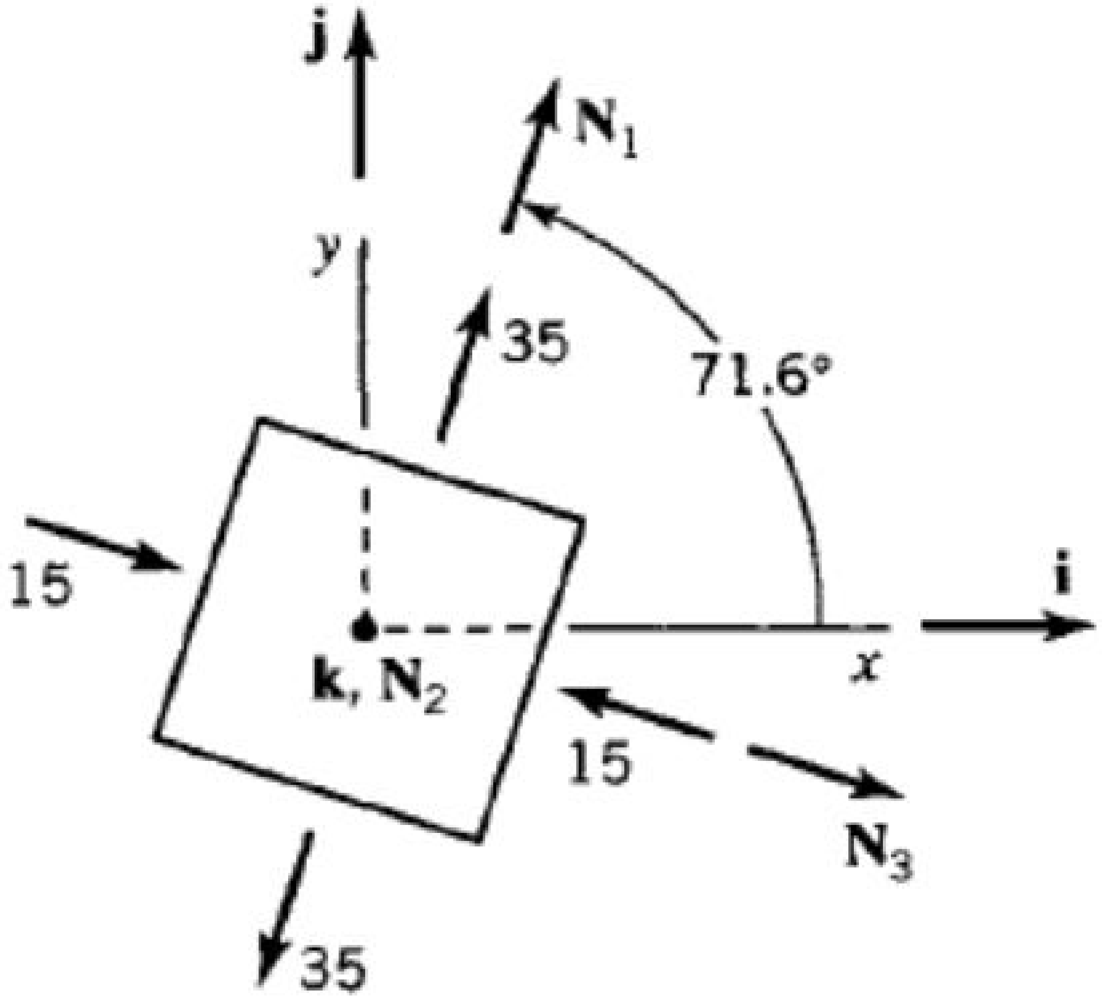
$$\mathbf{N}_3 = \mathbf{N}_1 \times \mathbf{N}_2$$

or

$$\mathbf{N}_3 = (m_1 n_2 - m_2 n_1)\mathbf{i} + (l_2 n_1 - l_1 n_2)\mathbf{j} + (l_1 m_2 - l_2 m_1)\mathbf{k} \qquad \text{(g)}$$

In our example, if we arbitrarily select $\mathbf{N}_1$ from Eq. (e) and $\mathbf{N}_2 = +\mathbf{k}$, we obtain $\mathbf{N}_3$ from Eq. (g) as

$$\mathbf{N}_3 = 0.9487\mathbf{i} - 0.3162\mathbf{j}$$



Alternate way to calculate DCs (lmn)

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. <u>Step 4</u>: Obtain the direction cosines (direction σ_1) by first solving for the determinates:

$$A = \begin{vmatrix} \sigma_{yy}-\sigma_1 & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz}-\sigma_1 \end{vmatrix} \Rightarrow \begin{vmatrix} 34.8-36.6 & 6.0 \\ 6.0 & 16.1-36.6 \end{vmatrix} \Rightarrow A = 0.90$$

$$B = -\begin{vmatrix} \sigma_{xy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zz}-\sigma_1 \end{vmatrix} \Rightarrow -\begin{vmatrix} -0.6 & 6.0 \\ -2.1 & 16.1-36.6 \end{vmatrix} \Rightarrow B = -24.90$$

$$C = \begin{vmatrix} \sigma_{xy} & \sigma_{yy}-\sigma_1 \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix} \Rightarrow \begin{vmatrix} -0.6 & 34.8-36.6 \\ -2.1 & 6.0 \end{vmatrix} \Rightarrow C = -7.38$$

For Direction Cosines

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = -0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = -2.1 MPa

Determine the principal stresses and their direction cosines.

A. Step 5: Substituting the determinates into the equations for the direction cosines for σ_1 gives:

$l_1 = \lambda_{x1} = A/(A^2 + B^2 + C^2)^{1/2}$

$m_1 = \lambda_{y1} = B/(A^2 + B^2 + C^2)^{1/2}$

$n_1 = \lambda_{z1} = C/(A^2 + B^2 + C^2)^{1/2}$

$$\lambda_{x1} = 0.90 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = 0.035$$

$$\lambda_{y1} = -24.90 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = -0.958$$

$$\lambda_{z1} = -7.38 \Big/ \left((0.90)^2 + (-24.90)^2 + (-7.38)^2\right)^{0.5} = -0.284$$

For Direction Cosines

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = –0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = –2.1 MPa

Determine the principal stresses and their direction cosines.

A. **Step 6**: Repeating for σ_2 and σ_3 , gives the direction cosines:

$\lambda_{x2} = -0.668$ $\lambda_{x3} = 0.741$
$\lambda_{y2} = -0.246$ $\lambda_{y3} = -0.154$
$\lambda_{z2} = 0.702$ $\lambda_{z3} = 0.653$

For Direction Cosines

Q. Six components of stress are measured at a point:

σ_{xx} = 14.0 MPa τ_{xy} = −0.6 MPa
σ_{yy} = 34.8 MPa τ_{yz} = 6.0 MPa
σ_{zz} = 16.1 MPa τ_{xz} = −2.1 MPa

Determine the principal stresses and their direction cosines.

A. Thus:

σ_1 = 36.6 MPa	σ_2 = 16.0 MPa	σ_3 = 12.3 MPa
$\lambda_{x1} = 0.035$	$\lambda_{x2} = -0.668$	$\lambda_{x3} = 0.741$
$\lambda_{y1} = -0.958$	$\lambda_{y2} = -0.246$	$\lambda_{y3} = -0.154$
$\lambda_{z1} = -0.284$	$\lambda_{z2} = 0.702$	$\lambda_{z3} = 0.653$

2.3.3 Normal Stress and Shear Stress on an Oblique Plane

The normal stress σ_{PN} on the plane P is the projection of the vector $\boldsymbol{\sigma}_P$ in the direction of N; that is, $\sigma_{PN} = \boldsymbol{\sigma}_P \cdot \mathbf{N}$. Hence, by Eqs. 2.7, 2.9, and 2.10

$$\sigma_{PN} = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2mn\sigma_{yz} + 2ln\sigma_{xz} + 2lm\sigma_{xy} \quad (2.11)$$

Often, the maximum value of σ_{PN} at a point is of importance in design (see Section 4.1). Of the infinite number of planes through point 0, σ_{PN} attains a maximum value called the *maximum principal stress* on one of these planes. The method of determining this stress and the orientation of the plane on which it acts is developed in Section 2.4.

To compute the magnitude of the shear stress σ_{PS} on plane P, we note by geometry (Figure 2.7) that

$$\sigma_{PS} = \sqrt{\sigma_P^2 - \sigma_{PN}^2} = \sqrt{\sigma_{Px}^2 + \sigma_{Py}^2 + \sigma_{Pz}^2 - \sigma_{PN}^2}$$

$$\sigma_{Px} = l\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx}$$
$$\sigma_{Py} = l\sigma_{xy} + m\sigma_{yy} + n\sigma_{zy}$$
$$\sigma_{Pz} = l\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz}$$

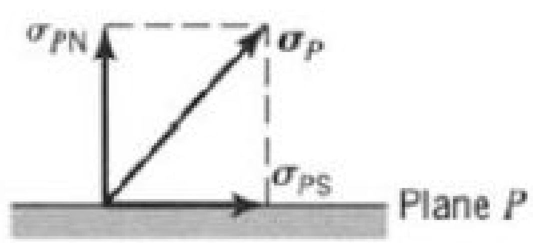


FIGURE 2.7 Normal and shear stress components of stress vector on an arbitrary plane.

Lecture.#.06

Once the state of stress at a point is expressed in terms of the principal stresses, three Mohr's circles of stress can be constructed as indicated in Figure 2.13. Consider plane *P* whose normal relative to the principal axes has direction cosines *l*, *m*, and *n*. The normal stress σ_{NN} on plane *P* is, by Eq. 2.11,

$$\sigma_{NN} = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 \tag{2.40}$$

Similarly, the square of the shear stress σ_{NS} on plane *P* is, by Eqs. 2.10 and 2.12,

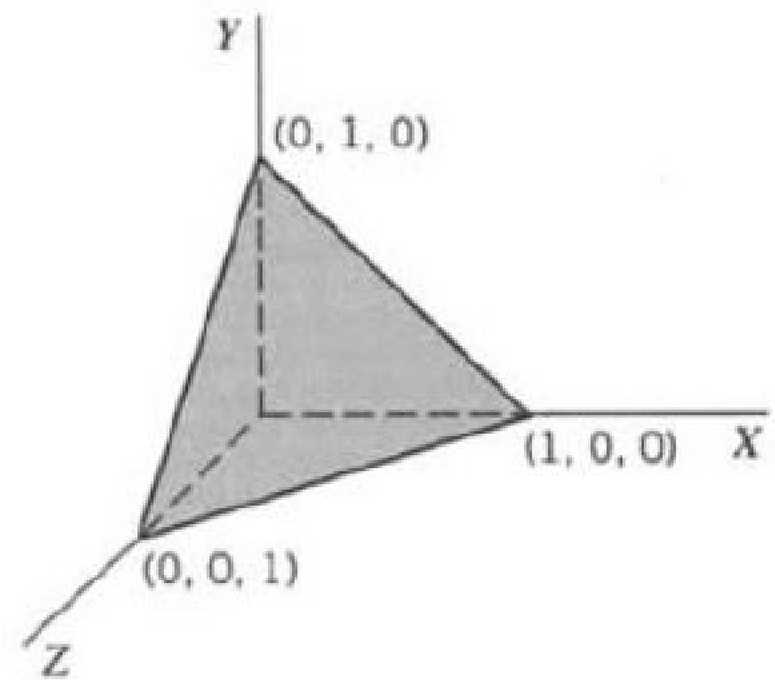
$$\sigma_{NS}^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2 - \sigma_{NN}^2 \tag{2.41}$$

2.4.4 Octahedral Stress

Let (X, Y, Z) be principal axes. Consider the family of planes whose unit normals satisfy the relation $l^2 = m^2 = n^2 = \frac{1}{3}$ with respect to the principal axes (X, Y, Z). There are eight such planes (the octahedral planes, Figure 2.9) that make equal angles with respect to the (X, Y, Z) directions. Therefore, the normal and shear stress components associated with these planes are called the *octahedral normal stress* σ_{oct} and *octahedral shear stress* τ_{oct}. By Eqs. 2.10–2.12, we obtain

$$\sigma_{oct} = \frac{1}{3}I_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\tau_{oct} = \sqrt{\frac{2}{9}I_1^2 - \frac{2}{3}I_2} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \quad (2.22)$$



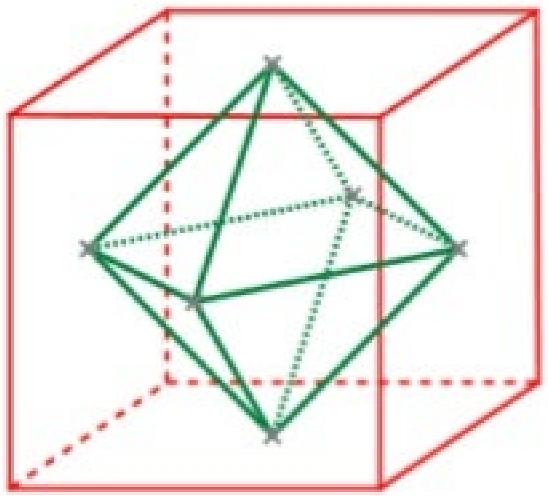


FIGURE 2.9 Octahedral plane for $l = m = n = 1/\sqrt{3}$, relative to principal axes (X, Y, Z).

Octahedral Stress

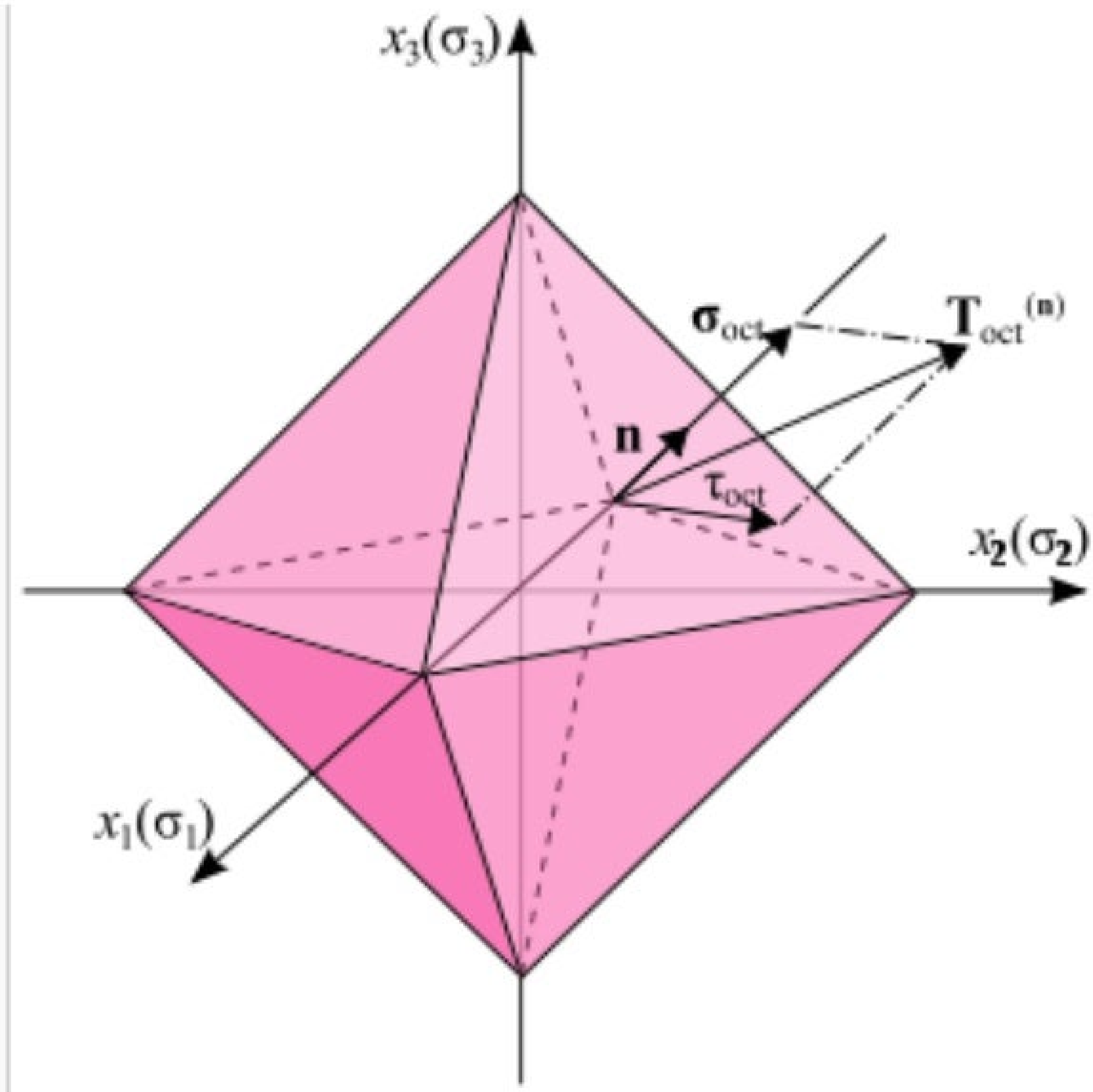


Figure 6. Octahedral stress planes

2.4.5 Mean and Deviator Stresses

Experiments indicate that yielding and plastic deformation of ductile metals are essentially independent of the mean normal stress σ_m, where

$$\sigma_m = \frac{1}{3}I_1 = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{2.24}$$

Comparing Eqs. 2.22–2.24, we note that the mean normal stress σ_m is equal to σ_{oct}. Most plasticity theories postulate that plastic behavior of materials is related primarily to that part of the stress tensor that is independent of σ_m. Therefore, the stress array (Eq. 2.5) is rewritten in the following form:

$$\mathbf{T} = \mathbf{T}_m + \mathbf{T}_d \tag{2.25}$$

where **T** symbolically represents the stress array, Eq. 2.5, and

$$\mathbf{T}_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

Tm = Mean Stress Tensor

and

$$\mathbf{T}_d = \begin{bmatrix} \sigma_{xx} - \sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_m & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_m \end{bmatrix}$$

Td = Deviatic Stress Tensor

EXAMPLE 2.5 ***Three-Dimensional Stress Quantities***

At a certain point in a drive shaft coupling, the stress components relative to axes (x, y, z) are σ_{xx} = 80 MPa, σ_{yy} = 60 MPa, σ_{zz} = 20 MPa, σ_{xy} = 20 MPa, σ_{xz} = 40 MPa, and σ_{yz} = 10 MPa.

(a) Determine the stress vector on a plane normal to the vector $\mathbf{R} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) Determine the principal stresses σ_1, σ_2, and σ_3 and the maximum shear stress τ_{max}.

(c) Determine the octahedral shear stress τ_{oct} and compare it to the maximum shear stress.

Solution

(a) The direction cosines of the normal to the plane are

$$l = \frac{1}{\sqrt{6}}, \quad m = \frac{2}{\sqrt{6}}, \quad n = \frac{1}{\sqrt{6}}$$

By Eqs. 2.10, the projections of the stress vector are

$$\sigma_{Px} = \left(\frac{1}{\sqrt{6}}\right)(80) + \left(\frac{2}{\sqrt{6}}\right)(20) + \left(\frac{1}{\sqrt{6}}\right)(40) = 65.320 \text{ MPa}$$

$$\sigma_{Py} = \left(\frac{1}{\sqrt{6}}\right)(20) + \left(\frac{2}{\sqrt{6}}\right)(60) + \left(\frac{1}{\sqrt{6}}\right)(10) = 61.237 \text{ MPa}$$

$$\sigma_{Pz} = \left(\frac{1}{\sqrt{6}}\right)(40) + \left(\frac{2}{\sqrt{6}}\right)(10) + \left(\frac{1}{\sqrt{6}}\right)(20) = 32.660 \text{ MPa}$$

Hence,

$$\boldsymbol{\sigma}_P = 65.320\mathbf{i} + 61.237\mathbf{j} + 32.660\mathbf{k}$$

(b) For the given stress state, the stress invariants are (by Eq. 2.21)

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} = 5500$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = 0$$

Hence, by Eq. 2.20,

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = \sigma^3 - 160\sigma^2 + 5500\sigma = 0$$

or the principal stresses are

$$\sigma_1 = 110, \quad \sigma_2 = 50, \quad \sigma_3 = 0$$

By Eq. 2.39, the maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(110 - 0) = 55$$

(c) By Eq. 2.22

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} = 44.969$$

Comparing τ_{oct} and τ_{max}, we see that

$$\tau_{max} = 1.223\,\tau_{oct}$$

EXAMPLE 2.4 ***Mohr's Circles in Three Dimensions***

The state of stress at a point in a machine component is given by $\sigma_{xx} = 120$ MPa, $\sigma_{yy} = 55$ MPa, $\sigma_{zz} = -85$ MPa, $\sigma_{xy} = -55$ MPa, $\sigma_{xz} = -75$ MPa, and $\sigma_{yz} = 33$ MPa. Construct the Mohr's circles of stress for this stress state and locate the coordinates of points A: $(\sigma_{NN1}, \sigma_{NS1})$ and B: $(\sigma_{NN2}, \sigma_{NS2})$ for normal and shear stress acting on the cutting planes with outward normal vectors given by $\mathbf{N}_1 : (1/\sqrt{3}, 1/\sqrt{3}, 1\sqrt{3})$ and $\mathbf{N}_2 : (1/\sqrt{2}, 1/\sqrt{2}, 0)$ relative to the principal axes of stress.

Solution

Substituting the given stress components into Eq. 2.20, we obtain

$$\sigma^3 - 90\sigma^2 - 18{,}014\sigma + 471{,}680 = 0$$

The three principal stresses are the three roots of this equation. They are

$$\sigma_1 = 176.80 \text{ MPa}, \quad \sigma_2 = 24.06 \text{ MPa}, \quad \sigma_3 = -110.86 \text{ MPa}$$

The center and radius of each circle is found directly from the principal stresses as

$$C_1 : \left(\frac{\sigma_2 + \sigma_3}{2}, 0\right) = (-43.40 \text{ MPa}, 0), \quad R_1 = \frac{\sigma_2 - \sigma_3}{2} = 67.46 \text{ MPa}$$

$$C_2 : \left(\frac{\sigma_1 + \sigma_3}{2}, 0\right) = (32.97 \text{ MPa}, 0), \quad R_2 = \frac{\sigma_1 - \sigma_3}{2} = 143.83 \text{ MPa}$$

$$C_3 : \left(\frac{\sigma_1 + \sigma_2}{2}, 0\right) = (100.43 \text{ MPa}, 0), \quad R_3 = \frac{\sigma_1 - \sigma_2}{2} = 76.37 \text{ MPa}$$

Figure E2.4 illustrates the corresponding circles with the shaded area indicating the region of admissible stress states. The normal and shear stresses acting on the planes with normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ are found from Eqs. 2.40 and 2.41:

$$\sigma_{NN1} = 30 \text{ MPa}, \quad \sigma_{NS1} = 117.51 \text{ MPa} \quad \text{Point A;}$$

$$\sigma_{NN2} = 100.43 \text{ MPa}, \quad \sigma_{NS2} = 76.37 \text{ MPa} \quad \text{Point B;}$$

These points are also shown in Figure E2.4. By this method, the correct signs of σ_{NS1} and σ_{NS2} are indeterminate. That is, this method does not determine if σ_{NS1} and σ_{NS2} are positive or negative. They are plotted in Figure E2.4 as positive values. Note that, since $\mathbf{N}_2 : (1/\sqrt{2}, 1/\sqrt{2}, 0)$, the third direction cosine is zero and point B lies on the circle with center C_3 and radius R_3.

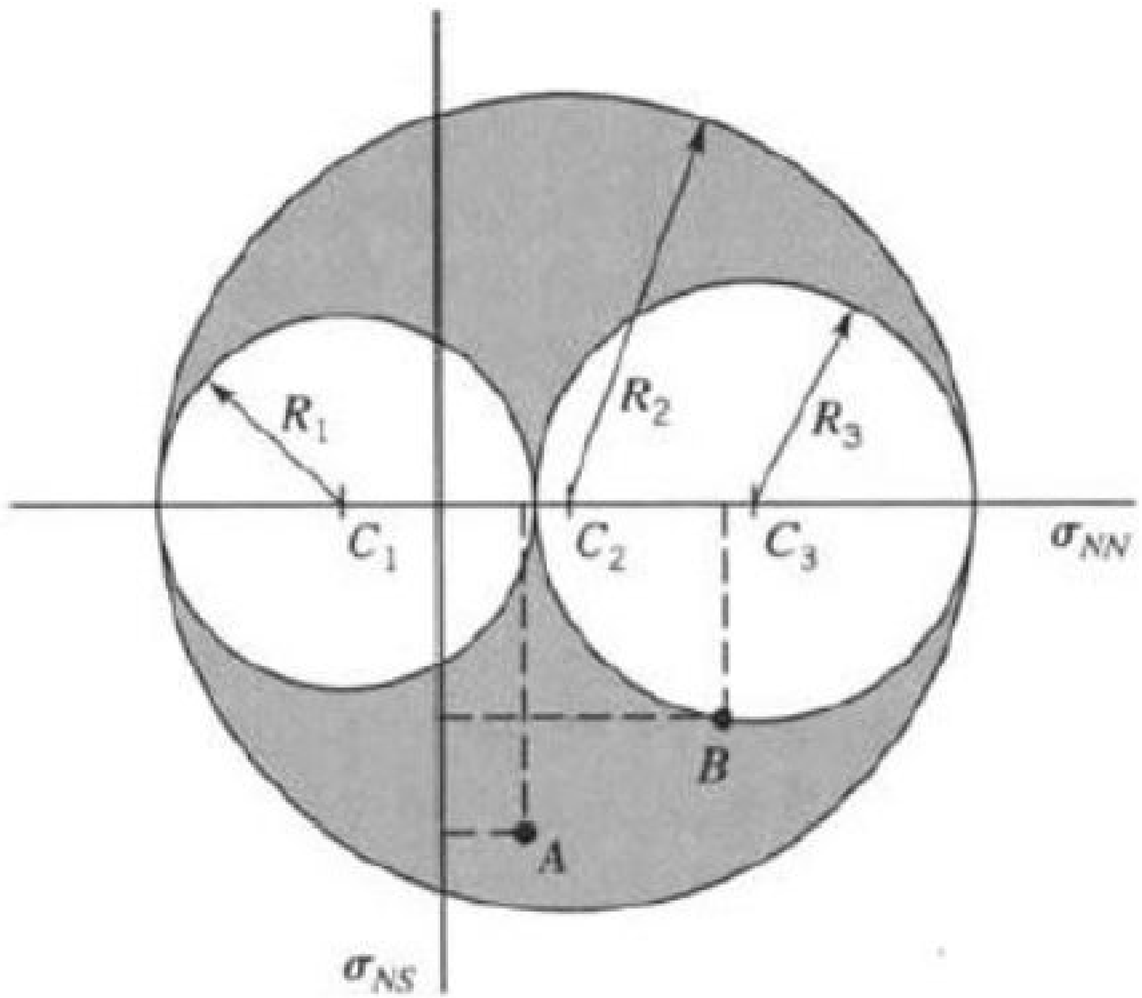


FIGURE E2.4

Yield Criteria for Ductile Material

An axially loaded member consider to be safe if the applied stress is less than the yield strength (σy) of the material.

Since this state is different from the state of uniaxial stress, it is not possible to predict from such a test whether or not the structural element under investigation will fail.

Some criterion regarding the actual mechanism of failure of the material must be established that will make it possible to compare the effects of both states of stress.

To this effect in upcoming slides to present the two yield criteria most frequently used for Ductile materials.

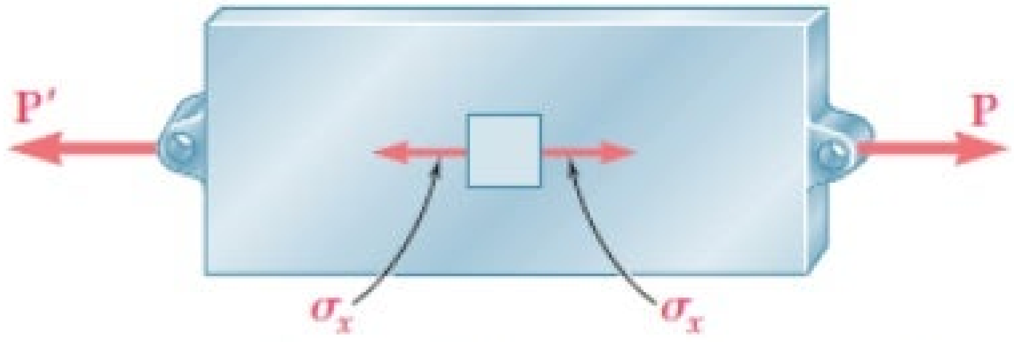


Fig-01 Structural element under uniaxial stress.

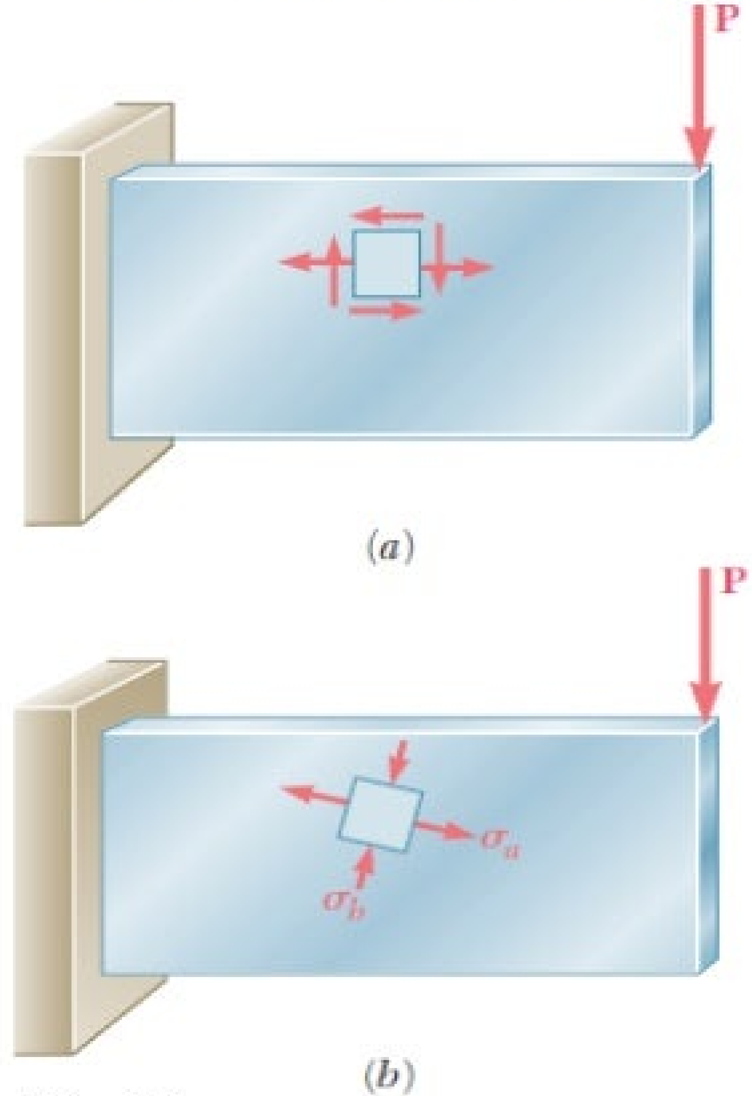
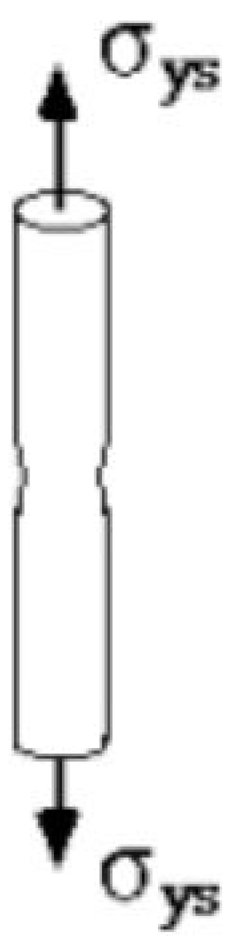


Fig-02 Structural element in a state of plane stress. (a) Stress element referred to coordinate axes. (b) Stress element referred to principal axes.

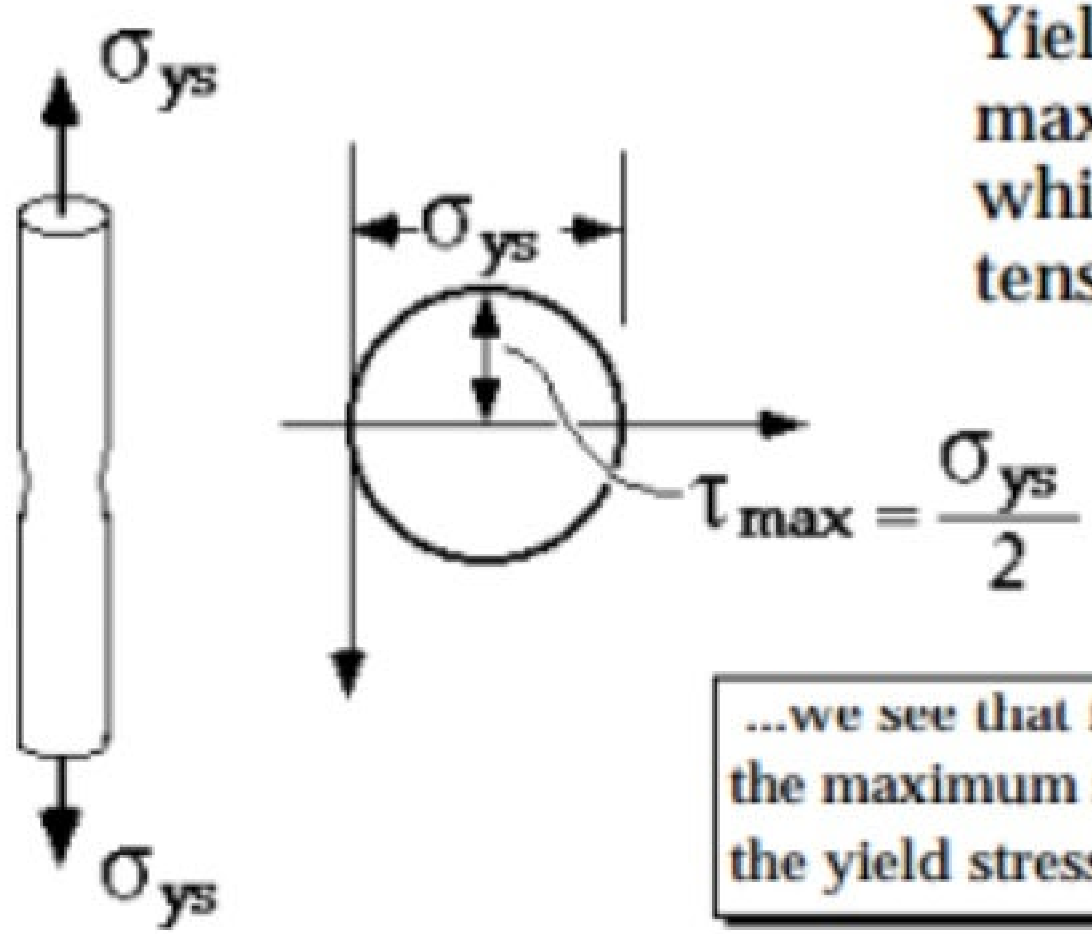
Maximum Shear Stress
H. Tresca 1868



Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.

A simple tension test is performed on the ductile specimen, and the yield stress is noted.

Maximum Shear Stress
H. Tresca 1868

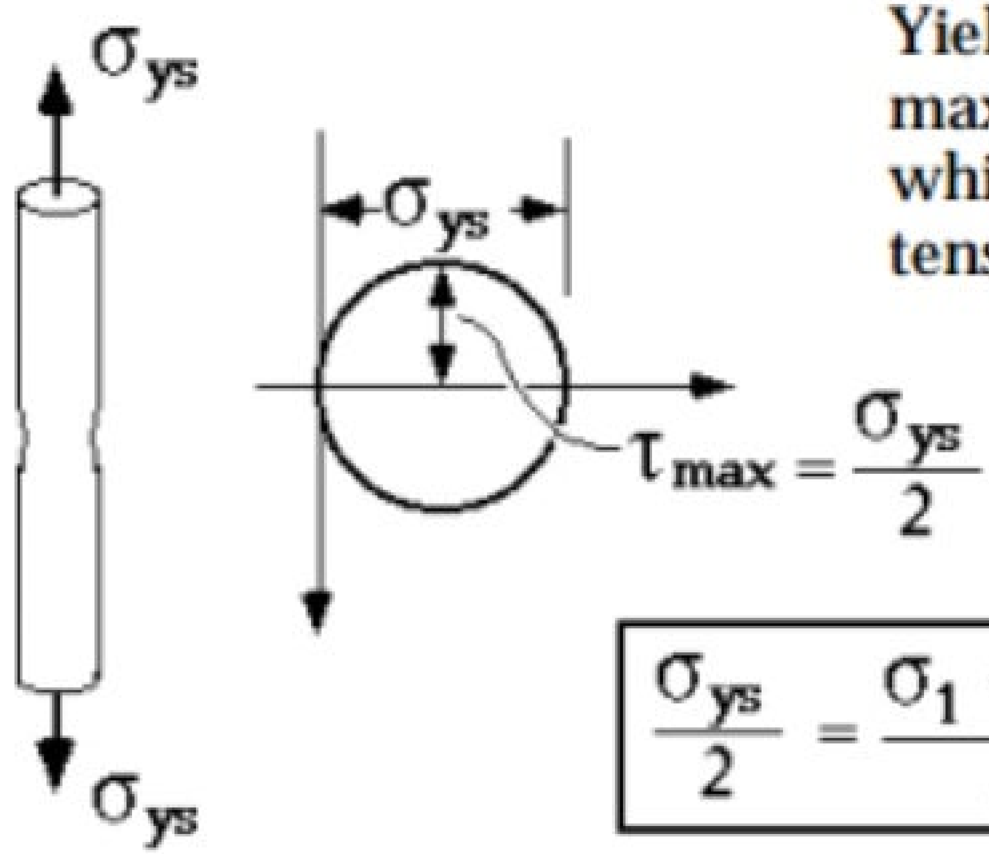


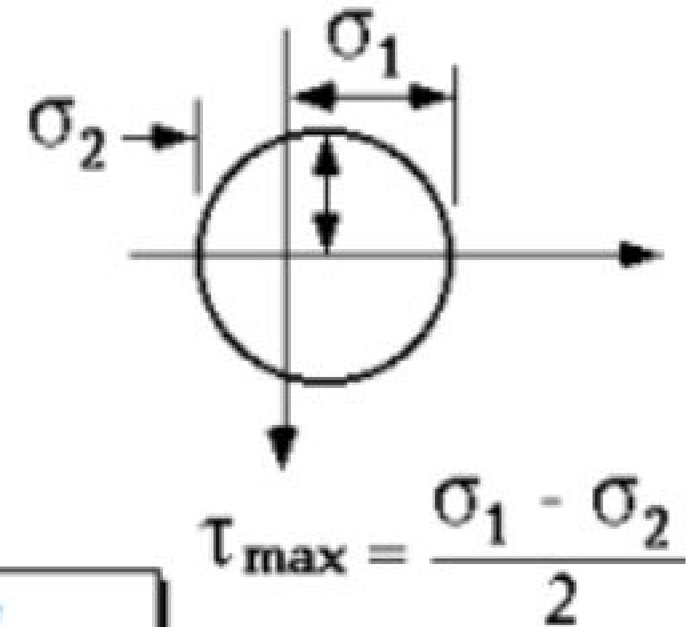
Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.

...we see that for a simple tension test the maximum shear stress is one half of the yield stress.

Maximum Shear Stress
H. Tresca 1868

Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.

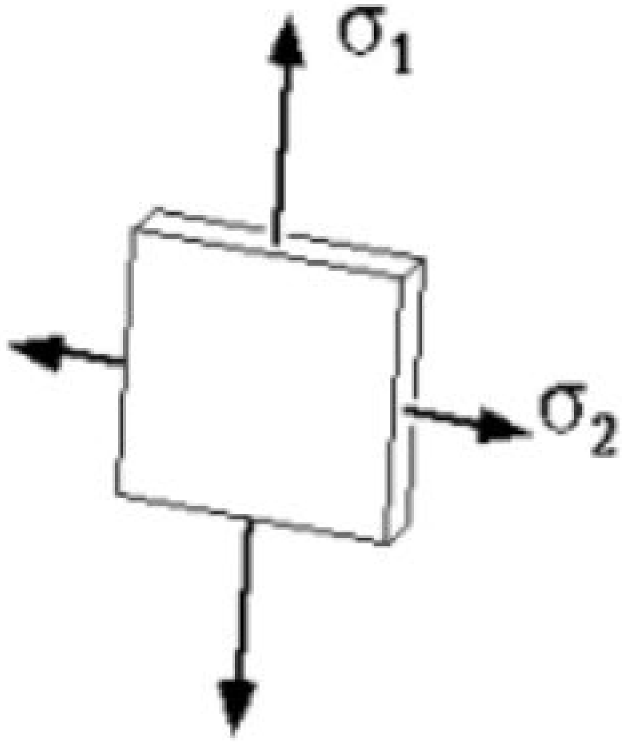




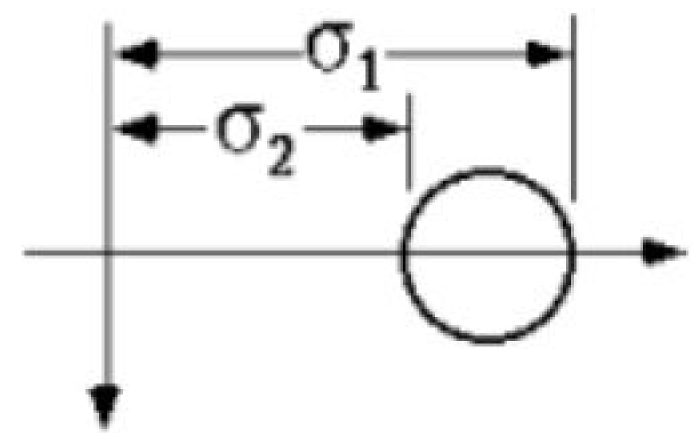
$$\frac{\sigma_{ys}}{2} = \frac{\sigma_1 - \sigma_2}{2}$$

The condition for yield is that the difference between the smallest and largest principal stresses equals or exceeds the yield stress.

Maximum Shear Stress
H. Tresca 1868



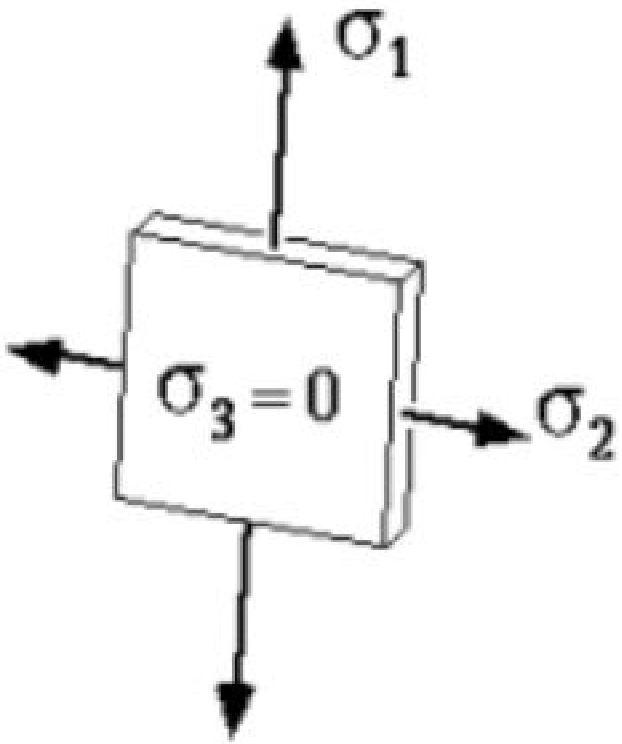
Given this case of plane strain, will the material yield according to the Tresca theory?
If we simply plug in the values for σ_1 and σ_2 it would appear that the material is safe.



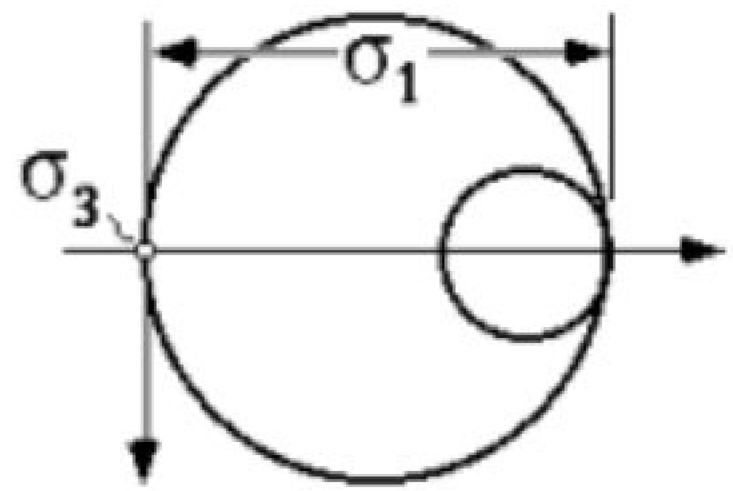
No Failure???

$$\frac{\sigma_1 - \sigma_2}{2} \leq \frac{\sigma_{ys}}{2}$$

Maximum Shear Stress
H. Tresca 1868



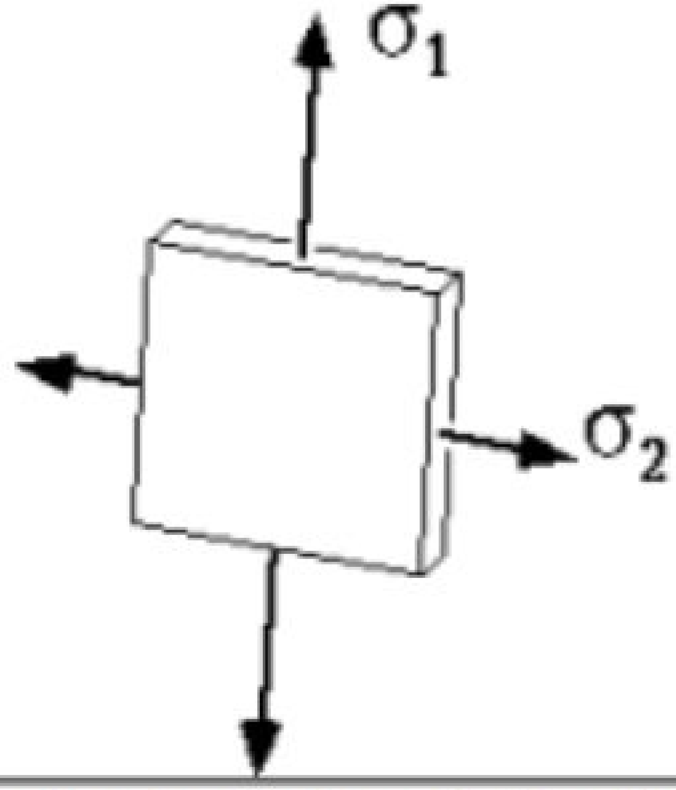
But we aren't that simple !!! We know that the maximum shear stress will occur out of plane if the sign of the two principal stresses is the same. In this case the maximum shear stress is actually about three times as great as we originally estimated.



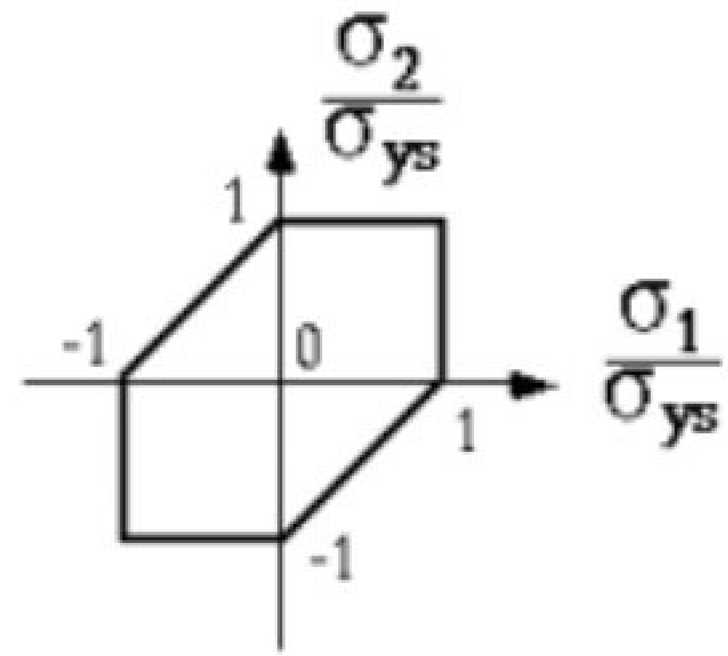
Failure !!

$$\frac{\sigma_1}{2} \geq \frac{\sigma_{ys}}{2}$$

Maximum Shear Stress
H. Tresca 1868



Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.

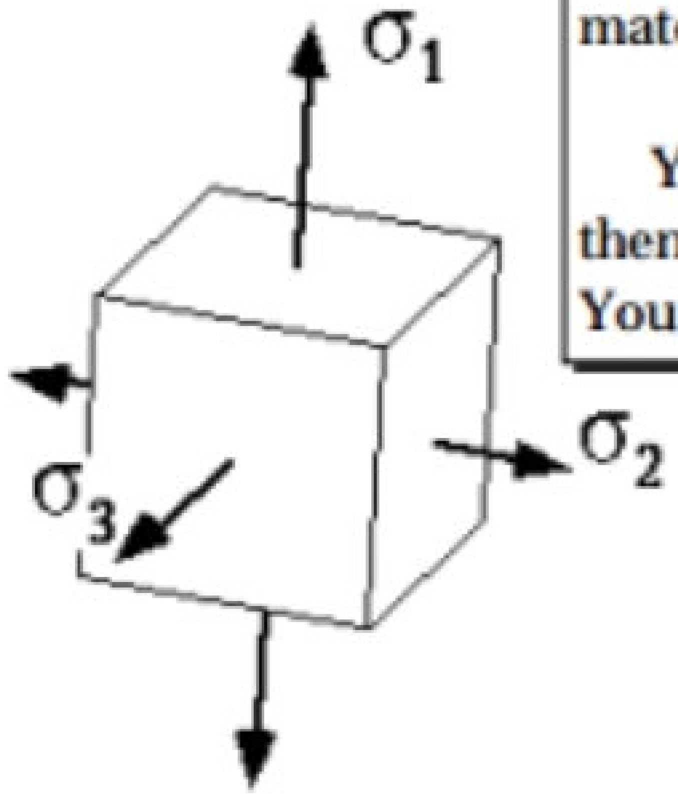


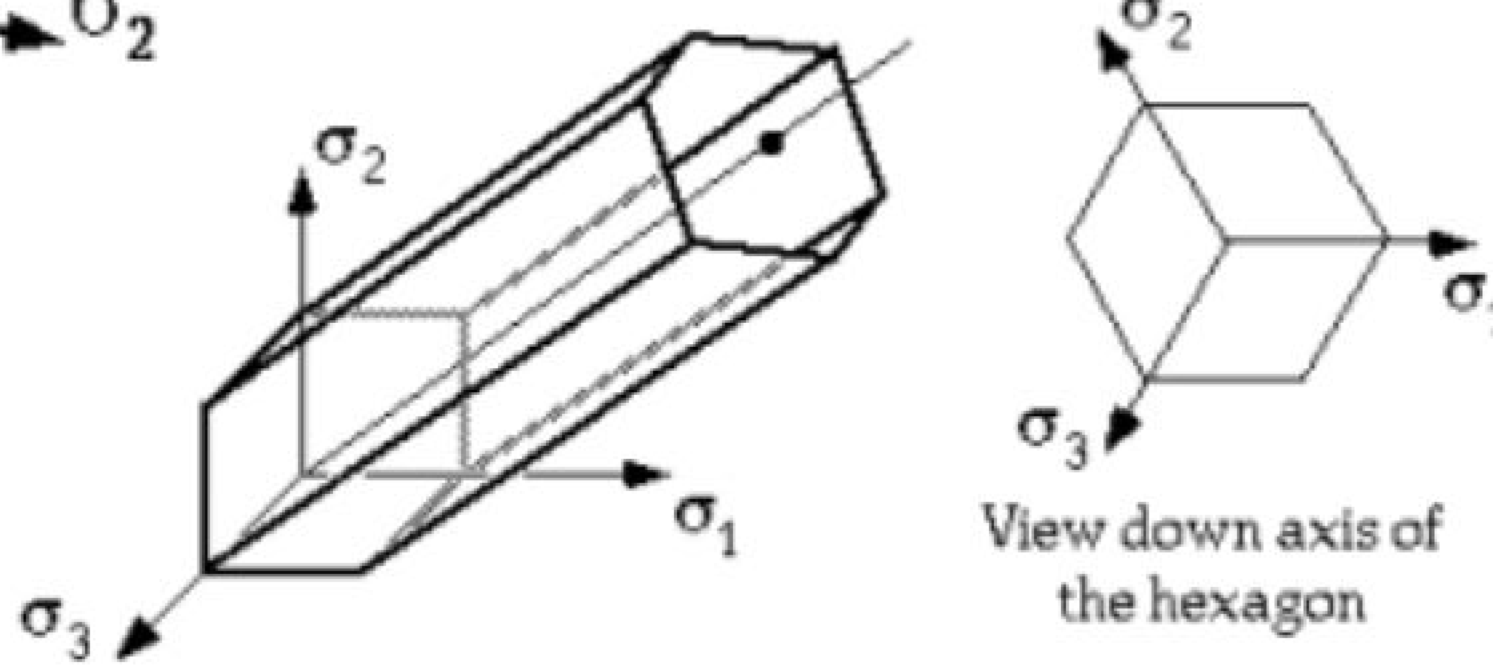
Yield Envelope

Similar to what we did for the maximum normal stress theory, we can plot a yield envelope representing the maximum shear stress failure criterion. Note how this envelope deviates from the previous one in the second and fourth quadrants.

The yield envelope in three dimensions appears as a hexagon projected down the hydrostatic axis, $\sigma_1 = \sigma_2 = \sigma_3$. This means that the theory predicts no change in material response with the addition of hydrostatic stresses.

You can confirm this by plotting 3-D Mohr's circle and then increasing each principal stress by a constant value. You should observe that the maximum shear stress does





3-D Yield Envelope

Maximum Distortional Energy (R. von Mises, 1913)

Yielding will occur when the distortional strain energy reaches that value which causes yielding in a simple tension test.

- **Ductile Materials**

Like the maximum shear stress or Tresca failure theory, the maximum distortional energy failure theory addresses ductile, isotropic materials.

Maximum Distortional Energy
(R. von Mises, 1913)

Yielding will occur when the distortional strain energy reaches that value which causes yielding in a simple tension test.

- **Ductile Materials**

$$u_d = \frac{1}{4G}[s_x{}^2 + s_y{}^2 + s_z{}^2 + 2\tau_{xy}{}^2 + 2\tau_{xz}{}^2 + 2\tau_{yz}{}^2]$$

Recall the expression for the distortional strain energy. Note that it is a function of the deviatoric portion of the normal stresses, s, and the shear stresses, τ.

We need to express the distortional strain energy in terms of principal stresses, so lets give ourselves some space...

Maximum Distortional Energy
(R. von Mises, 1913)

$$u_d = \frac{1}{4G}[s_1^2 + s_2^2 + s_3^2]$$

Deviatoric Normal Stress

$$s_1 = \sigma_1 - 1/3(\sigma_1 + \sigma_2 + \sigma_3)$$
$$s_2 = \sigma_2 - 1/3(\sigma_1 + \sigma_2 + \sigma_3)$$
$$s_3 = \sigma_3 - 1/3(\sigma_1 + \sigma_2 + \sigma_3)$$

Recall that normal components of the deviatoric stress tensor are calculated as each normal stress minus the average value of all three normal stresses.

Maximum Distortional Energy
(R. von Mises, 1913)

$$u_d = \frac{1}{4G}[s_1^{\,2} + s_2^{\,2} + s_3^{\,2}]$$

> At this point we are going to narrow the scope of our theory to the case of plane stress.
>
> Again, we are remaining fairly general, because plane stress problems, such as pressure vessels, beams, shafts, plates, etc.

Assume Plane Stress

$\sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \sigma_3 = 0$

Deviatoric Normal Stress

$$s_1 = \sigma_1 - 1/3(\sigma_1 + \sigma_2 + \cancel{\sigma_3})$$
$$s_2 = \sigma_2 - 1/3(\sigma_1 + \sigma_2 + \cancel{\sigma_3})$$
$$s_3 = \cancel{\sigma_3} - 1/3(\sigma_1 + \sigma_2 + \cancel{\sigma_3})$$

Maximum Distortional Energy
(R. von Mises, 1913)

Yielding will occur when the distortional strain energy reaches that value which causes yielding in a simple tension test.

$$u_d = \frac{1}{6G}[\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2] = \frac{1}{6G}[\sigma_{ys}^2]$$

$$\boxed{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{ys}^2}$$

Multiplying both sides of the equation by 6G we arrive at the condition for yield under combined stress.
Let's save this result, and then interpret what it means.

Maximum Distortional Energy
(R. von Mises, 1913)

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{ys}^2$$

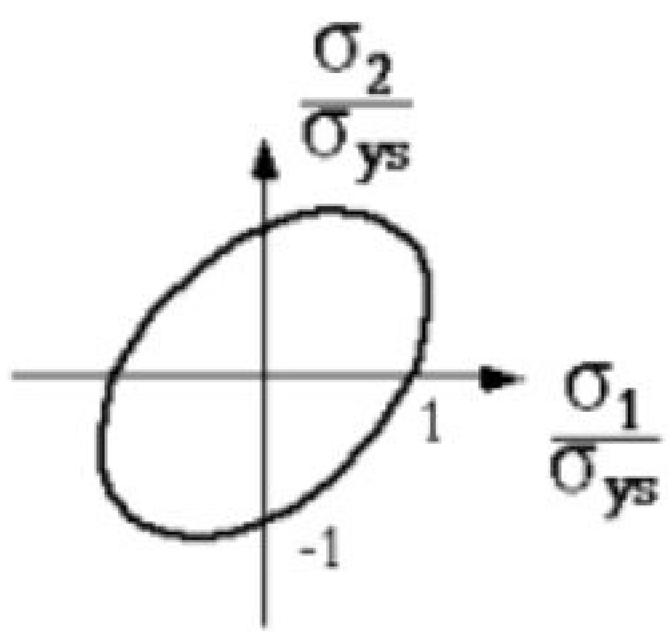
Yielding will occur when the distortional strain energy reaches that value which causes yielding in a simple tension test.

> Recall, for the previous two failure theories we plotted a "yield envelope" in the σ_1, σ_2 space. Assuming that σ_{ys} is constant, do you recognize how the equation above will plot?

Maximum Distortional Energy
(R. von Mises, 1913)

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{ys}^2$$

Yielding will occur when the distortional strain energy reaches that value which causes yielding in a simple tension test.



Yield Envelope

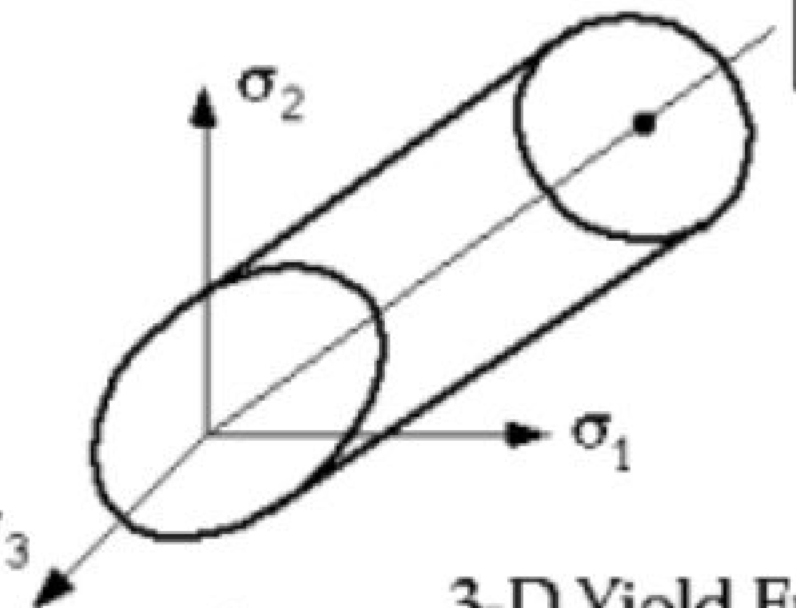
The yield envelope for the maximum distortional energy plots as an ellipse for plane stress.

How does this compare to the yield envelope for the maximum shear stress theory?

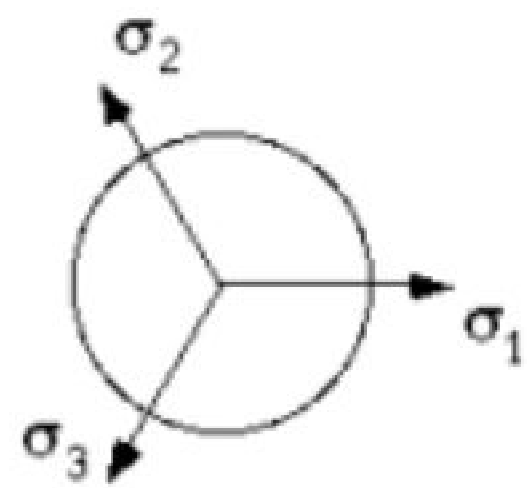
Maximum Distortional Energy
(R. von Mises, 1913)

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{ys}^2$$

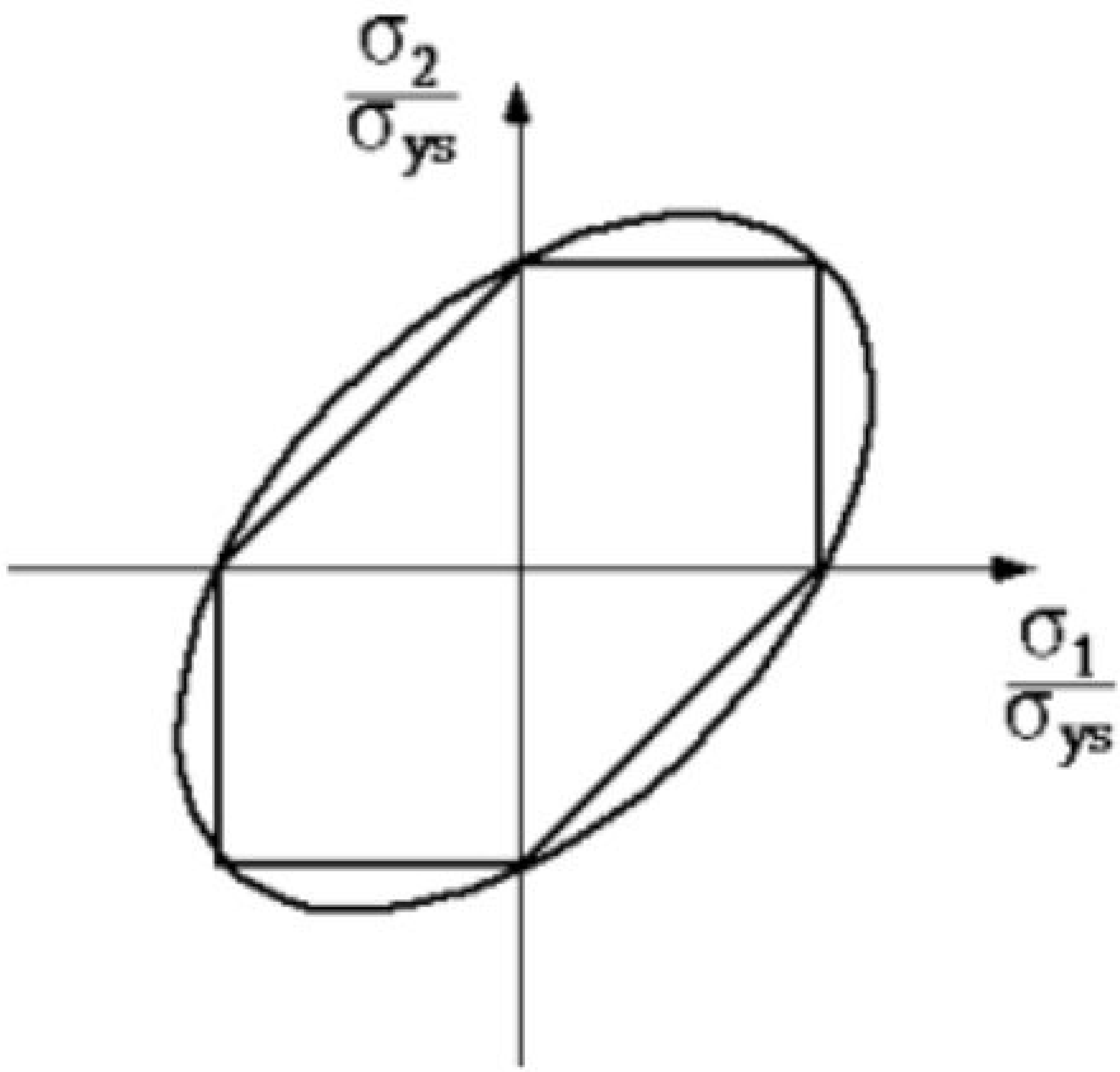
Looking at the 3-D case of stress, we see that the addition of a hydrostatic stress ($\sigma_1 = \sigma_2 = \sigma_3$) does not contribute to the yield of a material. The 3-D yield envelope plots as a cylinder centered along the "hydrostatic axis" ($\sigma_1 = \sigma_2 = \sigma_3$). Note that where the cylinder intersects the σ_1, σ_2 plane, the failure surface becomes an ellipse as previously shown.



3-D Yield Envolope



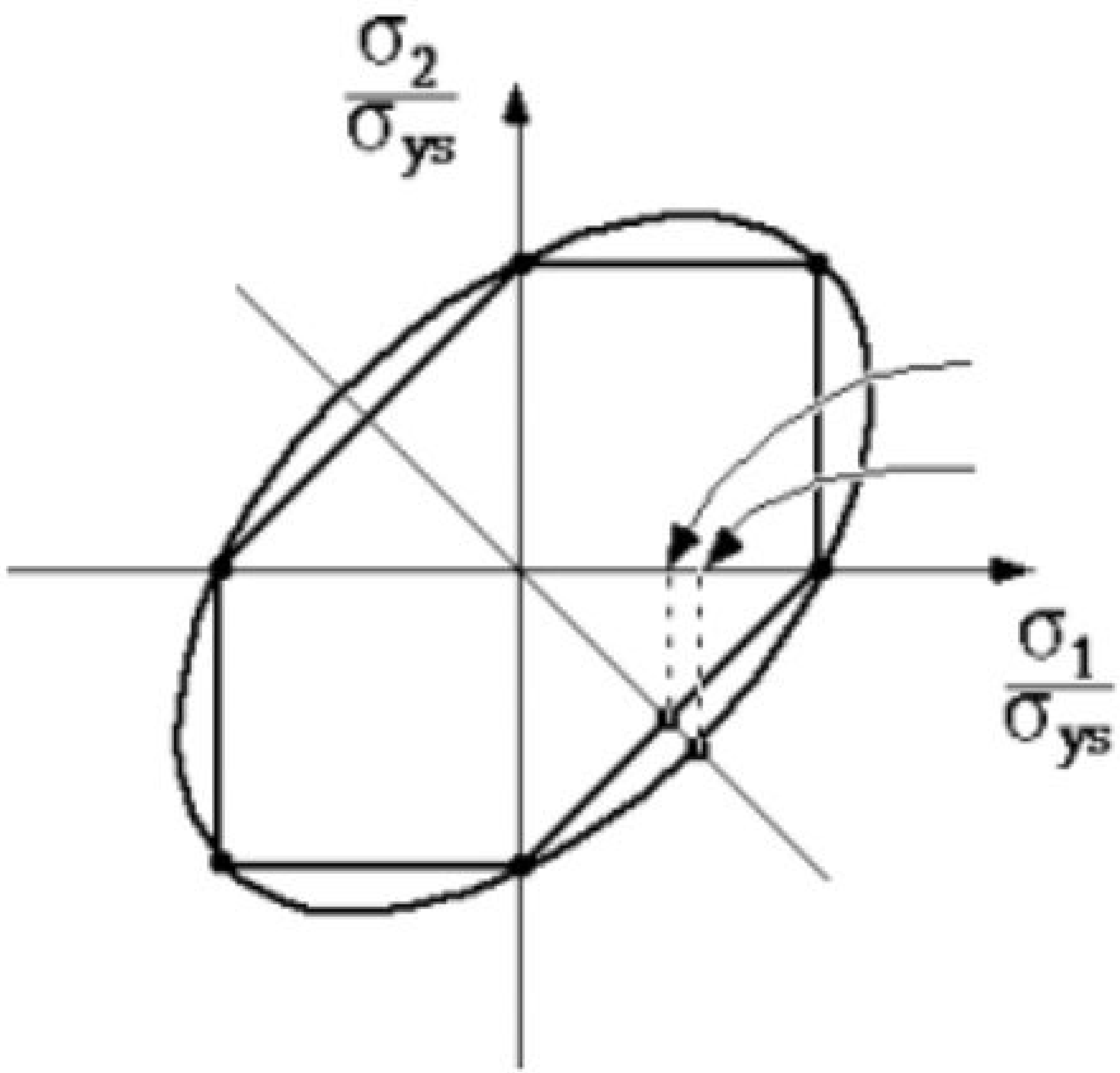
Maximum-Shear vs Maximum Distortional Energy for Plane Stress



How does the maximum shear stress yield theory compare to the maximum distortional energy yield theory?

The first relies strictly on the maximum shear stress in an element. The distortional energy criterion is more comprehensive, by considering the energy caused by shear deformations in three dimensions. Since shear stresses are the major parameters in both approaches, the differences are not great.

Maximum-Shear vs Maximum Distorsional Energy for Plane Stress



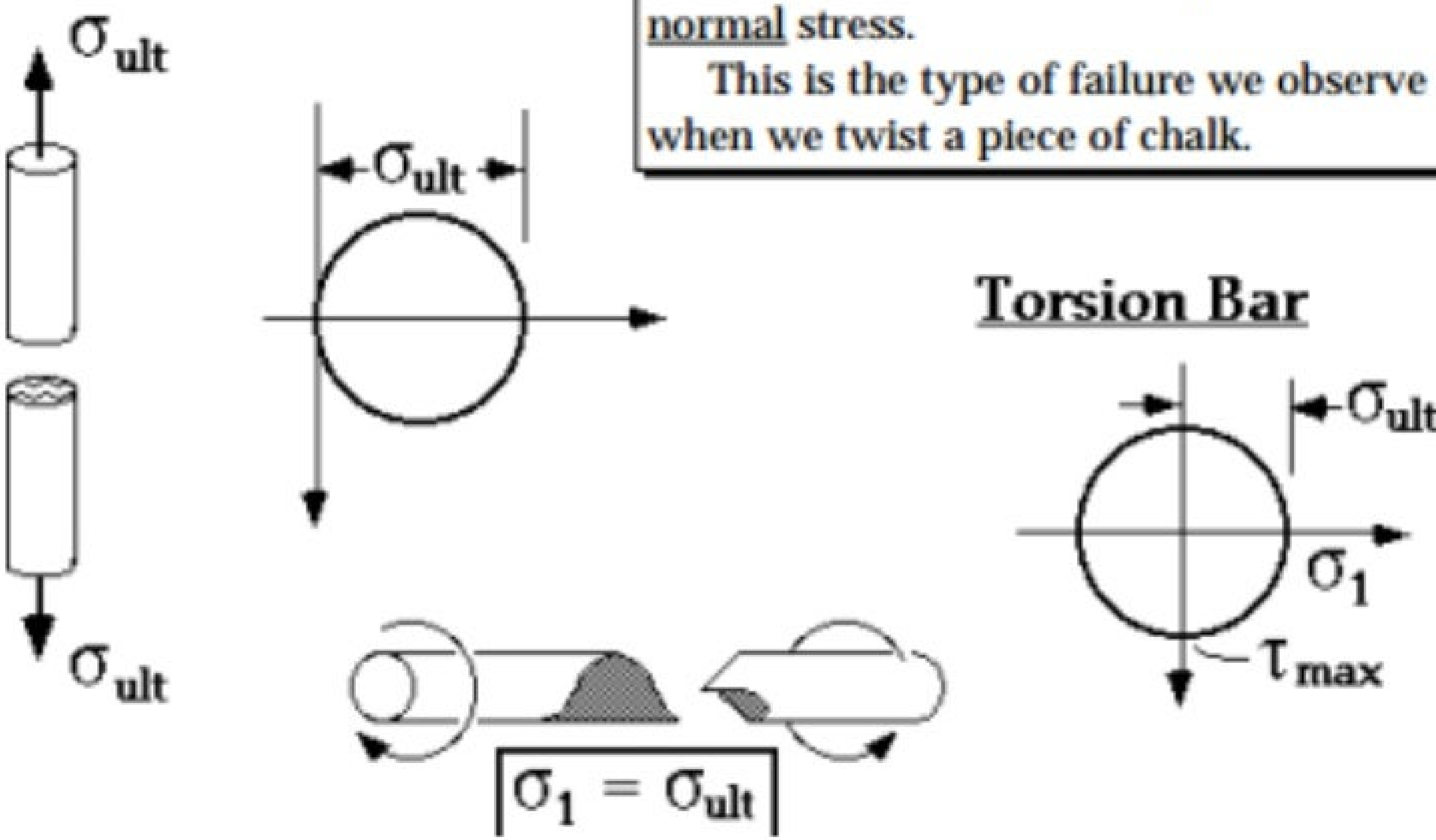
In general, the maximum shear stress theory is more conservative than the distortion energy theory.

At the point where two of the principal stresses are equal but of opposite sign (pure shear) the maximum shear stress theory predicts yield when the principal stresses equal $\sigma_{ys}/2$. The maximum distortion energy increases the limit to ~$0.557\sigma_{ys}$.

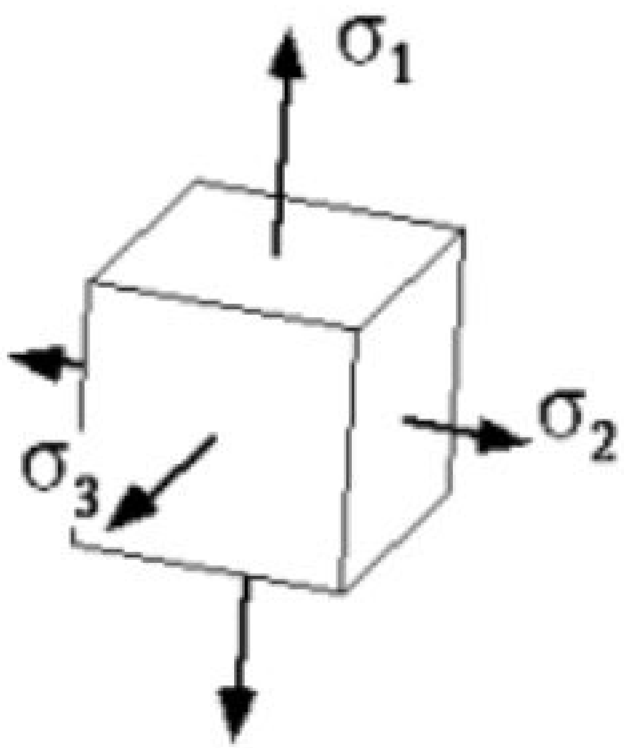
Experiments with many ductile materials tend to plot closer to the

Maximum Normal Stress Theory
W. Rankine ~1850

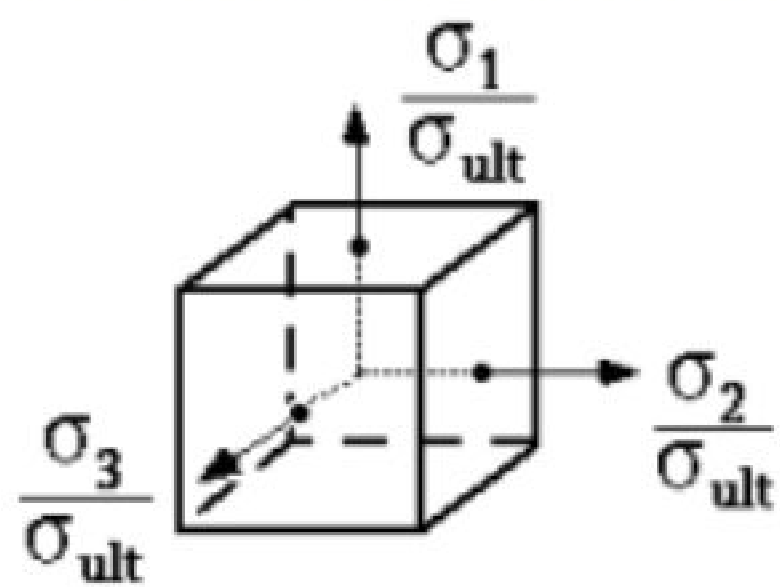
...we see that the torsion bar will fail when the maximum shear stress equals the ultimate normal stress.
This is the type of failure we observe when we twist a piece of chalk.



Maximum Normal Stress Theory
W. Rankine ~1850



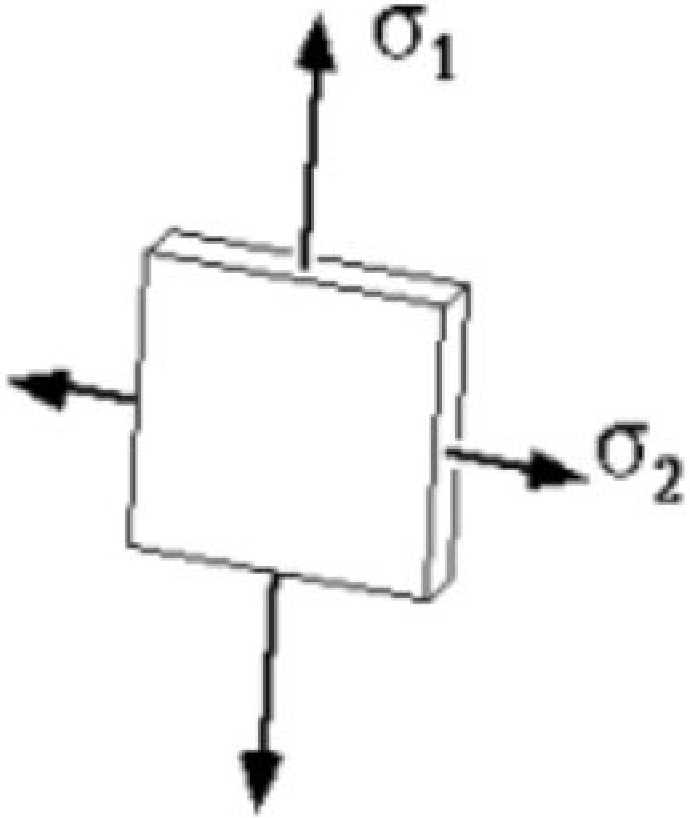
Failure will occur when the magnitude of the major principal stress reaches that which caused fracture in a simple tension test.



3-D Fracture Envelope

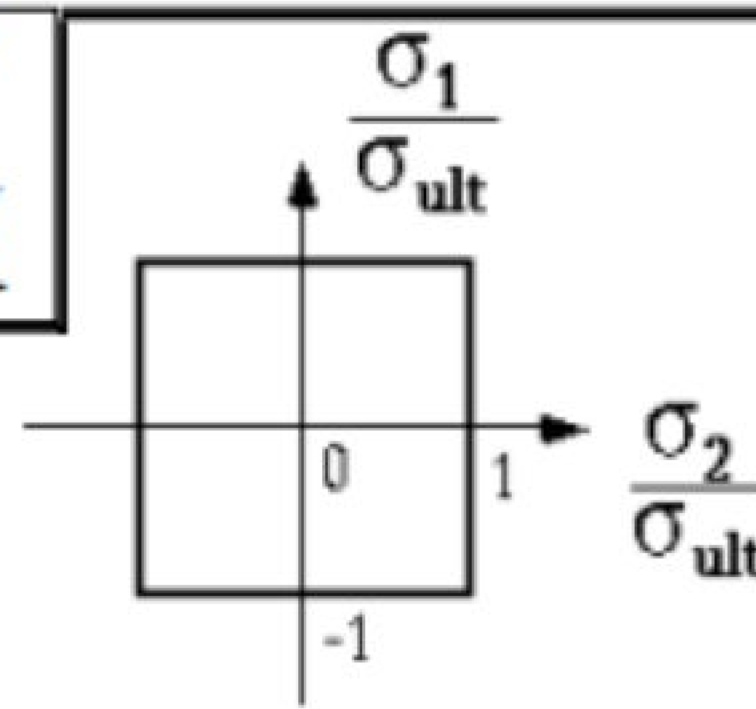
In the case of 3-D stress, the fracture envelope becomes a cube. Again, any stress state which plots outside of the fracture envelope represents a point where the material

Maximum Normal Stress Theory
W. Rankine ~1850



To help us visualize the maximum normal stress failure criterion, we plot a figure known as the fracture envelope. The edges of the envelope reflects the points at which the material fails, i.e. $\sigma_1 = \pm\ \sigma_{ult}$, $\sigma_2 = \pm\ \sigma_{ult}$.

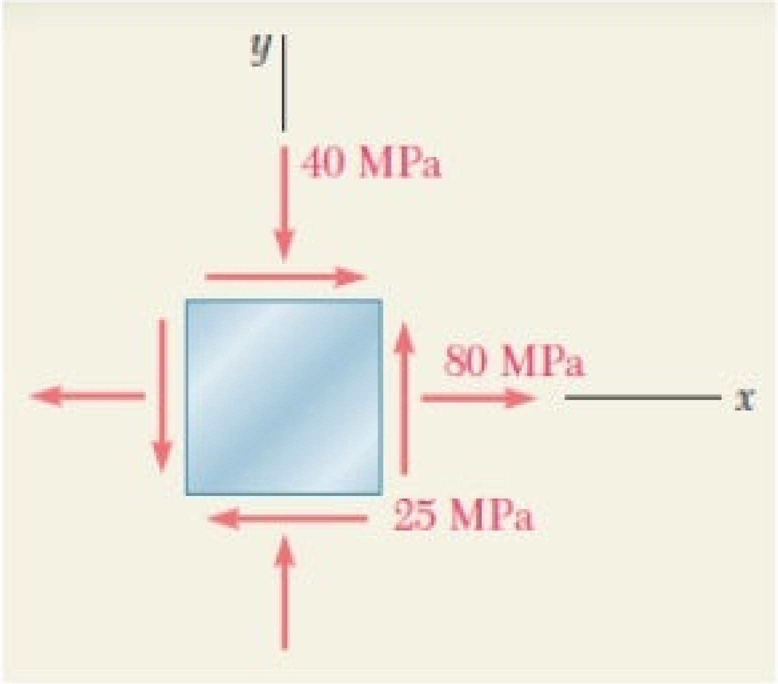
some material, and it lies on or outside of the envelope, we say that the material fails.

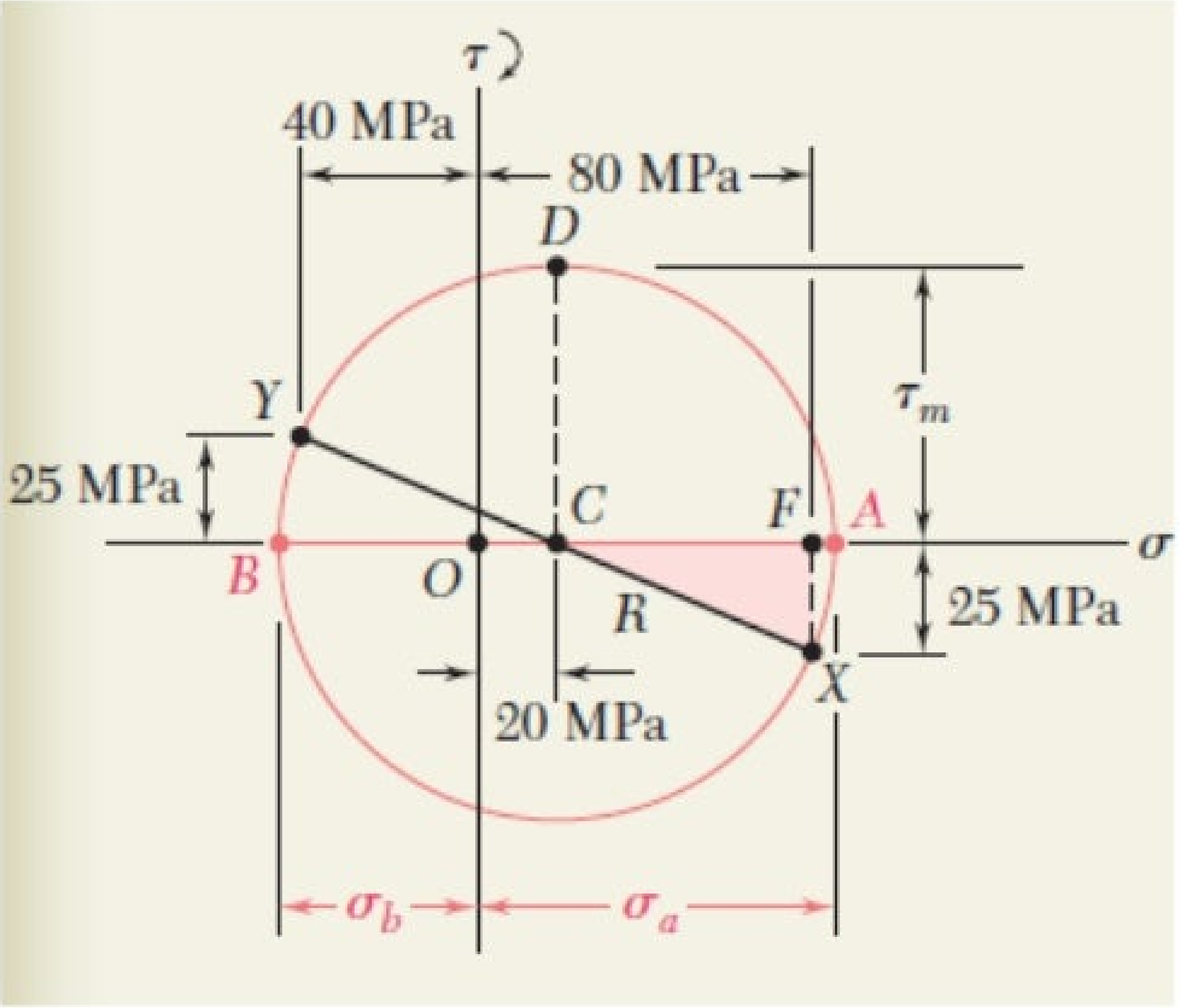


Fracture Envelope

Example-#-01

The state of plane stress shown occurs at a critical point of a steel machine component. As a result of several tensile tests, it has been found that the tensile yield strength is σ_Y = *250 MPa for the grade of steel used. Determine* the factor of safety with respect to yield, using (*a*) *the maximum-shearing stress* criterion, and (*b*) *the maximum-distortion-energy criterion?.*





Mohr's Circle. We construct Mohr's circle for the given state of stress and find

$$\sigma_{ave} = OC = \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(80 - 40) = 20 \text{ MPa}$$
$$\tau_m = R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(60)^2 + (25)^2} = 65 \text{ MPa}$$

Principal Stresses

$$\sigma_a = OC + CA = 20 + 65 = +85 \text{ MPa}$$
$$\sigma_b = OC - BC = 20 - 65 = -45 \text{ MPa}$$

a. Maximum-Shearing-Stress Criterion. Since for the grade of steel used the tensile strength is $\sigma_Y = 250$ MPa, the corresponding shearing stress at yield is

$$\tau_Y = \tfrac{1}{2}\sigma_Y = \tfrac{1}{2}(250 \text{ MPa}) = 125 \text{ MPa}$$

For $\tau_m = 65$ MPa: $$F.S. = \frac{\tau_Y}{\tau_m} = \frac{125 \text{ MPa}}{65 \text{ MPa}}$$ $F.S. = 1.92$ ◀

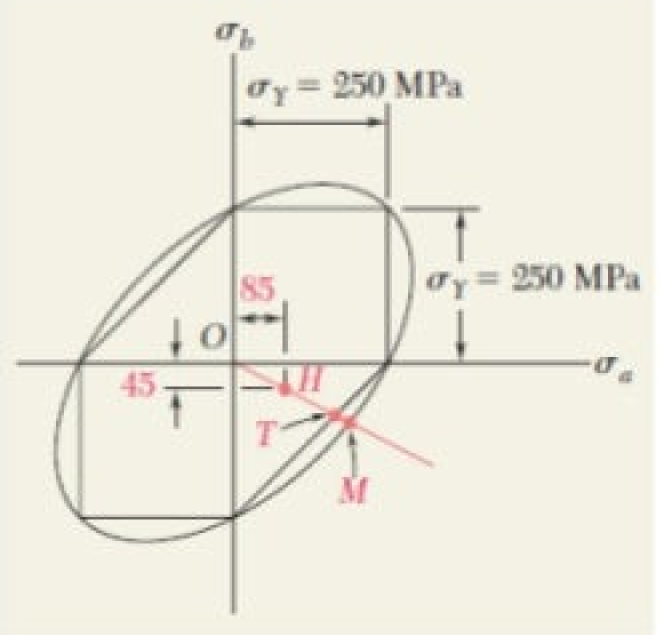
b. Maximum-Distortion-Energy Criterion. Introducing a factor of safety into Eq. (7.26), we write

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = \left(\frac{\sigma_Y}{F.S.}\right)^2$$

For $\sigma_a = +85$ MPa, $\sigma_b = -45$ MPa, and $\sigma_Y = 250$ MPa, we have

$$(85)^2 - (85)(-45) + (45)^2 = \left(\frac{250}{F.S.}\right)^2$$

$$114.3 = \frac{250}{F.S.} \qquad F.S. = 2.19 \blacktriangleleft$$



Comment. For a ductile material with $\sigma_Y = 250$ MPa, we have drawn the hexagon associated with the maximum-shearing-stress criterion and the ellipse associated with the maximum-distortion-energy criterion. The given state of plane stress is represented by point H of coordinates $\sigma_a = 85$ MPa and $\sigma_b = -45$ MPa. We note that the straight line drawn through points O and H intersects the hexagon at point T and the ellipse at point M. For each criterion, the value obtained for $F.S.$ can be verified by measuring the line segments indicated and computing their ratios:

$$(a)\ F.S. = \frac{OT}{OH} = 1.92 \qquad (b)\ F.S. = \frac{OM}{OH} = 2.19$$

UNIT-6: BENDING AND SHEAR STRESSES IN BEAMS

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UNIT-6

BENDING AND SHEAR STRESSES IN BEAMS

Syllabus

Introduction, Theory of simple bending, assumptions in simple bending, Bending stress equation, relationship between bending stress, radius of curvature, relationship between bending moment and radius of curvature, Moment carrying capacity of a section. Shearing stresses in beams, shear stress across rectangular, circular, symmetrical I and T sections. (composite / notched beams not included).

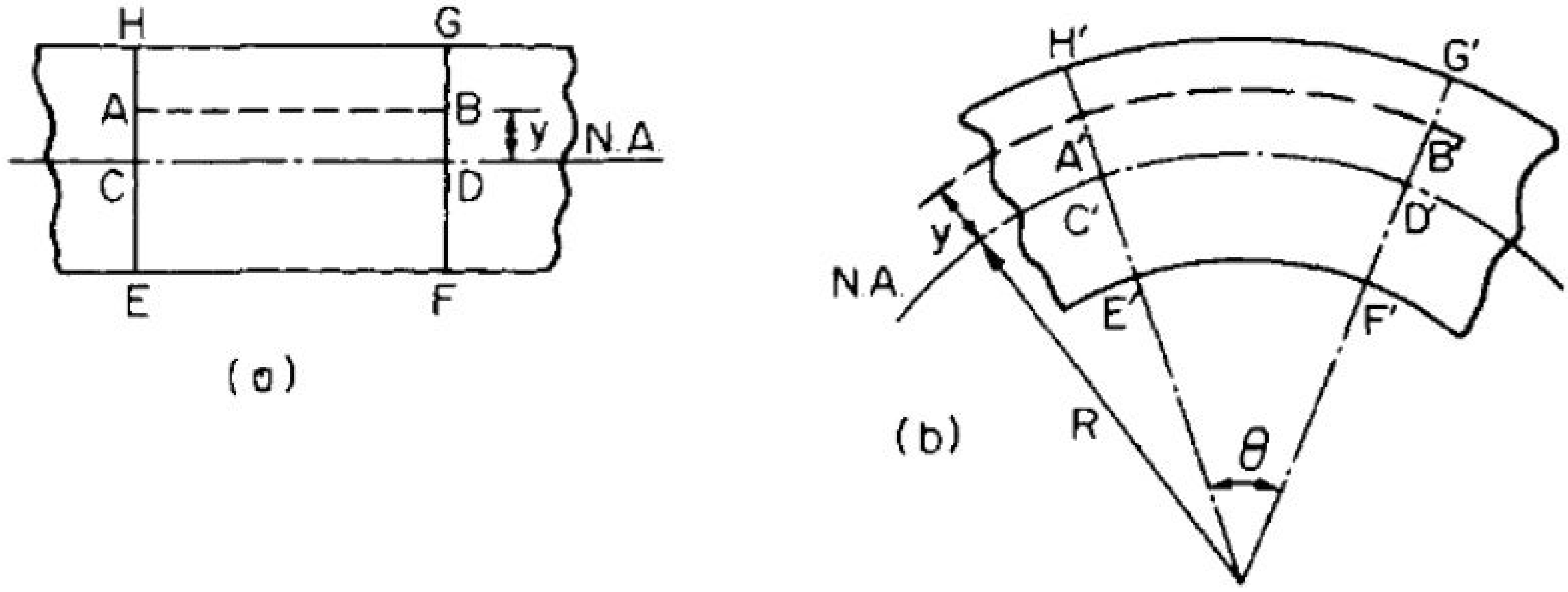
6.1. INTRODUCTION

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The material of the beam will offer resistance or stresses against these deformations. These stresses with certain assumptions can be calculated. The stresses introduced by bending moment are known as *bending stresses.* In this chapter, the theory of pure bending, expression for bending stresses, bending stress in symmetrical and unsymmetrical sections, strength of a beam and composite beams will be discussed.

E.g., Consider a piece of rubber, most conveniently of rectangular cross-section, is bent between one's fingers it is readily apparent that one surface of the rubber is stretched, i.e. put into tension, and the opposite surface is compressed.

6.2. SIMPLE BENDING

A theory which deals with finding stresses at a section due to pure moment is called bending theory. If we now consider a beam initially unstressed and subjected to a constant B.M. along its length, it will bend to a radius R as shown in Fig. b. As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. Somewhere between the two surfaces, there are points at which the stress is zero. The locus of all such points is termed the neutral axis (N.A). The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or centroid.



Beam subjected to pure bending (a) before, and (b) after, the moment *M* has been applied.

In simple bending the plane of transverse loads and the centroidal plane coincide. The theory of simple bending was developed by Galelio, Bernoulli and St. Venant. Sometimes this theory is called Bernoulli's theory of simple bending.

6.3. ASSUMPTIONS IN SIMPLE BENDING

The following assumptions are made in the theory of simple bending:

1 The beam is initially straight and unstressed.
2 The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
3 The elastic limit is nowhere exceeded.
4 Young's modulus for the material is the same in tension and compression.
5 Plane cross-sections remain plane before and after bending.
6 Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an axis perpendicular to the N.A.
7 There is no resultant force perpendicular to any cross-section.
8 The radius of curvature is large compared to depth of beam.

6.4. DERIVATION OF BENDING EQUATION

Consider a length of beam under the action of a bending moment *M* as shown in Fig. 6.2a. *N-N* is the original length considered of the beam. The neutral surface is a plane through *X-X*. In the side view *NA* indicates the neutral axis. *O* is the centre of curvature on bending (Fig. 6.2b).

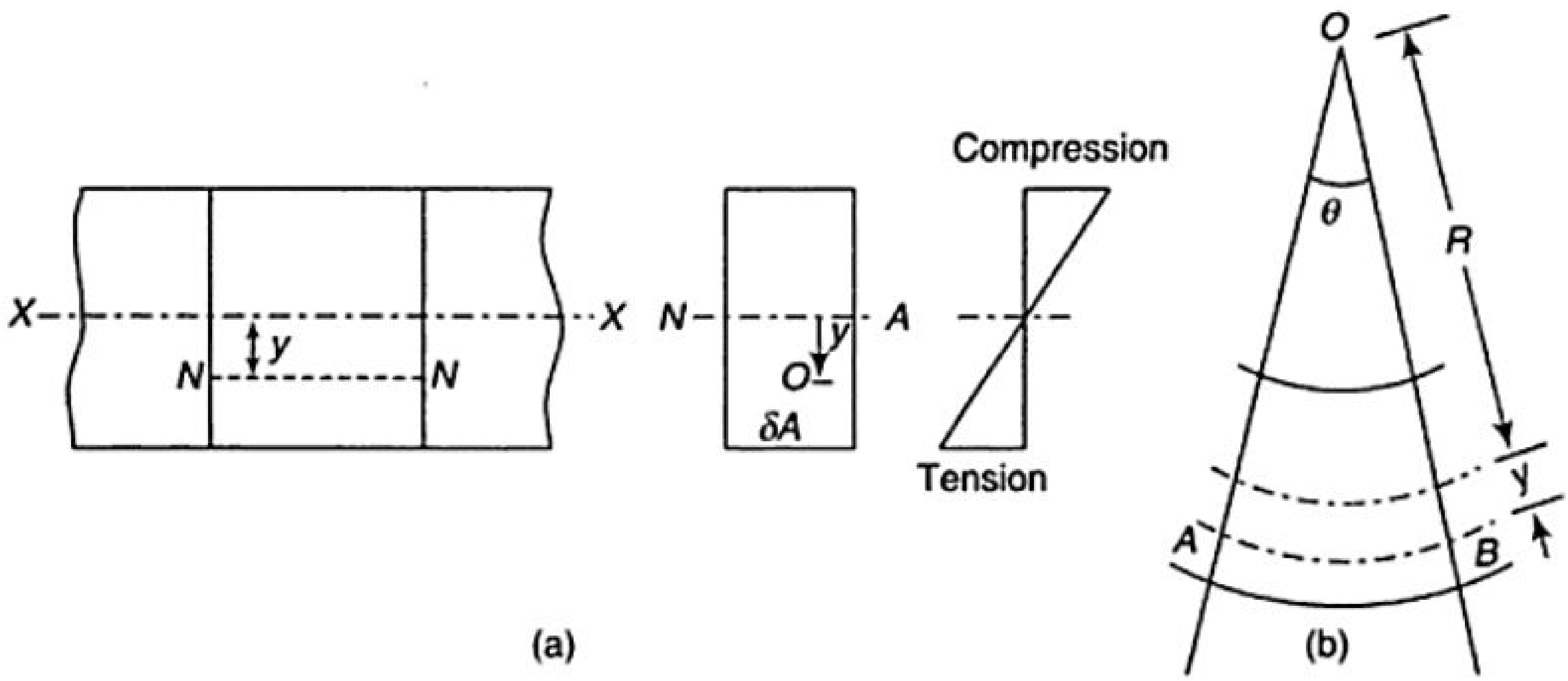


Fig. 6.2

Let R = radius of curvature of the neutral surface

θ = angle subtended by the beam length at centre O

σ = longitudinal stress

A filament of original length NN at a distance v from the neutral axis will be elongated to a length AB

$$\text{The strain in } AB = \frac{AB - NN}{NN}$$

$$\frac{\sigma}{E} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\sigma = y\frac{E}{R} \propto y \quad ..(i)$$

Thus stress is proportional to the distance from the neutral axis NA. This suggests that for the sake of weight reduction and economy, it is always advisable to make the cross-section of beams such that most of the material is concentrated at the greatest distance from the neutral axis. Thus there is universal adoption of the I-section for steel beams. Now let δA be an element of cross-sectional area of a transverse plane at a distance v from the neutral axis NA (Fig. 6.2).

For pure bending, Net normal force on the cross-section = 0

$$\int \sigma \cdot dA = 0$$

$$\int \frac{E}{R} y \cdot dA = 0 \text{ or } \frac{E}{R}\int y \cdot dA = 0$$

$$\int y \cdot dA = 0$$

This indicates the condition that the neutral axis passes through the centroid of the section. Also, bending moment = moment of the normal forces about neutral axis

$$M = \int(\sigma \cdot dA)y = \int \frac{E}{R} y \cdot dA \cdot y = \frac{E}{R}\int y^2 \cdot dA$$

$$= \frac{EI}{R}.$$

Or $$\frac{M}{I} = \frac{E}{R} \qquad \text{(ii)}$$

Where $I = \int y^2 dA$ and is known as the *moment of inertia* or *second moment of area* of the section. From (i) and (ii),

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

Where,

M = Bending Moment at a section (N-mm).

I = Moment of Inertia of the cross section of the beam about Neutral axis (mm^4).

σ = Bending stress in a fibre located at distance y from neutral axis (N/mm^2). This stress could be compressive stress or tensile stress depending on the location of the fibre.

y = Distance of the fibre under consideration from neutral axis (mm).

E = Young's Modulus of the material of the beam (N/mm^2).

R = Radius of curvature of the bent beam (mm).

6.5. SECTION MODULUS

The maximum tensile and compressive stresses in the beam occur at points located farthest from the neutral axis. Let us denote y_1 and y_2 as the distances from the neutral axis to the extreme fibres at the top and the bottom of the beam. Then the maximum bending normal stresses are

$$\sigma_{bc} = \frac{My_1}{I} = \frac{M}{I/y_1} = \frac{M}{Z_t},$$ σ_{bc} is bending compressive stress in the topmost layer.

Similarly,

$$\sigma_{bt} = \frac{My_2}{I} = \frac{M}{I/y_2} = \frac{M}{Z_b},$$ σ_{bt} is bending compressive stress in the topmost layer.

Here, Z_t and Z_b are called **section moduli** of the cross sectional area, and they have dimensions of length to the third power (ex. mm^3). If the cross section is symmetrical (like rectangular or square sections), then $Z_t = Z_b = Z$, and Z is called as section modulus. *Section modulus is defined* *as the ratio of rectangular moment of inertia of the section to the distance of the remote layer from the neutral axis.* Thus, general expression for bending stress reduces to

$$\sigma = \frac{M}{Z}$$

It is seen from the above expression that for a given bending moment, it is in the best interests of the designer of the beam to procure high value for section modulus so as to minimise the bending stress. More the section modulus designer provides for the beam, less will be the bending stress generated for a given value of bending moment.

6.6. MOMENT CARRYING CAPACITY OF A SECTION

From bending equation we have

$$\sigma = \frac{My}{I}$$

It shows bending stress is maximum on the extreme fibre where y is maximum. In any design this extreme fibre stress should not exceed maximum permissible stress. If σ_{per} *is* the permissible stress, then in a design

$$\sigma_{\max} \leq \sigma_{per}$$

$$\frac{M}{I} y \leq \sigma_{per}$$

Or if *M* is taken as maximum moment carrying capacity of the section,

$$\frac{M}{I} y_{\max} = \sigma_{per}$$

$$\text{Or } M = \frac{I}{y_{\max}} \sigma_{per}$$

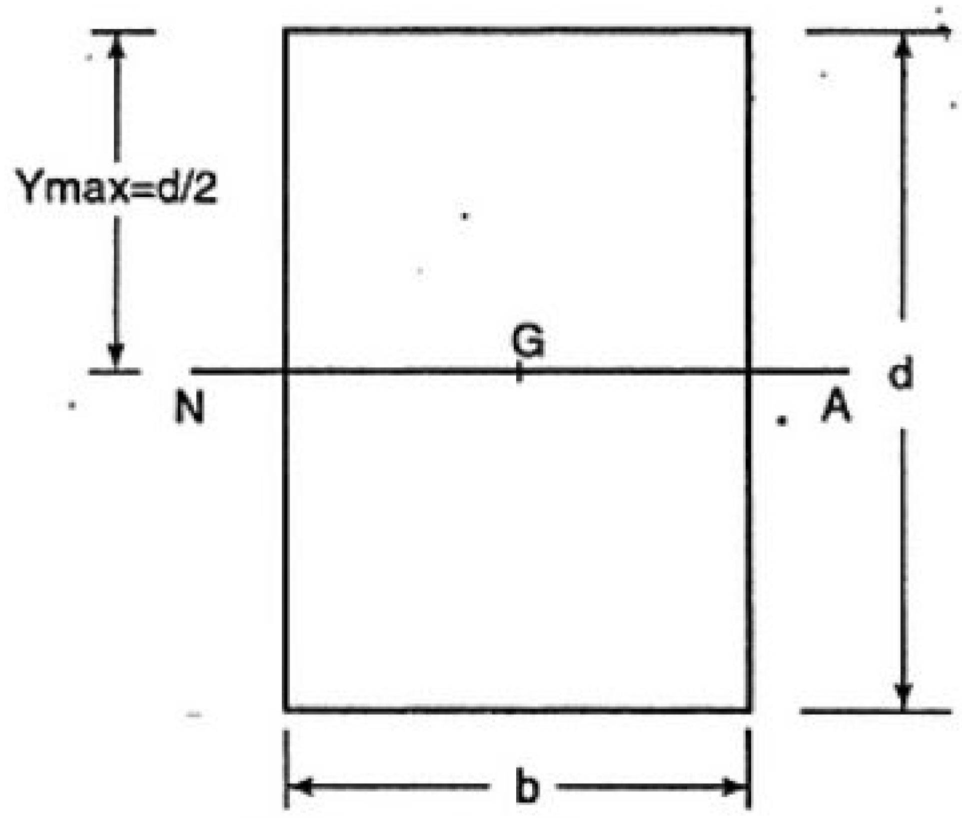
The moment of inertia I and extreme fibre distance y_{max} are the properties of cross-section. Hence, I/y_{max} is the property of cross-sectional area and is termed as section modulus and is denoted by Z. Thus the moment carrying capacity of a section is given by

$$M = \sigma_{per} Z$$

If permissible stresses in tension and compression are different, moment carrying capacity in tension and compression are found separately by considering respective extreme fibres and the smallest one is taken as moment carrying capacity of the section.

Expressions for section modulus of various standard cross-sections are derived below.

Rectangular section of width *b* and depth *d:*



$$I = \frac{1}{12}bd^3$$

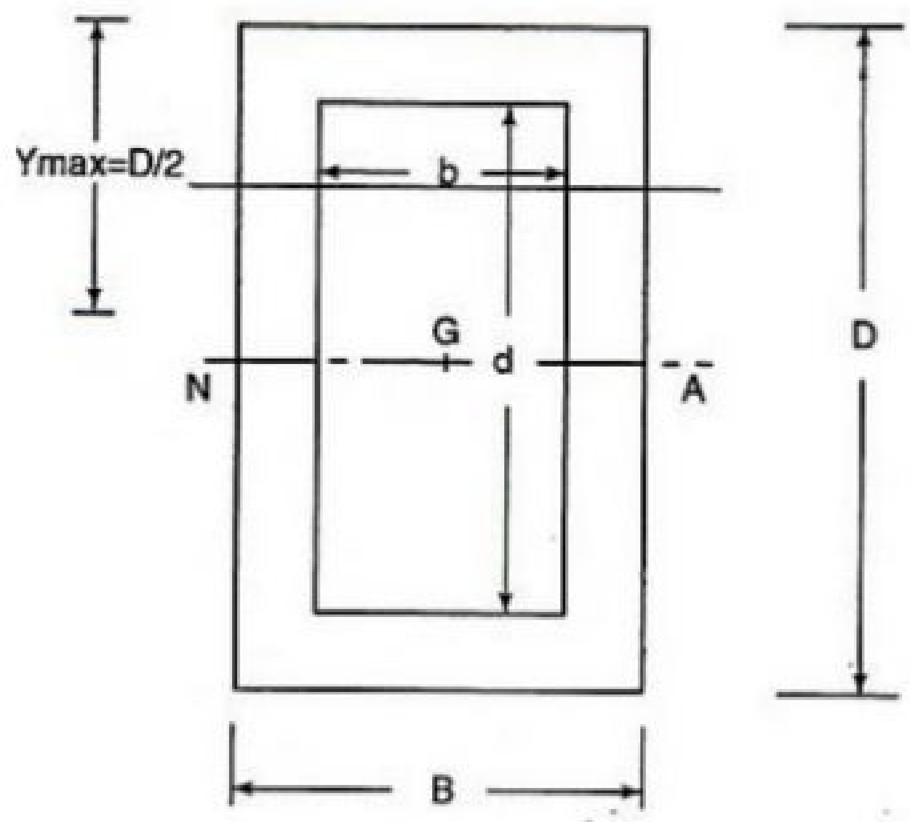
$$y_{max} = \frac{d}{2}$$

$$Z = \frac{I}{y_{max}} = \frac{\frac{bd^3}{12}}{\frac{d}{2}}$$

$$Z = \frac{1}{6}bd^2$$

Hollow rectangular section with symmetrically placed opening:

Consider the section of size *B* x *D* with symmetrical opening *bx d as* shown in Fig..



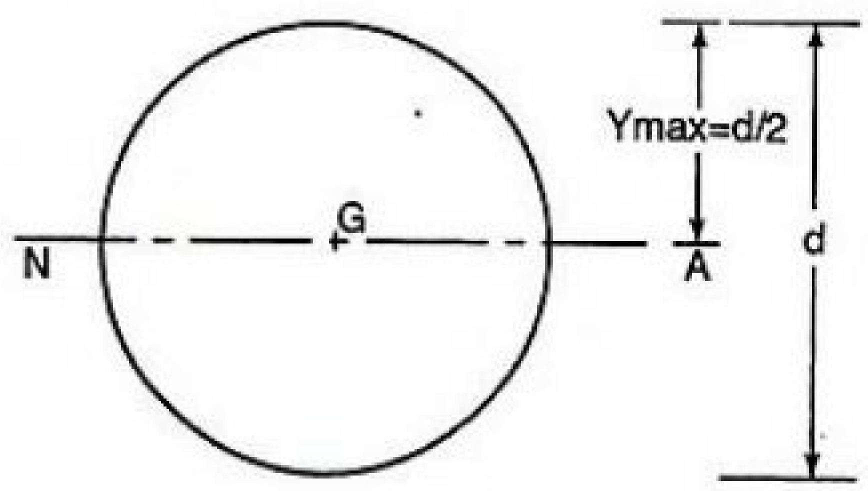
$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{1}{12}(BD^3 - bd^3)$$

$$y_{max} = \frac{D}{2}$$

$$Z = \frac{I}{y_{max}}$$

$$= \frac{1}{6}\frac{(BD^3 - bd^3)}{D}$$

Circular section of diameter *d*

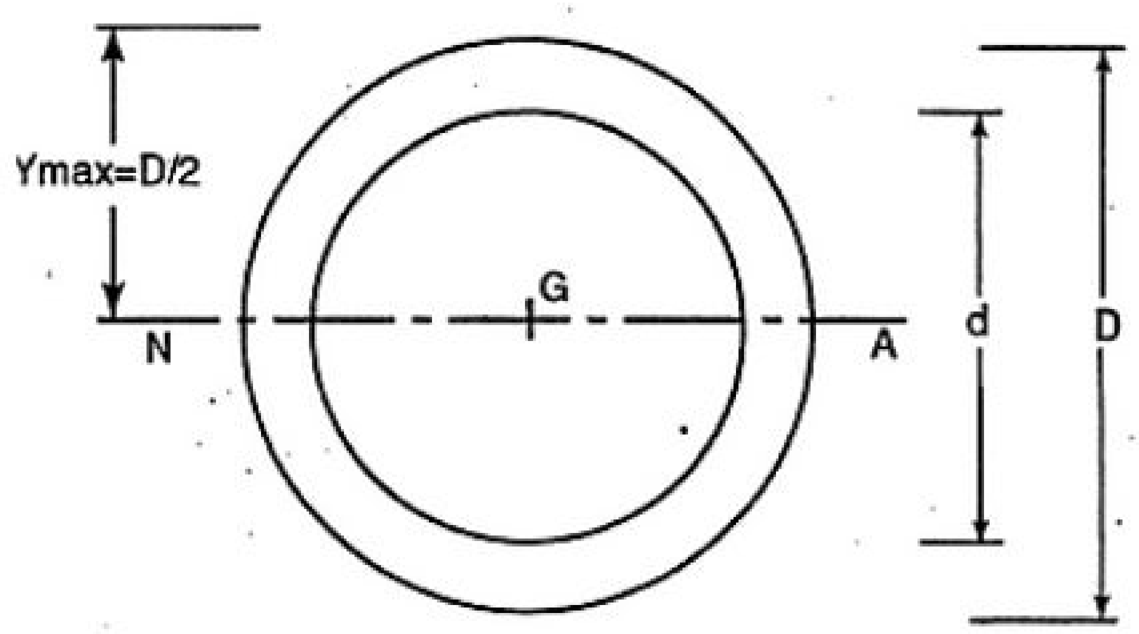


$$\text{For circular section} = I = \frac{\pi d^4}{64}$$

$$y_{max} = \frac{d}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\pi d^3}{32}$$

Hollow circular section of uniform thickness:



$$I = \frac{\pi}{64}D^4 - \frac{\pi}{64}d^4$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

$$\text{Extreme fibre distance} = \frac{D}{2}$$

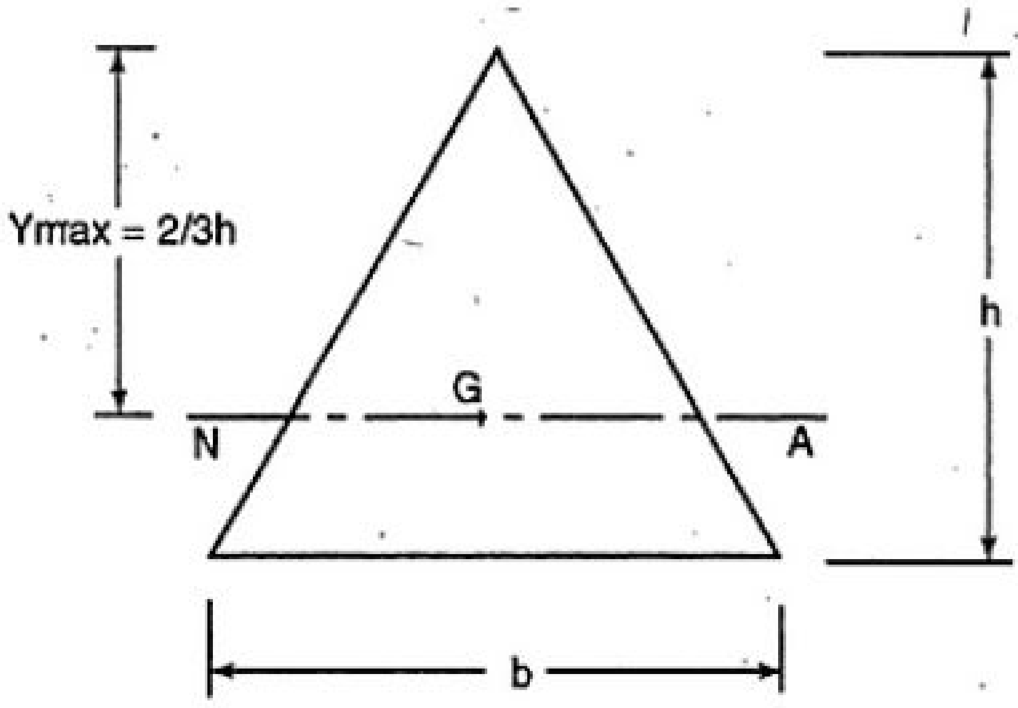
$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\pi}{32}\frac{(D^4 - d^4)}{D}$$

Triangular Section

$$I = \frac{bh^3}{36}$$

$$y_{max} = \frac{2}{3}h$$

$$Z = \frac{I}{y_{max}} = \frac{bh^2}{24}$$



6.7. SHEARING STRESSES IN BEAMS

we know that beams are usually subjected to varying bending moment and shearing forces. The relation between bending moment M and shearing force F is dM/dx=F. Bending stress act longitudinally and its intensity is directly proportional to its distance from neutral axis. Now we will find the stresses induced by shearing force.

Consider an elemental length of beam between the sections A-A and B- B separated by a distance dx as shown in Fig. 6.3a. Let the moments acting at A- A and B-B be *M* and *M+dM*.

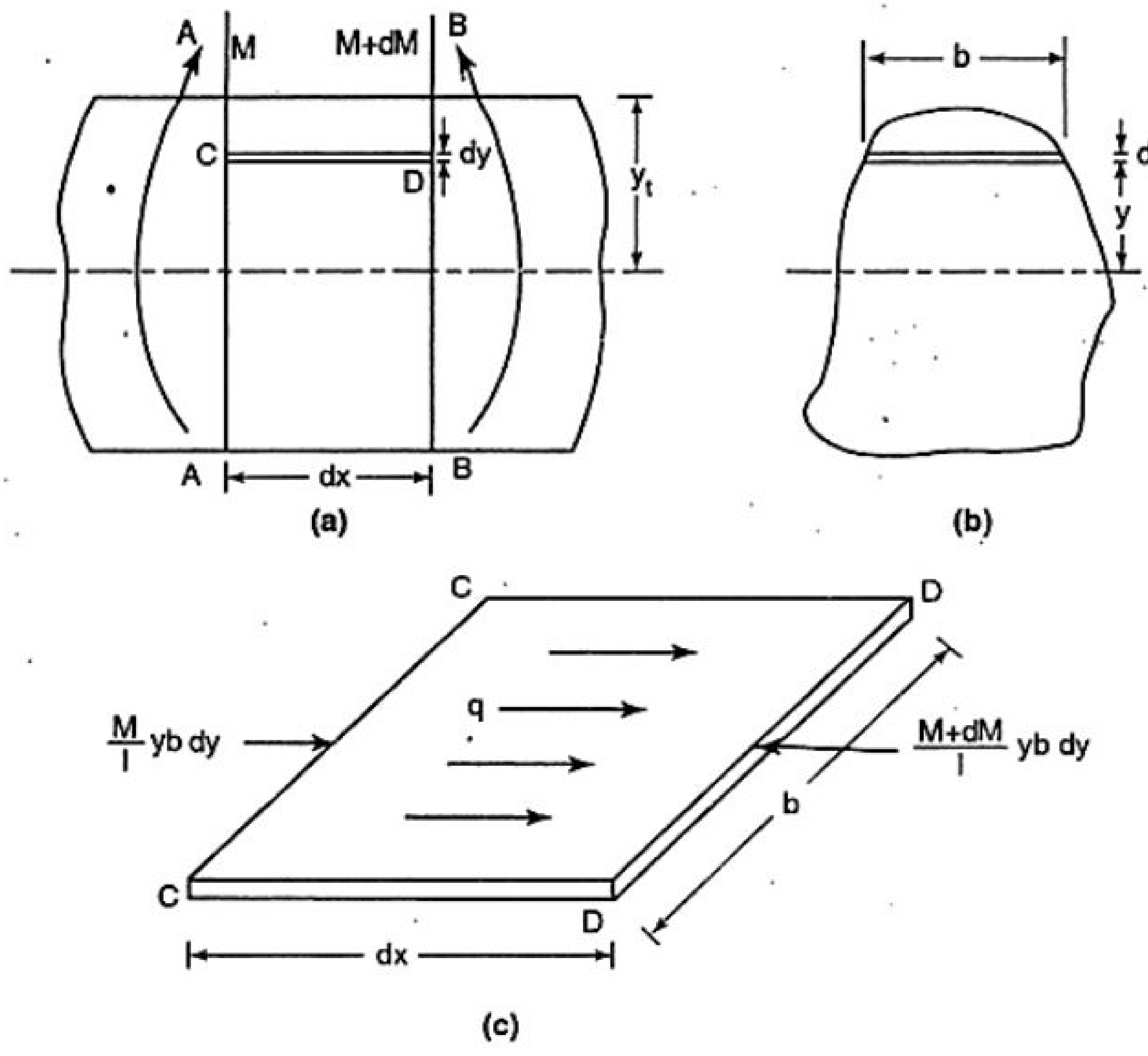


Fig. 6.3

Let CD be a fibre at a distance y-from neutral axis. Then bending stress at left side of the element

$$= \frac{M}{I} y$$

The force on the element on left side

$$= \frac{M}{I} ybdy$$

Similarly due to bending, force on the right side of the element

$$= \frac{M + dM}{I} ybdy$$

Unbalanced force towards right in element

$$= \frac{M + dM}{I} ybdy - \frac{M}{I} ybdy = \frac{dM}{I} ybdy$$

There are a number of such elements above section CD. Hence unbalance horizontal force above section *CD*

$$= \int_{y}^{y'} \frac{dM}{I} ybdy$$

This horizontal force is resisted by shearing stresses acting horizontally on plane at *CD.* Let intensity of shearing stress be *q.* Then equating shearing force to unbalanced horizontal force we get

$$= \tau bdx = \int_{y}^{y'} \frac{dM}{I} ybdy$$

Or $$\tau = \frac{dM}{dx} \cdot \frac{1}{bI} \int_{y}^{y'} ya$$

Where $a = b\ dy$ is area of element.

The term $\int_{y}^{y'} ya$ can be looked as

$$\sum_{y}^{y'} ay = a\overline{y}$$

Where $a\overline{y}$ is the moment of area above the section under consideration about neutral axis.

From equation, dM/dx=F

$$\tau = \frac{F}{bI} a\overline{y}$$

From the above expression it may be noted that shearing stress on extreme fibre is zero.

6.8 SHEAR STRESSES ACROSS RECTANGULAR SECTIONS

Consider a rectangular section of width *b* and depth *d* subjected to shearing force *F*. Let *A-A* be the section at distance *y* from neutral axis as shown in Fig. 6.4.

We know that shear stress at this section.

$$\tau = \frac{F}{bI} a\overline{y}$$

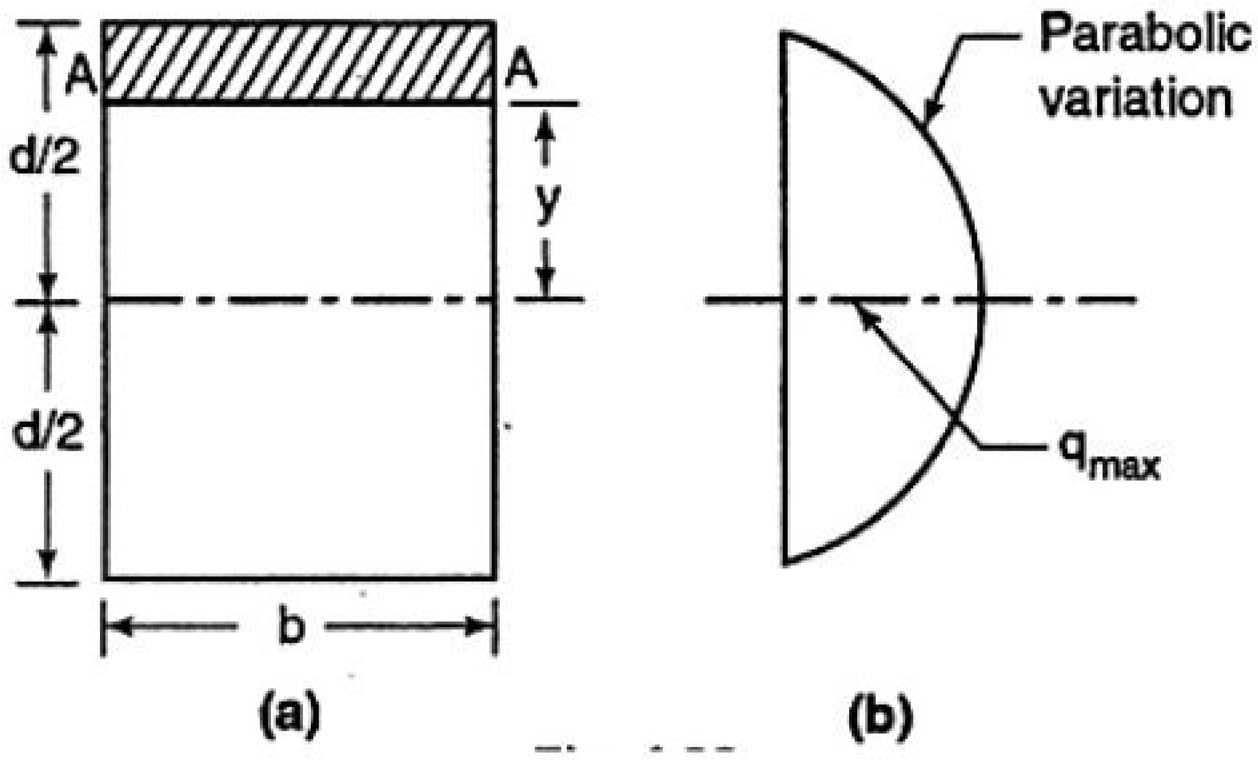


Fig. 6.4

where $a\bar{y}$ is the moment of area above this section (shown shaded) about the neutral axis.

Now $$a = b\left(\frac{d}{2} - y\right)$$

$$\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

and $$I = \frac{1}{12} bd^3$$

$\therefore$ $$q = \frac{F}{b \times \frac{1}{12} bd^3} \times b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

$$= \frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)$$

i.e., shear stress varies parabolically.

When y=d/2, $\tau = 0$

y=d/2, $\tau = 0$

y = 0, τ is maximum and its value is

$$\tau_{max} = \frac{6F}{bd^3}\frac{d^2}{4}$$

$$= 1.5\frac{F}{bd} = 1.5\tau_{avg}$$

Where

$$\tau_{avg} = \frac{ShearingForce}{Area} = \frac{F}{bd}$$

Thus, maximum shear stress is 1.5 times the average shear stress in rectangular section and occurs at the neutral axis. Shear stress variation is parabolic. Shear stress variation diagram across the section is shown in Fig.6.4b.

WORKED EXAMPLES

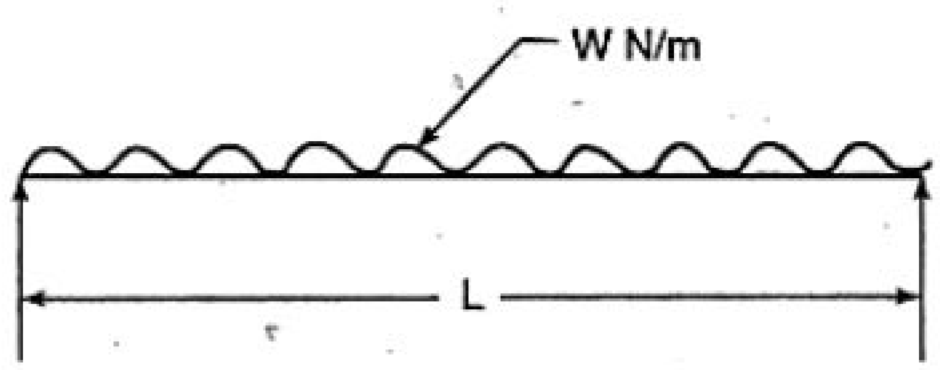
1) A simply supported beam of span 5 m has a cross-section 150 mm * 250 mm. If the permissible stress is 10 N/mm^2, find *(a)* maximum intensity of uniformly distributed load it can carry. *(b)* maximum concentrated load *P* applied at 2 m from one end it can carry.

Solution:

$$I = \frac{1}{12}bd^3$$

$$y_{max} = \frac{d}{2}$$

$$Z = \frac{1}{6}bd^2 = \frac{1}{6}\times 150 \times 250^2 = 1562500\,\text{mm}^3$$



Moment carrying capacity $M = \sigma\ Z = 10 \times 1562500$ N - mm

(a) If *w* is the intensity of load in N/m units, then maximum moment

$$= \frac{wL^2}{8} = \frac{w\times 5^2}{8} = \frac{w\times 25}{8}\ \text{N}-\text{m}$$

$$= \frac{w}{8}\times 25\times 1000\,\text{N}-\text{mm}$$

Equating it to moment carrying capacity, we get maximum intensity of load as

$$\frac{w\times 25}{8}\times 1000 = 10\times 1562500$$

$$w = 5000\,\text{N/m}$$

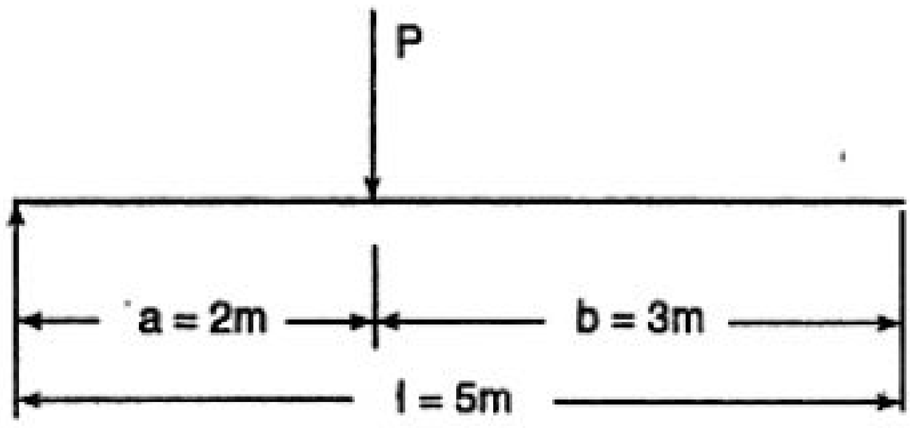
$$= 5\ \text{kN/m}$$

(b) If P is the concentrated load as shown in Fig., then maximum moment occurs under the load and its value

$$M = \frac{P\times a\times b}{L} = \frac{P\times 2\times 3}{5}$$

$$= 1.20P\ \text{kN-m}$$

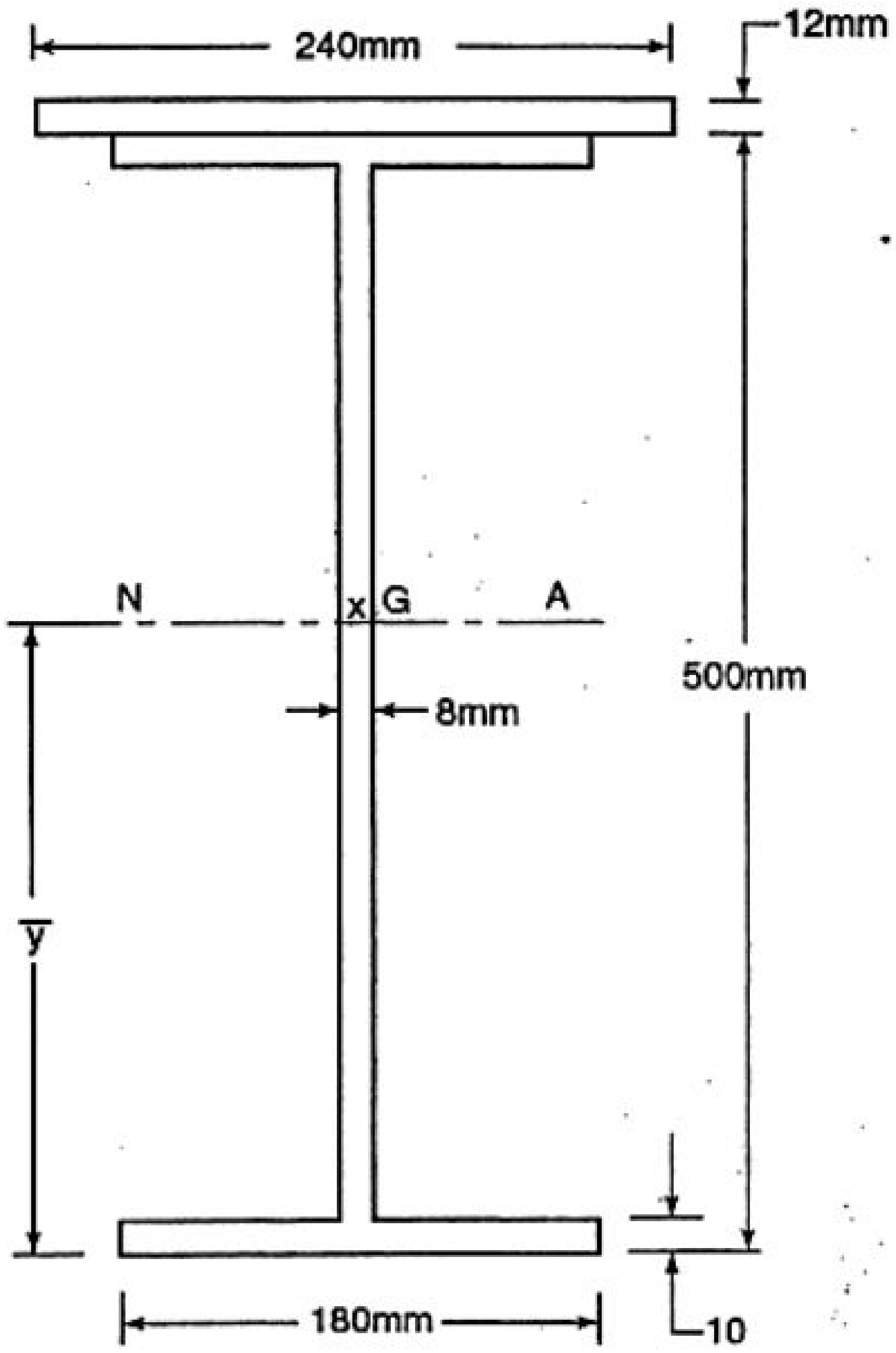
$$= 1200P\ \text{N-mm}$$



2) **A** symmetric I-section has flanges of size 180 mm x 10 mm and its overall depth is 500 mm. Thickness of web is 8 mm. It is strengthened with a plate of size 240 mm x 12 mm on compression side. Find the moment of resistance of the section, if permissible stress is 150 N/mm^2. How much uniformly distributed load it can carry if it is used as a cantilever of span 3 m?

Solution

The section of beam is as shown in Fig. Let $\bar{y}$ be the distance of centroid from the bottom-most fibre.



$$\bar{y} = \frac{\text{Moment of area about botton fibre}}{\text{Total area}}$$

$$= \frac{240 \times 12 \times 506 + 180 \times 10 \times 495 + 180 \times 10 \times 5 + 480 \times 8 \times 250}{240 \times 12 + 180 \times 10 + 180 \times 10 + 480 \times 8}$$

$$= \frac{3317280}{10320} = 321.442 \text{ mm}$$

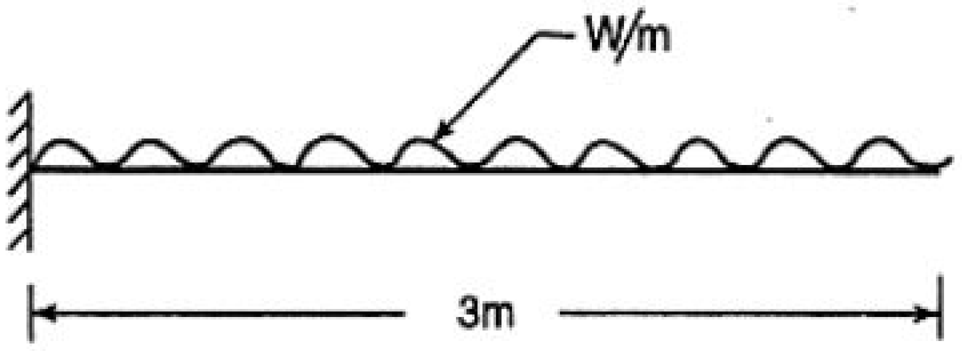
$$I = \frac{1}{12} \times 240 \times 12^3 + 240 \times 12(506 - 321.442)^2$$
$$+ \frac{1}{12} \times 180 \times 10^3 + 180 \times 10(495 - 321.442)^2$$
$$+ \frac{1}{12} \times 180 \times 10^3 + 180 \times 10(5 - 321.442)^2$$
$$+ \frac{1}{12} \times 8 \times 480^3 + 8 \times 480(250 - 321.442)^2$$
$$= 4.25952 \times 10^8 \text{ mm}^4$$

$$y_{top} = 512 - 321.442 = 190.558 \text{ mm}$$
$$y_{max} = \bar{y} = 321.442 \text{ mm}$$

Moment of resistance (Moment carrying capacity)

$$= f_{per} \times Z$$
$$= 150 \times \frac{4.25952 \times 10^8}{321.442} = 1.98769 \times 10^8 \text{ N} - \text{mm}$$
$$= 198.769 \text{ kN} - \text{mm} \qquad \textbf{(Ans)}$$

Let the load on cantilever be w/m length as shown in Fig.



Then maximum moment produced $= \frac{wL^2}{2}$ kN − m (where w is in kilo Newtons)

$$= w \times \frac{3^2}{2} = 4.5\ w\ \text{kN} - \text{m}$$

Equating moment of resistance to maximum moment, we get maximum load w

$$4.5w = 198.769$$

$$w = 44.171\ \text{kN/m} \qquad \textbf{(Ans)}$$

3) A T-section is formed by cutting the bottom flange of an I-section. The flange is 100 mm x 20 mm and the web is 150 mm x 20 mm. Draw the bending stress distribution diagrams if bending moment at a section of the beam is 10 kN-m (hogging).

Solution

M = 10 kN-m = 10 x 10^6 N mm (hogging)

Maximum bending stresses occur at extreme fibres, i.e. at the top bottom fibres which can be computed as

$$\sigma = \frac{My}{I} \qquad \text{(i)}$$

$$\bar{y} = \frac{(100 \times 20)(150 + 10) + \left((20 \times 150)\left(\frac{150}{2}\right)\right)}{(100 \times 20) + (20 \times 150)} = 109\ \text{mm}$$

Moment of inertia is given by

$$I = \left[\frac{(100 \times 20^3)}{12} + (100 \times 20)(109 - (150 + 10))^2\right]$$

$$+ \left[\frac{(20 \times 150^3)}{12} + (20 \times 150)\left(109 - \frac{150}{2}\right)^2\right] = 14.36167 \times 10^6\ \text{mm}^4$$

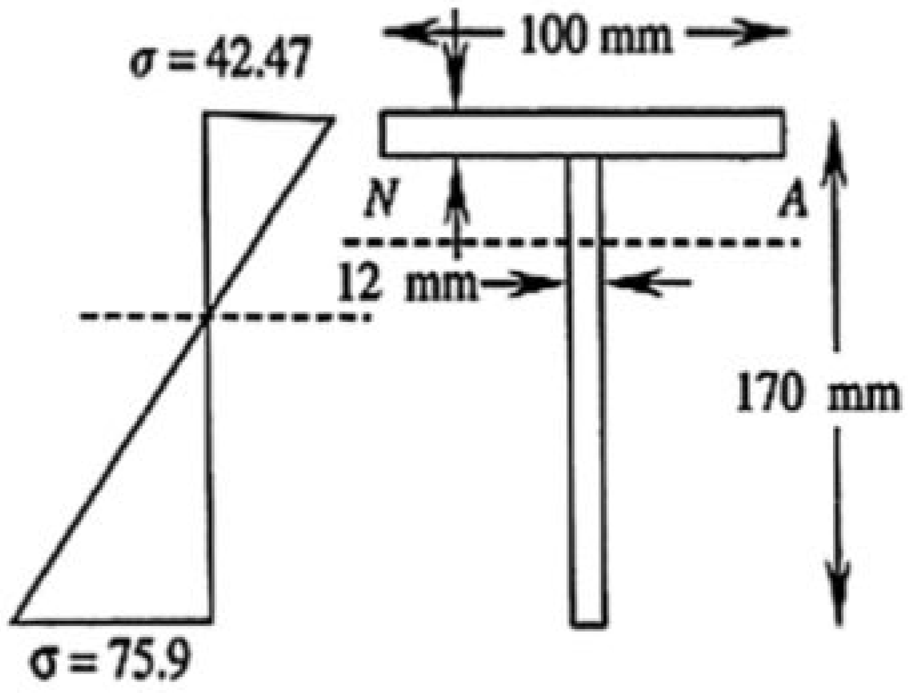
Substituting these values in Eq. (1),
Stress in the top fibre =

$$\sigma_{top} = \frac{M \times y_t}{\bar{I}} = \frac{(10 \times 10^6)(61)}{14.36167 \times 10^6} = 42.4742\ \text{N/mm}^2$$

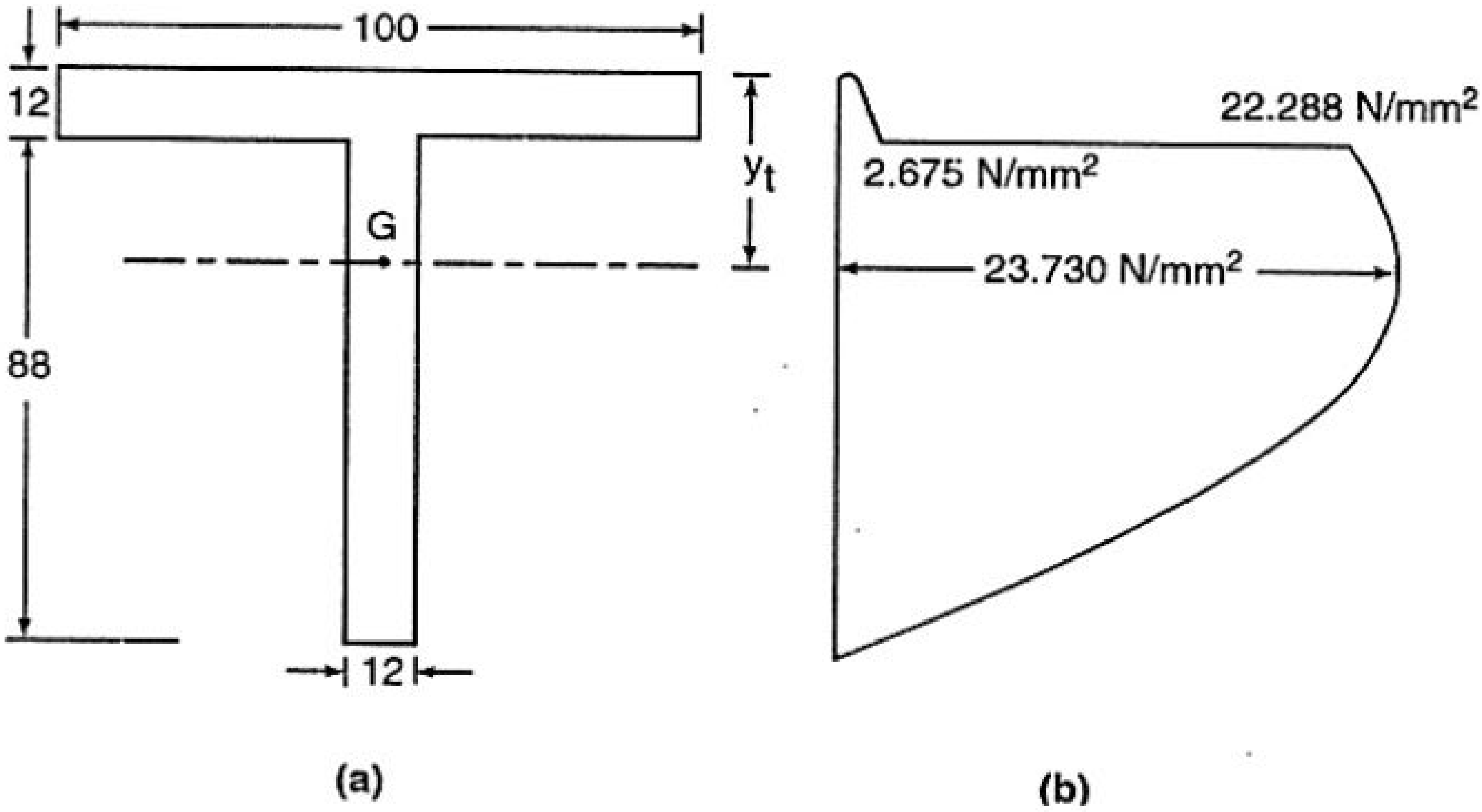
Stress in the bottom fibre =

$$\sigma_{bottom} = \frac{M \times y_b}{\bar{I}} = \frac{(10 \times 10^6)(109)}{14.36167 \times 10^6} = 75.8965\ \text{N/mm}^2$$

The given bending moment is hogging and hence negative and the tensile stresses occur at top fibre and compressive stresses in bottom fibres.



4) Fig. shows the cross-section of a beam which is subjected to a shear force of 20 kN. Draw shear stress distribution across the depth marking values at salient points.



Solution

Let y, be the distance of C.G form top fibre. Then taking moment of area about top fibre and dividing it by total area, we get

$$y_t = \frac{100\times12\times6+12\times88(44+12)}{100\times12+88\times12} = 29.404\text{ mm}$$

Moment of inertia about NA,

$$I = \frac{1}{12}\times100\times12^3+100\times12(29.404-6)^2$$

$$+\frac{1}{12}\times12\times88^3+12\times88(56-29.404)^2$$

$$= 2100127.3\text{ mm}^4$$

Shear stress at bottom of flange:

Area above this level $= 100 \times 12 = 1200 \text{ mm}^2$

C.G of this area from $N - A$ $\bar{y} = y_t - 6 = 29.404 - 6$

$= 23.404$ mm

Width at this level $= 100$ mm

$$\therefore \quad (q)_{\text{bottom of flange}} = \frac{20 \times 10^3}{100 \times 2100127.3} \times (1200 \times 23.404)$$

$$= 2.675 \text{ N/mm}^2.$$

q at same level but in web where width is 12 mm

$$= \frac{20 \times 10^3}{12 \times 2100127.3} \times (1200 \times 23.404)$$

$$= 22.288 \text{ N/mm}^2$$

To find shear stress at neutral axis:

$a\bar{y}$ above this level $= a\bar{y}$ of flange $+ a\bar{y}$ of web above this level

$$= 12 \times 100 \times (29.404 - 6) + 12 \times (29.404 - 12) \times \frac{(29.404 - 12)}{2}$$

$$= 29902.195 \text{ mm}^3$$

$$\tau = \frac{F}{bI} a\bar{y}$$

$$= \frac{20 \times 10^3}{12 \times 2100127.3} \times 29902.195$$

$$= 23.730 \text{ N/mm}^2$$

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