FOURIER SERIES

Let $f(x)$ be defined in the interval $(-l,l)$ and outside the interval by $f(x+2l)=f(x)$ i.e assume that $f(x)$ has the period $2l$ .The Fourierseries corresponding to $f(x)$ is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

where the Fourier coeffecients are

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

$$n = 1,2,3,....$$

If $f(x)$ is defined in the interval $(c, c+2l)$, the coefficients can be determined equivalently from

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

DIRICHLET CONDITIONS

Suppose that

1. $f(x)$ is defined and single valued except possibly at finite number of points in $(-l,+l)$
2. $f(x)$ is periodic outside $(-l,+l)$ with period $2l$
3. $f(x)$ and $f'(x)$ are piecewise continuous in $(-l,+l)$

Then the Fourier series of $f(x)$ converges to

a) $f(x)$ if x is a point of continuity

b) $[f(x+0)+f(x-0)]/2$ if x is a point of discontinuity

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METHOD OF OBTAINING FOURIER SERIES OF $f(x)$

1. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

2. $$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)\,dx$$

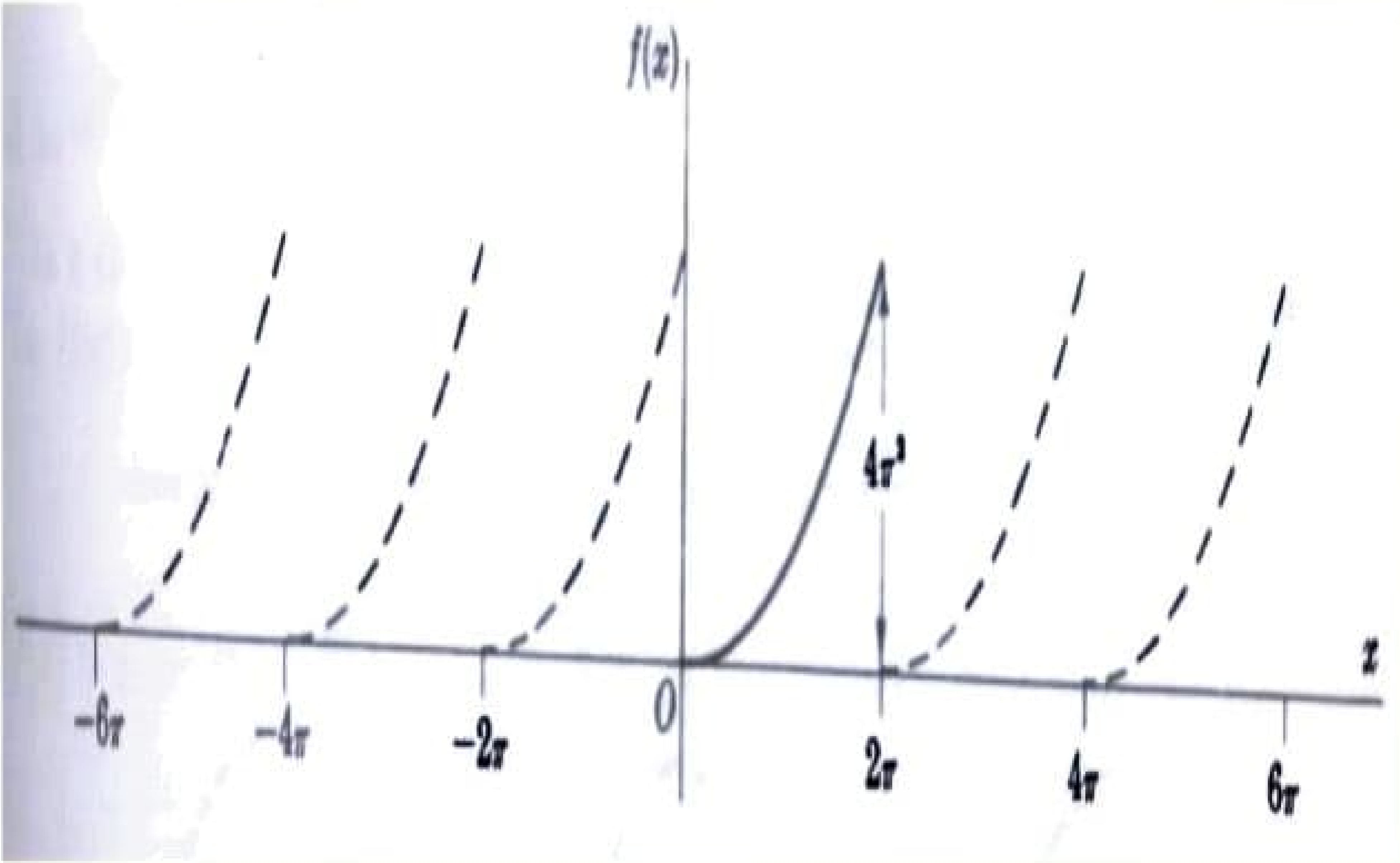
3. $$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx$$

4. $$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$n = 1,2,3,\ldots.$$

SOLVED PROBLEMS

1. Expand $f(x)=x^2, 0<x<2\pi$ in Fourier series if the period is 2π. Prove that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$


f(x)
4π²
x
−6π
−4π
−2π
0
2π
4π
6π

SOLUTION

Period = 2 l = 2 π thus l = π and choosing c=0

$$a_n = \frac{1}{l}\int_c^{c+2\pi} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \qquad n \neq 0$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{l}\int_c^{c+2\pi} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{-4\pi}{n}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$f(x)=x^2=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$$

At x=0 the above Fourier series reduces to $\dfrac{4\pi^2}{3}+\sum_{n=1}^{\infty}\dfrac{4}{n^2}$

X=0 is the point of discontinuity

By Dirichlet conditions, the series converges at x=0 to $(0+4\pi^2)/2 = 2\pi^2$

$$\Rightarrow 2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2. Find the Fourier series expansion for the following periodic function of period 4.

$$f(x)=\begin{cases}2+x & -2\le x<0\\ 2-x & 0<x\le 2\end{cases}$$

Solution

$$f(x+4)=f(x)$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx$$

$$=\frac{1}{2}\left[\int_{-2}^{0}(2+x)dx+\int_{0}^{2}(2-x)dx\right]$$

$$=\frac{1}{2}\left[\left(2x+\frac{x^2}{2}\right)_{-2}^{0}+\left(2x-\frac{x^2}{2}\right)_{0}^{2}\right]$$

$$=2$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(2+x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2}(2-x)\cos\frac{n\pi}{2}x\,dx\right]$$

$$= \frac{1}{2}\left[\left((2+x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (1)\frac{-\cos\frac{n\pi x}{2}}{\frac{(n\pi)^2}{4}}\right)_{-2}^{0}\right.$$

$$\left. + \left((2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\frac{-\cos\frac{n\pi x}{2}}{\frac{(n\pi)^2}{4}}\right)_{0}^{2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 \; for\, n \;\; even \\ \dfrac{8}{n^2\pi^2} \; for\, n \;\; odd \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(2+x)\sin\frac{n\pi x}{l}\,dx + \int_{0}^{2}(2-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{(2+x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_{-2}^{0}\right.$$

$$\left.\left\{(2-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_{0}^{2}\right] = 0$$

$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos(2n-1)\frac{\pi x}{2}\right]$$

EVEN AND ODD FUNCTIONS

A function $f(x)$ is called odd if $f(-x)=-f(x)$

Ex: x^3, $\sin x$, $\tan x$, x^5+2x+3

A function $f(x)$ is called even if $f(-x)=f(x)$

Ex: x^4, $\cos x$, e^x+e^{-x}, $2x^6+x^2+2$

EXPANSIONS OF EVEN AND ODD PERIODIC FUNCTIONS

If $f(x)$ is a periodic function defined in the interval $(-l,l)$, it can be represented by the Fourier series

Case1. If $f(x)$ is an even function

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$= \frac{2}{l}\int_{0}^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \quad \left(\because f(x)\cos\frac{n\pi x}{l}\ is\ also even\ function\right)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

$$= 0 \left(\because f(x)\sin\frac{n\pi x}{l}\ is\ odd\ function\right)$$

If a periodic function $f(x)$ is even in $(-l,l)$, its Fourier series expansion contains only cosine terms

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$a_0=\frac{2}{l}\int_0^l f(x)dx$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Case 2. When $f(x)$ is an odd function

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx=0$$

SOLVED PROBLEMS

1.For a function defined by $f(x)=|x|, -\pi < x < \pi$ obtain a Fourier series. Deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$$

Solution

$f(x)=|x|$ is an even function

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

SOLUTION

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |x| dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2} \cos nx$$

At x=0 the above series reduces to

$$\frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2}$$

x=0 is a point of continuity, by Dirichlet condition the Fourier series converges to f(0) and f(0)=0

$$0=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2}\right]$$

$$0=\frac{\pi}{2}-\frac{2}{\pi}\left(\frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+....\right)$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\frac{\pi^2}{8}$$

PROBLEM 2 $f(x)=\begin{cases}-k & when\ -3<x<0\\ k & when\ 0<x<3\end{cases}$

Is the function even or odd. Find the Fourier series of f(x)

$$= \frac{2}{3}\left[\frac{-k\ \cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right]_0^3$$

$$= \frac{2k}{n\pi}[1-(-1)^n]$$

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\frac{n\pi x}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0\left(\because f(x)\cos\frac{n\pi x}{l}\ is\ odd\right)$$

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\left(\because f(x)\sin\frac{n\pi x}{l}\ is\ even\right)$$

If a periodic function $f(x)$ is odd in $(-l,l)$,its Fourier expansion contains only sine terms

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

SOLUTION

$f(x)$ is odd function

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 k\sin\frac{n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\frac{-k\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right]_0^3$$

$$= \frac{2k}{n\pi}[1-(-1)^n]$$

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\frac{n\pi x}{3}$$

$$= \frac{2}{3}\left[\frac{-k\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right]_0^3$$

$$= \frac{2k}{n\pi}\left[1-(-1)^n\right]$$

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\frac{n\pi x}{3}$$