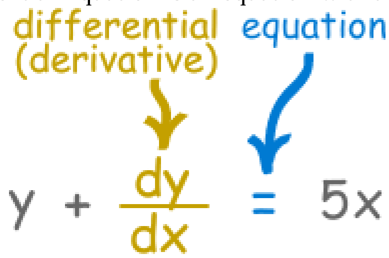
21BSS21 & ADVANCED CALCULUS AND COMPLEX ANALYSIS

Topic of Lecture : Introduction to Ordinary Differential Equation

Introduction: A Differential Equation is an equation with a function and one or more of its derivatives.



Example : An equation with the function y and its derivative $\frac{dy}{dx}$

Prerequisite knowledge for Complete understanding and learning of Topic :

Differential Equations can describe how populations change, how heat moves, how springs vibrate, how radioactive material decays and much more. They are a very natural way to describe many things in the universe.

The first major grouping is:

- Ordinary Differential Equations (ODEs) have **a single independent variable** (like **y**).
- Partial Differential Equations (PDEs) have two or more independent variables.

We are learning about **Ordinary Differential Equations** here!

Detailed content of the Lecture:

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$$

Where P and Q are constants and R is the function of x or constant.

Differential operators

The symbol D stands for the operation of differential

(i.e) $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$,

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$ can be written in the operator form $D^2y + P\,Dy + Qy$ = R (or)

$(D^2 + P\,D + Q)y$ = R

Complete Solution is y = Complementary function + Particular Integral

To find the Complementary functions

S.NO	Roots of Auxiliary Equation	Complementary function
1	**TYPE – I** Roots are real and different	$Ae^{m_1x} + Be^{m_2x}$
2	**TYPE – 2** Roots are real and equal	$(Ax + B)e^{mx}$
3	**TYPE – 3**	$e^{\alpha x}(Acos\beta x + Bsin\beta x)$

	Roots are imaginary	

To find the Particular Integral

$$\text{P.I}= \frac{1}{f(D)}X$$

S. NO	X	P.I
1	e^{ax}	P.I $=\frac{1}{f(D)}e^{ax}=e^{ax}\frac{1}{f(D)},\ \ f(a)\neq 0$ $=xe^{ax}\frac{1}{f'(a)}, f(a)=0, f'(a)\neq 0$ $=x^2e^{ax}\frac{1}{f''(a)}, f(a)=0, f'(a)=0, f''(a)\neq 0$
2	x^n	P.I $=\frac{1}{f(D)}\ x^n=[f(D)]^{-1}\ x^n$ Expand $[f(D)]^{-1}$ and then operate.
3	$\sin ax$ (or) $\cos ax$	P.I=$\frac{1}{f(D)}[\sin ax$ (or) $\cos ax]$ Replace D^2 by $-a^2$
4	$e^{ax}\phi(x)$	P.I=$\frac{1}{f(D)}e^{ax}\phi(x)$= $e^{ax}\frac{1}{f(D+a)}\ \phi(x)$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=HKvP2ESjJbA

Important Books/Journals for further learning including the page nos.:

1. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 243-265**
2. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.5**

Topic of Lecture : Higher Order Linear Differential Equations with Constant Coefficients

Introduction : The solutions of linear differential equations with constant coefficients of the third order or higher can be found in similar ways as the solutions of second order linear equations.

Prerequisite knowledge for Complete understanding and learning of Topic :

Second-order linear differential equations are used to model many situations in physics and engineering. For Example

- Mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series is modeled as differential equations.
- Bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Detailed content of the Lecture:

1. **Solve (D^2+D +1) y = 0**

Given (D^2+D +1) y = 0

The auxiliary equation is $m^2+m+1 = 0$.

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1^2-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

Here, the roots are imaginary.

$$\text{C.F} = e^{\frac{-1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

The general solution is given by

$$y = e^{\frac{-1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

2. **Solve $(D^2 - 5D + 6)y = 0$.**

Given$(D^2 - 5D + 6)y = 0$

The auxiliary equation is $m^2-5m+6 = 0$. i.e., $m = 2, m = 3$

The roots are real and distinct.

C. F=$Ae^{2x} + Be^{3x}$

y=$Ae^{2x} + Be^{3x}$

3. **Solve $(D^2 + 6D + 9)y = 0$.**

Given $(D^2 + 6D + 9)y = 0$.

The auxiliary equation is $m^2+6m+9 = 0$. i.e., $m = -3, m = -3$

The roots are equal.

C. F=$(Ax + B)e^{-3x}$

y=$(Ax + B)e^{-3x}$

4. Find the P.I of (D^2-2D+5) y = e^x sin2x.

$$P.I = \frac{1}{D^2-2D+5}e^x \sin 2x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1)+5}\sin 2x$$

$$= e^x \frac{1}{D^2+1+2D-2D-2+5}\sin 2x$$

$$= e^x \frac{1}{D^2+4} \sin 2x$$

$$= e^x \frac{1}{-4+4} \sin 2x$$

$$= xe^x \frac{1}{2D} \sin 2x$$

$$= xe^x \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) = \frac{-xe^x \cos 2x}{4}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ODxP7BbqAjA
https://www.youtube.com/watch?v=V9bl02Ffo_o
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/17%3A_Second-Order_Differential_Equations/17.3%3A_Applications_of_Second-Order_Differential_Equations

Important Books/Journals for further learning including the page nos.:

1. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 243-265**
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Topic of Lecture : Higher Order Linear Differential Equations with Constant Coefficients

Introduction: The solutions of linear differential equations with constant coefficients of the third order or higher can be found in similar ways as the solutions of second order linear equations.

Prerequisite knowledge for Complete understanding and learning of Topic :

Second-order linear differential equations are used to model many situations in physics and engineering. For Example

- Mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series is modeled as differential equations.
- Bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Detailed content of the Lecture:

1. **Solve $(D^2\text{-}4D + 4)\, y = e^{2x} + \cos 2x$**

Given $(D^2\text{-}4D + 4)\, y = 0$

The auxiliary equation is $m^2 - 4m + 4 = 0$

$(m - 2)(m - 2) = 0$

$m = 2, 2$

The roots are real and distinct.

C. F= $(Ax+B)e^{2x}$

$$P.I_1 = \frac{1}{D^2 - 4D + 4} e^{2x}$$

$$= \frac{1}{2^2 - 8 + 4} e^{2x}$$

D=a=2 & Denominator=0

$$= \frac{1}{0} e^{2x}$$

$$= \frac{x}{2D - 4} e^{2x}$$

$$= \frac{x}{4 - 4} e^{2x}$$

$$= \frac{x^2}{2} e^{2x}$$

$$P.I_2 = \frac{1}{D^2 - 4D + 4} \cos 2x$$

$$= \frac{1}{-4 - 4D + 4} \cos 2x$$

$$= -\frac{1}{4} \frac{1}{D} \cos 2x$$

D^2 =- 2^2 =-4

$$= -\frac{1}{4} \int \cos 2x dx$$

$$= \frac{-1}{8} \sin 2x$$

The Solution is y= C.F + $P.I_1$ + $P.I_2$

= (Ax+B)e^{2x}+ $\frac{x^2}{2} e^{2x}$ - $\frac{1}{8} \sin 2x$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ODxP7BbqAjA
https://www.youtube.com/watch?v=V9bl02Ffo_o
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/17%3A_Second-Order_Differential_Equations/17.3%3A_Applications_of_Second-Order_Differential_Equations

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3. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 243-265**
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- Mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series is modeled as differential equations.
- Bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Detailed content of the Lecture:

1. **Solve $(D^2+3D-2)\, y=2\cos(2x+3)+2e^x$**

Given $(D^2+3D-2)\, y = 0$

The auxiliary equation is $m^2+3m-2=0$

$(m-1)(m-2)=0$

$m=1, 2$

The roots are real and distinct.

C. F= Ae^x+Be^{2x}

$$P.I = \frac{1}{D^2-3D+2}[2\cos(2x+3)+2e^x]$$

$$= 2\left(\frac{1}{D^2-3D+2}\cos(2x+3)+\frac{1}{D^2-3D+2}e^x\right)$$

$$= 2\left(P.I_1 + P.I_2\right)..........(1)$$

$$P.I_1 = \frac{1}{D^2-3D+2}\cos(2x+3) \qquad \left(D^2 \to -4\right)$$

$$= \frac{1}{-4-3D+2}\cos(2x+3)$$

$$= \frac{1}{-3D-2}\cos(2x+3)$$

$$= -\frac{1}{3D+2}\cos(2x+3) = -\frac{1}{3D+2}\frac{3D-2}{3D-2}\cos(2x+3)$$

$$= \frac{-3D+2}{9D^2-4}\cos(2x+3)\ \left(D^2 \to -4\right)$$

$$= \frac{-3D+2}{-36-4}\cos(2x+3)$$

$$= \frac{-1}{40}\left[-3D\left[\cos\left(2x+3\right)\right]+2\cos\left(2x+3\right)\right]$$

$$= \frac{-1}{40}\left[6\,Sin\,\left(2x+3\right)\;+\;2\cos\left(2x+3\right)\right]$$

$$P.I_1 = \frac{-1}{20}\left[3\,Sin\,\left(2x+3\right)\;+\;\cos\left(2x+3\right)\right]$$

$$P.I_2 = \frac{1}{D^2-3D+2}e^x\left(D\;\rightarrow\;1\right)$$

$$= \frac{1}{0}e^x = \frac{x}{2D-3}e^x$$

$$= -xe^x$$

$$P.I = \frac{-1}{10}\left[3\,Sin\,\left(2x+3\right)+\cos\left(2x+3\right)\right]-2xe^x$$

The Solution is y= C.F + P.I

$$= Ae^x+Be^{2x} - \frac{1}{10}\left[3\,Sin\,\left(2x+3\right)+\cos\left(2x+3\right)\right] - 2\,xe^x$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ODxP7BbqAjA
https://www.youtube.com/watch?v=V9bl02Ffo_o
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/17%3A_Second-Order_Differential_Equations/17.3%3A_Applications_of_Second-Order_Differential_Equations

Important Books/Journals for further learning including the page nos.:

5. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 243-265**
6. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.5**

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Detailed content of the Lecture:

1. **Solve $(D^2-4D+3)\,y=e^x \cos 2x$**

Given $(D^2-4D+3)\,y = 0$

The auxiliary equation is $m^2 - 4m + 3=0$

$m^2 - 3m - m + 3=0$

$m(m-3)-1(m-3)=0$

$(m-1)(m-3)=0$

$m=1, 3$

The roots are real and distinct.

C. F= Ae^x+Be^{3x}

$$P.I = \frac{1}{D^2-4D+3}e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$= e^x \frac{1}{D^2+1+2D-4D-4+3}\cos 2x$$

$$= e^x \frac{1}{D^2-2D}\cos 2x$$

$$= e^{x}\frac{1}{-4-2D}\cos 2x$$

$$= -e^{x}\frac{(2D-4)}{(2D+4)(2D-4)}\cos 2x$$

$$= -e^{x}\frac{(2D-4)}{(4D^{2}-16)}\cos 2x$$

$$= -e^{x}\frac{(2D-4)}{-32}\cos 2x$$

$$= e^{x}\frac{(2D-4)}{32}\cos 2x$$

$$= e^{x}\frac{[2D(\cos 2x)-4\cos 2x]}{32}$$

$$= e^{x}\frac{[D(\cos 2x)-2\cos 2x]}{16}$$

$$= e^{x}\frac{[-2\sin 2x-2\cos 2x]}{16}$$

$$= -e^{x}\frac{[\sin 2x+\cos 2x]}{8}$$

The Solution is y= C.F + P.I

$$= Ae^{x}+Be^{3x}-e^{x}\frac{[\sin 2x+\cos 2x]}{8}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ODxP7BbqAjA
https://www.youtube.com/watch?v=V9bl02Ffo_o
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/17%3A_Second-Order_Differential_Equations/17.3%3A_Applications_of_Second-Order_Differential_Equations

Important Books/Journals for further learning including the page nos.:

7. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 243-265**
8. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.5**

Topic of Lecture : Cauchy’s and Euler's equations

Introduction :

In mathematics, an Euler–Cauchy equation, or Cauchy–Euler equation, or simply Euler's equation is a linear homogeneous ordinary differential equation with variable coefficients. It is sometimes referred to as an equidimensional equation. Because of its particularly simple equidimensional structure the differential equation can be solved explicitly.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Differentiation
2. Complementary function
3. Particular integral

Detailed content of the Lecture:

1. Solve $\boldsymbol{x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = x^2\sin(\log x)}$

Given $(x^2D^2 - xD + 4)y = x^2\sin(\log x)$ (1) →

Step-1:

$x = e^t$ $\quad xD = \theta$

Let $t = \log x$ $\quad x^2D^2 = \theta(\theta-1)$

Using this in (1)

$(\theta(\theta-1) - \theta + 4)y = e^{2t}\sin t$

$(\theta^2 - 2\theta + 4)y = e^{2t}\sin t$ (2) →

Step-2:

Sub: $\theta = m$

Auxiliary equation: $m^2 - 2m + 4 = 0$

$m = 1 \pm \sqrt{3}$.

Hence the roots are complex.

C.F: $e^t(A\cos\sqrt{3}t + B\sin\sqrt{3}t)$

Step-3:

$$P.I = \frac{1}{(D' - 1)} e^{2t}\sin t$$

$$= e^{2t}\frac{1}{(\theta+2)^2 - 2(\theta+2) + 4}\sin t$$

$$= e^{2t}\frac{1}{(\theta^2+2\theta+4)}\sin t$$

$$= e^{2t}\frac{1}{(2\theta+3)}\sin t$$

$$= e^{2t}\frac{(2\theta-3)}{(2\theta+3)(2\theta-3)}\sin t$$

$$= e^{2t}\frac{[2\theta(\sin t)-3\sin t]}{4\theta^2-9}$$

$$= e^{2t}\frac{[2cos\,t-3\sin t]}{-13}$$

$$= e^{2t}\frac{[3\sin t-2cos\,t]}{13}$$

Step-4:

The Complete Solution = C.F + P.I

$$y= e^{t}(A\cos\sqrt{3}t + B\sin\sqrt{3}t) + e^{2t}\frac{[3\sin t-2cos\,t]}{13}$$

Step-5:

Using this $x = e^{t}$, $t = \log x$ in y

$$y= x(A\cos\sqrt{3}\log x + B\sin\sqrt{3}\log x) + x^2\frac{[3\sin(\log x)-2cos(\log x)]}{13}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=1HKn9IvYiEM
https://www.youtube.com/watch?v=H7Ji3m1wTSY
https://www.youtube.com/watch?v=MyBrs3l_-rw

Important Books/Journals for further learning including the page nos.:

9. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 266-285**
10. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.25**

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Introduction:

In mathematics, an Euler–Cauchy equation, or Cauchy–Euler equation, or simply Euler's equation is a linear homogeneous ordinary differential equation with variable coefficients. It is sometimes referred to as an equidimensional equation. Because of its particularly simple equidimensional structure the differential equation can be solved explicitly.

Prerequisite knowledge for Complete understanding and learning of Topic :

4. Differentiation
5. Complementary function
6. Particular integral

Detailed content of the Lecture:

1. **Solve** $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$ (1)

$x = e^z$ $xD = D'$

Let $z = \log x$ $x^2D^2 = D'(D'-1)$

Using this in (1)

$(D'(D'-1) - D' + 1)y = \left(\frac{z}{e^z}\right)^2$

$(D'^2 - 2D' + 1)y = z^2e^{-2z}$

Sub: $\theta = m$

Auxiliary equation: $m^2 - 2m + 1 = 0$

$m = 1,1.$

C.F = $(Az + B)\, e^z = (A \log x + B)\, x$

P.I $= 1/(D'-1)^2\, Z^2 e^{-2z} = e^{-2z}\, 1/[(D'-2)-1]^2\, Z^2$

$= e^{-2z} 1/(D'-3)^2\, Z^2 = e^{-2z}\, 1/(3-D')^2\, Z^2$

$= e^{-2z}/9\; 1/[1-D'/3]^2\, Z^2 = e^{-2z}/9\, [1-D'/3]^{-2}\, Z^2$

$= e^{-2z}/9\, [1 + 2D'/3 + (D'/3)^2 + \ldots\ldots]\, Z^2$

The Solution is y= C.F + P.I

$= = (A \log x + B)\, x + 1/27x^2[3(\log x)^2 + 4 \log x + 2]$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=1HKn9IvYiEM
https://www.youtube.com/watch?v=H7Ji3m1wTSY
https://www.youtube.com/watch?v=MyBrs3l_-rw

Important Books/Journals for further learning including the page nos.:

11. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 266-285**
12. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.25**

Topic of Lecture : Method of variation of parameter

Introduction : To Finding the general solution of the second order differential equation.
To determine whether the solution of a homogeneous differential equation are linearly independent. Wronskain: To determine whether the solution of a homogeneous differential equation are linearly independent.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Fundamental system
2. Cramer's rule
3. Wronskain determinant

Detailed content of the Lecture:

1. Solve $(D^2 + a^2)y = tanax$ by the method of variation of parameters

Solution :

Given :$(D^2 + a^2)$y= tanx

$M^2 + a^2$=0

$M^2 = -a^2$

M = $\pm$ai

C.F = $e^{0x}[c_{1\cos\,ax+c_2 sinax}]$

$= c_1 cosax + c_2 sinax$

Here $f_1 = cosax$ $\quad$ $f_2 = sinax$

$f_1^{'} = -sinaxa$ $\quad$ $f_2^{'} = cosaxa$

=-asinax $\quad$ =acosax

$f_1 f_2 - f_{1 f_2} = cosax(acosax) - (-asinax)(sinax)$

=a $cos^2 ax + a\, sin^2 ax = a(cos^2 ax + sin^2 ax) = a$

Here x = tanax

P.I=$[Pf_1 + Qf_2]$

$P = \int \frac{f2x}{f1f2^{'} - f1^{'}f2} dx$

$= -\int \frac{sinax\ \ tanax}{a} dx$

$= -\int \frac{sinax\ -\frac{sinax}{cosax}}{a} dx$

$= \frac{-1}{a} \int \frac{sin^2 ax}{cosax} dx$

$= \frac{-1}{a} \int \frac{1-cos^2 ax}{cosax} dx$

$= \frac{-1}{a} \int (\frac{1}{cosax} - \frac{cos^2 ax}{cosax}) dx$

$= \frac{-1}{a} (\int secax\ dx - \int cosax\ dx)$

$= \frac{-1}{a} [\frac{\log(secax + tanax)}{a} . \frac{sinax}{a}]$

P $= \frac{-1}{a^2} [\log(secax + tanxax) - sinax]$

Q$=\int \frac{f_1 x}{f_{1f_2} - f_{1f_2}} dx$

$=\int \frac{cosax + tanax}{cosax + acosax + sinax + sinax} dx$

$=\int \frac{cosax\ \ taxax}{acos^2 ax + asin^2 ax} dx$

$=\frac{1}{a} cosax \frac{sinax}{cosax}$ dx

$=\frac{1}{a} \int sinax\ dx$

$=\frac{-1}{a} \frac{cosax}{a} dx$

P.I$=Pf_1 + Qf_2$

P.I=$(\frac{-1}{a^2} [\log(secax + tanax) - sinax])$cosax$\frac{cosax}{a^2} sinax$

Y=C.F+P.I

Y$=c_1 cosax + c_2 sinax - \frac{1}{a^2} (\log[secx + tanax] - sinax) cosax - \frac{cosax}{a^2} sinax$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=kFto6HjIp0Q (Method of variation of parameter)

Important Books/Journals for further learning including the page nos.:

13. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 266-285**
14. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.29**

Topic of Lecture : Method of variation of parameter

Introduction : To Finding the general solution of the second order differential equation.
To determine whether the solution of a homogeneous differential equation are linearly independent. Wronskain: To determine whether the solution of a homogeneous differential equation are linearly independent.

Prerequisite knowledge for Complete understanding and learning of Topic :

4. Fundamental system
5. Cramer's rule
6. Wronskain determinant

Detailed content of the Lecture:

1. Solve $(D^2 + a^2)y = secax$ by the method of variation of parameters

Solution :

Given $(D^2 + a^2) = \sec ax$

Replace D=m

$m^2 + a^2 = 0$

$m^2 = -a^2 m = \pm ai$

C.F $= e^{ax}(c_1 \cos ax + c_2 \sin ax)$

$= c_1 \cos ax + c_2 \sin ax$

f1=cosax f2=sinax

$f_1^{'} = -\sin ax \; a \; f_2^{'} = \cos ax \; a$

= - a sin ax = a co sax

P.I $= Pf_1 + Qf_2$

$$P = \int \frac{f_2 x}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= -\int \frac{\sin ax - \sec ax}{\cos ax . a \cos ax + a . \sin ax . \text{Sin } ax} \; dx$$

$$= -\int \frac{\sin ax . \sec ax}{a} dx$$

$$= -\frac{1}{a} \int sinax . \frac{1}{\cos ax} dx$$

$$= -\frac{1}{a} \int \tan ax \; dx$$

$$= -\frac{1}{a} \left[\frac{\log a(\sec ax)}{a} \right]$$

$$P = \left[\frac{-\log ax \; (\sec ax)}{a} \right]$$

$$Q = \int \frac{f_1 x}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= \int \frac{\cos ax . \sec ax}{a} \; dx$$

$$= \frac{1}{a} \int dx$$

$$Q = \frac{x}{a}$$

$$P.I = p\, f_1 + q\, f_2$$

$$= \frac{-\log(\sec ax)\cos ax}{a^2} + \frac{x}{a}\sin ax$$

$$Y = C.F + P.I$$

$$= c_1 \cos ax + c_2 \sin ax - \frac{\log(\sec ax)\cos ax}{a^2} + \frac{x}{a}\sin ax$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=kFto6HjIp0Q (Method of variation of parameter)

Important Books/Journals for further learning including the page nos.:

15. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 266-285**
16. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.29**

Topic of Lecture : Legendre's Linear Equation

Introduction :

The equation of the form $(ax+b)^n \frac{d^n y}{dx^n} + k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots..K_n y = f(x)$

Can be reduce to homogeneuse linear differential equation by putting the substitution $ax + b = e^z \; or \; z = \log(ax+b)$

Put $(ax+b)D = aD'y$

$(ax+b)^2 D^2 = a^2 D'(D'-1)y$

$(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2)y$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Frobenius Method
2. Legendre's Polynomial
3. Linear operator

Detailed content of the Lecture:

1. **Transform the equation** $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$ **into a differential equation with constant coefficients**

$$(2x+3)^2\frac{d^2y}{dx^2}-(2x+3)\frac{dy}{dx}12y=6x\to(1)$$

put

$$(2x+3)=e^z,z=\log(2x+3)$$

$$(2x+3)^2D^2=4\theta(\theta-1)y$$

$$(2x+3)D=2\theta y,\theta=\frac{d}{dz}$$

$$(1)\Rightarrow(4\theta^2-6\theta-12)y=3e^z-9$$

2. **Solve** $(x+1)^2\frac{d^2y}{dx}+(x+1)\frac{dy}{dx}+y=4\cos[\log(x+1)]$

Solution :

Given$((x+1)^2D^2+(x+1)D+1)y=4\cos\log(x+1)$------------(1)

$x+1=e^t$ $x=e^t-1$ $(x+1)D=\Theta$

Let $(x+1)^2D^2=1^2\Theta(\Theta-1)=\Theta^2-\Theta$

$\text{Log}(x+1)=t$

$(\Theta^2-\Theta+\Theta+1)y=4\cos t$

$(\Theta^2+1)y=4\cos t$ ----------------(2)

sub$\Theta=m$

Auxiliary Equation:$m^2+1=0$

$M=\pm i$ Roots are complex

Complementary function:

$$C.F=e^{ot}[A\cos t+B\sin t]$$

Particular integral

P.I=$\frac{1}{\theta^2+1}4\cos t$

=$4\frac{1}{-1+1}\cos t$

=$4t\frac{1}{2\Theta}\cos t$

=$2t\int\cos t\,dt$

Particular integral : P.I =2tsin t

Complete solution : C.S=C.F+P.I

Using this $e^t=(x+1)$ and $t=\log(x+1)$ in y

Y=A coslog(x+1)+B sinlog(x+1)+2log(x+1)sinlog(x+1)

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=FN7ysBTIqwM (Legendre's linear equations)

Important Books/Journals for further learning including the page nos.:

17. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 266-285**
18. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.25**

Topic of Lecture : Simultaneous differential equations with constant coefficient

Introduction :

Solving of a simultaneous differential equation means we have to find the dependent variables in terms of independent variable . To get complete solution , a simultaneous equation must contain as many equations as there are number of dependent variable .

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Classification of differential equation.

2.Separable differential equation.

3.Exact differential equation.

4. Homogeneous differential equation.

Detailed content of the Lecture:

1. **Eliminate y from the system** $\frac{dx}{dt}+2y=-\sin t; \frac{dy}{dt}-2x=\cos t$

Given $\frac{dx}{dt}+2y=-\sin t$ $\qquad$ $\frac{dy}{dt}-2x=\cos t$

Dx+2y=-sint ……. (1) $\qquad$ Dy-2x= cost….. (2)

$(1)\times D \Rightarrow D^2x+2Dy=-D(\sin t)$ $\qquad$ $D^2x+2Dy=-\cos t$ ……… (3)

$(2)\times 2 \Rightarrow 2Dy-4x=2\cos t$ $\qquad$ $2Dy-4x=2\cos t$ ………. (4)

(3) - (4) $\Rightarrow (D^2+4)x=-3\cos t$

After eliminating y we get $(D^2+4)x=-3\cos t$

2. **Solve** $\frac{dx}{dy}+2x+3y=0$, $3x+\frac{dy}{dx}+2y=2e^{2t}$

Solution :

Solution :

Dx+2y+3y=0

(D+2)x+3y=0 → (1)

3x+Dy+2y=0

3x+(D+2)y=0 → (2)

$(1) \times (D+2) \Longrightarrow (D+2)^2 x + 3(D+2)y = 0$

$(2) \times 3 \quad \Longrightarrow \quad \underline{9x + 3(D+2)y = 6e^{2t}}$

$(D+2)^2 x - 9x = -6e^{2t}$

$(D^2 + 4D + 4)$ -9x= $-6e^{2t}$

$D^2 x + 4D + 4x - 9x = -6e^{2t}$

$D^2 x$+4Dx-5x=$-6e^{2t}$

$(D^2 + 4D - 5)x = m$

Replace D=m

$m^2 + 4m - 5 = 0$

(m-1)(m+5)=0

m=1 , m=-5

C.F= $Ae^t + Be^{-5t}$

P.I=$\frac{1}{D^2+4D-5} - 6e^{2t}$

=$\frac{-6}{D^2+4\,D-5} e^{2t}$

Hear a=2

Replace D=a=2

=$\frac{-6}{(2)^2+4(2)-5} e^{2t}$

P.I=$\frac{-6}{7} e^{2t}$

x=$Ae^t + Be^{-5t} - \frac{6}{7} e^{2t}$

3y=-2x$- \frac{dy}{dt}$

$\frac{dx}{dt}$ =$Ae^t + Be^{-5t}(-5) - \frac{6}{7} e^{2t}(2)$

3y= $-2[Ae^t + Be^{-5t} - \frac{6}{7} e^{2t}] - [Ae^t + 5Be^{-5t} + \frac{12}{7} e^{2t} -$

$Ae^t \quad + 5Be^{-5t} + \frac{12}{7} e^{2t}$

3y=$-3Ae^t - 2Be^{-5t} + \frac{24}{7} e^{2t}$

3y=$-3[Ae^t + Be^{-5t} + \frac{8}{7} e^{2t}]$

y=$-Ae^t - Be^{-5t} + \frac{8}{7} e^{2t}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=-k4L4dYQb8o (simultaneous first order linear equations with constant co-efficient.)

Important Books/Journals for further learning including the page nos.:

19. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 286-298**
20. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.25**

Topic of Lecture : Simultaneous differential equations with constant coefficient

Introduction :

Solving of a simultaneous differential equation means we have to find the dependent variables in terms of independent variable . To get complete solution , a simultaneous equation must contain as many equations as there are number of dependent variable .

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Classification of differential equation.

2.Separable differential equation.

3.Exact differential equation.

4. Homogeneous differential equation.

Detailed content of the Lecture:

3. **Solve** $\frac{dx}{dt}+y=\sin t, x+\frac{dy}{dt}=\cos t$ **, given that x = 2 and y = 0 at t = 0.**

Solution :

Solution : Given

$Dx + y = sint$ -------(1)

$x + Dy = cost$ -------(2)

$(1) \quad \Longrightarrow Dx + y = sint$

$(2) \times D \Longrightarrow Dx + D^2y = -sint$

$(D^2 - 1)y = -2sint$

A.E is Replace D=m

$m^2 - 1 = 0$

m=1 , m=-1

C.F= A$e^t + Be^{-t}$

P.I=$\frac{1}{D^2-1} - 2sint$

Hear a=1

Replace $D^2 = -a^2 = -1$

$$=\frac{1}{-1-1}-2sint$$

$$P.I = sint$$

$$y = Ae^{t} + Be^{-t} + sint \text{--------------}(3)$$

$$\frac{dy}{dt} = Ae^{t} + Be^{-t}(-1) + cost \text{---------------}(4)$$

Substituting (3)&(4) in (2)

$$x = -Dy + cost$$

$$x(t) = -Ae^{t} + Be^{-t} - cost + cost \text{ ------------}(5)$$

$$y(t) = Ae^{t} + Be^{-t} + sint \quad \text{--------------------}(6)$$

Given x(o)=2 and y(0)=0

$$x(0) = -A + B = 2 \text{ -------------}(7)$$

$$y(0) = A + B = 0 \text{ -----------------}(8)$$

Adding (7)&(8)

$$2A = 2$$

$$A = 1$$

$$B = -1$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=-k4L4dYQb8o (simultaneous first order linear equations with constant co-efficient.)

Important Books/Journals for further learning including the page nos.:

21. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 286-298**
22. **B.V. Ramana, Higher Engineering Mathematics, Tata MC Graw Hill Education PVT Limited. Page No: 9.38**

Unit-II

Vector calculus

Topic of Lecture : Gradient , Divergence, Curl, Solenoidal, Irrotational

Introduction

Two type of functions in vector calculus.

- Scalar function
- Vector function

Vector differential operator $\nabla= \vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}$.

Where $\vec{i},\vec{j},\vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.
Velocity : Velocity is the rate of change of position with respect to time.
Acceleration: Acceleration is the rate of change of velocity with time.

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Slope of a vector point function
2.Partial Derivatives
3.Calulus

Detailed content of the Lecture:

Gradient: $\nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x}+\vec{j}\frac{\partial\emptyset}{\partial y}+\vec{k}\frac{\partial\emptyset}{\partial z}$

Where $\emptyset$ be a real valued function having continuous first order partial derivatives.

Divergence: $\nabla.\vec{F} = div\vec{F}$, $\vec{F}$ be a vector point function and $div\vec{F}$ is a scalar function.

Curl: $\nabla\times\vec{F} = curl\ \vec{F}$, The curl is a vector operator that describes the infinitesimal rotation of vector field in three dimensional Euclidean spaces.

1. Find grad $\emptyset$ if $\emptyset = xyz$ at (1,1,1)

   Solution:

   Given $\emptyset = xyz$

$$\text{grad } \emptyset = \nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x}+\vec{j}\frac{\partial\emptyset}{\partial y}+\vec{k}\frac{\partial\emptyset}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz)+\vec{j}\frac{\partial}{\partial y}(xyz)+\vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}(yz)+\vec{j}(xz)+\vec{k}(xy)$$

$$\text{grad}\emptyset_{(1,1,1)} = \vec{i}+\vec{j}+\vec{k}$$

2. If $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ then find div $\vec{F}$

   Solution:

   Given $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$

$$\text{div } \vec{F} = \nabla\vec{F} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(x^2\vec{i}+y^2\vec{j}+z^2\vec{k}\ \right)$$

$$= \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(z^2)$$

$$= 2x+2y+2z$$

$$= 2(x+y+z)$$

3. If $\vec{F} = x^2y\vec{i}+y^2z\vec{j}+z^2x\vec{k}$ then find curl $\vec{F}$

   Solution:

   Given $\vec{F} = x^2y\vec{i}+y^2z\vec{j}+z^2x\vec{k}$

$$\text{curl } \vec{F} = \nabla x\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \vec{i}(0-y^2)-\vec{j}(z^2-0)+\vec{k}(0-x^2)$$

$= -y^2\vec{i} - z^2\vec{j} - x^2\vec{k}$

4. Find $Curl\vec{F}$ if $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$
   Solution:

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}(-y) - z\vec{j} - x\vec{k}$$

5. Show that $\vec{F} = (x^2\vec{i} + y^2\vec{j} + z^2)\vec{k}$ is a conservative vector field.
   Solution:

If $\hat{F}$ is conservative then $\nabla \times \hat{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$

Therefore $\hat{F}$ is a conservative vector field.

Solenoidal : If $\nabla.\vec{F} = div\vec{F} = 0$, then $\vec{F}$ is Solenoidal
Irrotational : If $\nabla \times \vec{F} = curl\ \vec{F} = 0$, then $\vec{F}$ is Irrotational

6. Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is Solenoidal
   Solution:
   Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$
   $\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(z\vec{i} + x\vec{j} + y\vec{k}\ \ )$
   $= (0\vec{i} + 0\vec{j} + 0\vec{k}$
   Hence $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is Solenoidal

7. Find λ so that $\vec{F} = (3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2k)\vec{k}$ is solenoidal
   Solution:

Given $\vec{F}$ is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\vec{\partial}}{\partial y}j + \frac{\partial}{\partial y}\vec{k}\right) \cdot \left((3x - 2y + z)\vec{i} + (4x + \lambda y - z)\vec{j} + (x - y + 2k)\vec{k}\right)$$

$3 + \lambda + 2 = 0$

$\lambda = -5$

8. Prove that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational.
   Solution:

$$\nabla X \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ yz & zx & xy \end{vmatrix}$$

$$\vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=csCskd01jwE (Gradient , Divergence, Curl& Line integral)

Important Books/Journals for further learning including the page nos.:

23. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 525-546**

Topic of Lecture : Directional derivative ,Unit normal**,** Line integral,

Introduction :

Curvature: The magnitude of the rate of change of the unit vector with respect to the arc length.

Tangent line: The derivative of a function at a point is the slope of the tangent line at this point

Normal line: The Normal line is defined as the line that is perpendicular to the tangent line to the point of tangency.

Directional derivative: Directional derivative = $\nabla \phi \cdot \hat{n}$

Unit normal: $\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$

Line intergral: A line integral is an integral where the function to be integrated is evaluated along a curve.

$$I = \int \vec{F} \, . d\vec{r}$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Calculus
2.Line integral of scalar function
3. Line integral of Vecter fields

Detailed content of the Lecture:

1. **Find the unit normal vector to the surface $x^2+y^2=z$ at (1,-2,5)**

   ϕ = x^2+y^2-z

Solution:

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial \vec{z}} \vec{k} \right) x^2+y^2-z$$

$= 2x\vec{i} + 2y\vec{j} - \vec{k}$

$(\nabla\phi)_{1,-2,5)} = (2\vec{i} - 4\vec{j} - \vec{k}), |\nabla\phi| = \sqrt{21}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$

2. Find the directional derivative $\phi = x^2 + y^2 + z^2$ in the direction of the vector $\vec{F} = \vec{i} + 2\vec{j} + 2\vec{k}$ at (1,1,1)

Solution:

Unit normal vector $\hat{n}$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$ **is** $\left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\right)$

Directional derivative $= \nabla\phi \cdot \hat{n}$

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \vec{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \vec{k}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$grad\,\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{at(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

Directional derivative = $\nabla\phi \cdot \hat{n} = \left(2\vec{i} + 2\vec{j} + 2\vec{k}\right) \cdot \left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}\right) = \frac{10}{3}$

3. If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_c \vec{F}.d\vec{r}$ from (0,0) to (1,1) along the line y = x.

Solution:

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

Along the line y = x, dy = dx

$\therefore \vec{F} = x^2\vec{i} + x.x\vec{j}, \quad d\vec{r} = dx\vec{i} + dyj = dx\vec{i} + dx\vec{j}$

$\vec{F}.d\vec{r} = (x^2\vec{i} + x^2\vec{j}).(dx\vec{i} + dy\vec{j})$

$= x^2dx + x^2dx = 2x^2dx$

$$\int_c \vec{F}.d\vec{r} = \int_0^1 2x^2dx = \frac{2}{3}$$

4. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_c \vec{F}.dr$ Where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$.

Solution:

$\vec{F}.d\vec{r} = (5xy\vec{i} + 2y\vec{j}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$

= 5xydx+2ydy

The curve C: $y = x^2$

dy=2xdx

x varies from 1 to 2

$$\int_c \vec{F}.d\vec{r} = \int_1^2 5x(x^2)dx + 2x^2 2xdx = \left[5\frac{x^4}{4} + \frac{4x^4}{4}\right]_1^2$$

$$= 36 - \frac{9}{4} = \frac{135}{4}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=csCskd01jwE (Gradient , Divergence, Curl& Line integral)

Important Books/Journals for further learning including the page nos.:

24. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 565-575**

Topic of Lecture : Surface,Volume integrals

Introduction: **Surface integral:** A surface integral is a generalization of multiple integrals to integration over surfaces.

$\iint_S \vec{F}\,\hat{n}ds = \iint_{R_1} \vec{F}\,\hat{n}\,\frac{dx\,dy}{|\hat{n}.\vec{k}|}$ (the projection of S on XY plane)

$\iint_S \vec{F}\,\hat{n}ds = \iint_{R_2} \vec{F}\,\hat{n}\,\frac{dydz}{|\hat{n}.\vec{i}|}$ (the projection of S on YZ plane)

$\iint_S \vec{F}\,\hat{n}ds = \iint_{R_3} \vec{F}\,\hat{n}\,\frac{dz\,dx}{|\hat{n}.\vec{j}|}$ (the projection of S on ZX plane)

Volume Integral :

$$\iiint_V f(x,y,z)dV = \iiint_V f(x,y,z)dx\,dy\,dz$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. **partial Derivatives**
2. **Differential calculus**
3. **Volume of the solid with cross sectional area.**

Detailed content of the Lecture:

1. Find the surface,Where $\vec{F} = z\vec{i} + x\vec{j} - y^2z\vec{k}$ and S is surface of the Cylinder $x^2 + y^2 = 1$ Included in the 1st octant between the plane z=0 and z=2

Solution:

$$\vec{F} = z\vec{i} + x\vec{j} - y^2z\vec{k}$$

Given $\varphi = x^2 + y^2 - 1$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial \vec{z}}\vec{k}\right)$$

$$= \vec{i}(2x) + \vec{j}(2y)$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2}$$

$$= 2\sqrt{x^2 + y^2} = 2$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$= \frac{\vec{i}(2x)+\vec{j}(2y)}{2}$

$= \vec{i}x + \vec{j}y$

$\vec{F}.\hat{n} = z\vec{i} + x\vec{j} - y^2 z\vec{k}\ .\vec{i}x + \vec{j}y$

$= xz + xy$

$$\iint_S \vec{F}\,\hat{n}ds = \iint_{R_2} \vec{F}\,\hat{n}\,\frac{dydz}{|\hat{n}.\vec{i}|}$$

$$= \iint_R xz + xy\frac{dydz}{x}$$

$$= \iint_R (z+y)dydz$$

$$= \iint_{00}^{21} (z+y)dydz$$

$$= \int_0^2 \left[zy + \frac{y^2}{2}\right]\begin{matrix}1\\0\end{matrix}dydz$$

$$= \int_0^2 \left[z + \frac{1}{2}\right]dydz$$

$$= \left[\frac{z^2}{2} + \frac{z}{2}\right]\begin{matrix}2\\0\end{matrix}$$

$$= 2 + 1$$

=3

2. **If** $\vec{F} = ax\vec{i} + by\,\vec{j} + cz\,\vec{k}$, **prove that** $\iiint_V \nabla.\vec{F}dV$ **=(a+b+c)V.**

Solution:

$$\iiint_V \nabla.\vec{F}dV = \iiint_V (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}).(ax\vec{i} + by\vec{j} + cz\vec{k})dv$$

$$= \iiint_V (a+b+c)dv$$

= (a+b+c) V

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=K37VbB5Ukxk(Surface,Volume integrals)

Important Books/Journals for further learning including the page nos.:

25. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 565-575**

Topic of Lecture : Angle between the surface

Introduction: The angle θ between the two surface φ_1 and φ_2 is given by $\cos\theta = \frac{\nabla\varphi_1 . \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.First fundamental form and angel between the surface

2.Angel between the normal curve

Detailed content of the Lecture:

1. **Find the** angle between the surface $x\log z = y^2 - 1$ and $x^2y = 2 - z$ *at the point* (1,1,1)

Solution :

Let $\varphi_1 = x\log z - y^2 + 1$

$\varphi_2 = x^2y - 2 + z$

$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{i}(logz) + \vec{j}(-2y) + \vec{k}(\frac{x}{z})$

$\nabla\varphi_1\ (1,1,1) = 0\vec{i} - 2\vec{j} + \vec{k}$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}(2xy) + \vec{j}(x^2) + \vec{k}(1)$

$\nabla\varphi_2\ (1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$

The angle θ between the two surface φ_1 and φ_2 is given by $\cos\theta = \frac{\nabla\varphi_1 . \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$\cos\theta = \frac{0\vec{i} - 2\vec{j} + \vec{k}\,.\,2\vec{i} + \vec{j} + \vec{k}}{\sqrt{0+4+1}\,.\sqrt{4+1+1}}$$

$$\cos\theta = \frac{-2+1}{\sqrt{5}\,.\sqrt{6}}$$

$$\cos\theta = \frac{-1}{\sqrt{5}\,.\sqrt{6}}$$

2. Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ *at the point* (2, -1,2)

Solution :

Let $\varphi_1 = x^2 + y^2 + z^2 - 9$

$\varphi_2 = x^2 + y^2 - z - 3$

$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$

$\nabla\varphi_1\ (2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}(2x) + \vec{j}(2y) - \vec{k}$

$\nabla\varphi_2\ (2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$

The angle θ between the two surface φ_1 and φ_2 is given by $\cos\theta = \frac{\nabla\varphi_1 . \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$\cos\theta = \frac{4\vec{i} - 2\vec{j} + 4\vec{k}\,.\,4\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{16+4+16}\,.\sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{\sqrt{36}.\sqrt{21}}$$

$$\cos\theta = \frac{8}{3.\sqrt{21}}$$

.

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=Zfo9Nld0Ano **(Angle between the surface)**

Important Books/Journals for further learning including the page nos.:

26. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 526-528**

Topic of Lecture : Scalar potential

Introduction: A scalar potential is a fundamental concept in vector analysis and physics (the adjective scalar is frequently omitted if there is no danger of confusion with vector potential). The scalar potential is an example of a scalar field.

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Vector potential
2.Electric potential

Detailed content of the Lecture:

1. Show that $\vec{F} = (Y^2+2xz^2)\vec{i} + (2xy-2)\vec{j}+(2x^2z-y+2z)\vec{k}$ is irrotional Hence find its scalar potential:

Solution:

$\vec{F} = (Y^2+2xz^2)\vec{i} + (2xy-2)\vec{j}+(2x^2z-y+2z)\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix}$$

$=\vec{i}\left(\frac{\partial}{\partial y}\right)(2x^2z-y+2z) - \left(\frac{\partial}{\partial z}\right)(2xy-z) - \vec{j}\left(\frac{\partial}{\partial x}\right)(2x^2z-y+2z) - (\frac{\partial}{\partial z})(y^2+2xz^2)+\vec{k}(\frac{\partial}{\partial x})(2xy-z)-(\frac{\partial}{\partial y})(y^2+2xz^2)$

$=\vec{i}(-1+1)-\vec{j}(4xz-4xz)+\vec{k}(2y-2y)$

$=0\vec{i}-0\vec{j}+0\vec{k}$

$=0$

$\nabla\times\vec{F}$ is irrotational, $\nabla\emptyset = \vec{F}$

$$\nabla\emptyset = \vec{F}$$

$$(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}) = [(y^2+2xz^2)\vec{i}+(2xy-z)\vec{j}+(2x^2z-y+2z)\vec{k}]$$

Equvating the cooeficient $\vec{i}, \vec{j}, \vec{k}$,

$\frac{\partial\emptyset}{\partial x} = y^2 + 2xz^2; \frac{\partial\emptyset}{\partial y} = 2xy - z; \frac{\partial\emptyset}{\partial z} = 2x^2z - y + 2z$

Let $\frac{\partial\emptyset}{\partial x} = y^2 + 2xz^2$

Integrating on both sides,

$$\int \partial\emptyset = \int (y^2 + 2xz^2)\partial x$$

$\emptyset = xy^2 + z^2x^2$ ………..(1)

Let $\frac{\partial\emptyset}{\partial y} = 2xy - z$

Integrating on both sides,

$$\int \partial\emptyset = \int (2xy - z)\delta y$$

$\emptyset = 2x\frac{y^2}{2} - yz = xy^2 - yz$…………(2)

Let $\frac{\partial\emptyset}{\partial z} = 2x^2z - y + 2z$

$$\int \partial\emptyset = \int (2x^2z - y + 2z)\,\partial z$$

$\emptyset = 2x^2\frac{z^2}{2} - yz + \frac{z^2}{2}$ ………(3)

The scalar potential is $\emptyset = xy^2 + z^2x^2 - yz + \frac{z^2}{2}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=Zfo9Nld0Ano (Angle between the surface)

Important Books/Journals for further learning including the page nos.:

27. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 526-528**

Topic of Lecture : Gauss divergence theorem

Introduction :

The volume integral of the divergence of a vector filed A taken over any volume, V is equal to the surface integral of the A taken over the closed surface surrounding the volume V.

$$\iint_S \vec{F}\,\hat{n}dS = \iiint_V \nabla\vec{F}\,dV$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Vector integral theorems
2.Vector calculus

Detailed content of the Lecture:

1 . Verify the gauss divergence theorem $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by x=o to 1, y=0 to 1, z=0 to 1

Solution:

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\iint_S \vec{F}\,\hat{n}dS = \iiint_V \nabla\vec{F}\,dV$

R.H.S

$$\nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

= 4z-2y+y

$$\nabla.\vec{F} = 4z - y$$

$$\iiint_v \nabla\vec{f}.dv = \iiint_{000}^{111}(4z-y)dz\,dy\,dx$$

$$=\int_0^1 \int_0^1 \left(\frac{4z^2}{2}-yz\right)_0^1 dydx$$

$$=\int_0^1 \int_0^1 (2(1)-y)dy\,dx$$

$$=\int_0^1 (2y-\frac{y^2}{2})_0^1 dx$$

$$=\int_0^1 \left(2(1)-\frac{1}{2}\right)dx$$

$$=\int_0^1 (2-\frac{1}{2})dx$$

$$=\int_0^1 \left(\frac{4-1}{2}\right)dx$$

$$=\frac{3}{2}\int_0^1 dx$$

$$=\frac{3}{2}(x)_0^1$$

$$=\frac{3}{2} \quad \rightarrow (A)$$

L.H.S

Now $\int_S \vec{F}\hat{n}ds = \iint_{s1} + \iint_{s2} + \iint_{s3} + \iint_{s4} + \iint_{s5} + \iint_{s6}$

SURFACE	FACE	UNIT OUTWARD VECTOR $\hat{n}$	$\overrightarrow{f.\hat{n}}$	AXIS	
S1	ADEF	$\vec{i}$	4xz	X=1	dx=0
S2	OCGB	$\overrightarrow{-i}$	-4xz	X=0	dx=0
S3	BDEG	$\vec{j}$	y^2	Y=1	dy=0
S4	OCFA	$\overrightarrow{-j}$	$-y^2$	Y=0	dy=0
S5	CFEG	$\vec{k}$	Yz	Z=1	dz=0
S6	OADB	$\overrightarrow{-k}$	-yz	Z=0	dz=0

Evalution of S_1:

$$\iint_{S_1}\vec{F}.\hat{n}ds = \int_0^1 \int_0^1 4xz\,dy\,dz$$

$$=\int_0^1 \int_0^1 4z\, dy\, dz$$

$$= 4\int_0^1 \left(\frac{z^2}{2}\right) dy$$

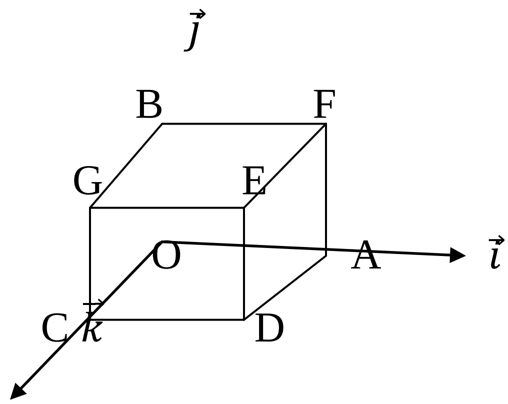
$$= 4\int_0^1 \left(\frac{1}{2} - 0\right) dy$$

$$=4\int_0^1 \frac{1}{2}\, dy$$

$$=2\int_0^1 dy$$

$$=2(y) =2(1-0)$$

$$= 2 \quad \rightarrow (1)$$



Evalution of S_2:

$$\iint_{s2} \vec{F}.\hat{n} ds = \int_0^1 \int_0^1 -4xz\ dy\, dz$$

$$=\int_0^1 \int_0^1 -4(0)(z)\, dy\, dz$$

$$= 0 \rightarrow (2)$$

Evalution of S_3:

$$\iint_{s3} \vec{F}\hat{n}\ ds=\int_0^1 \int_0^1 (-y^2)\, dx\, dz$$

$$= \int_0^1 \int_0^1 -1\, dxdz$$

$$=- \int_0^1 \int_0^1 dx\, dz$$

$$= - \int_0^1 dx\ (z)_0^1 =(x)_0^1\ (1-0)$$

$$= -1 \rightarrow (3)$$

Evalution of S_4:

$$\iint_{s4} \vec{F}.\hat{n} ds = \int_0^1 \int_0^1 (y^2)\, dx\, dz$$

$$=\int_0^1 \int_0^1 0\, dx\, dz$$

$$=0\rightarrow (4)$$

Evalution of S_5:

$$\int_{s5} \overrightarrow{F}.\hat{n} ds = \int_0^1 \int_0^1 yz\ dxdy$$

$$=\int_0^1 y \int_0^1 1\, dxdy$$

$$=\int_0^1 (1)\ \left(\frac{y^2}{2}\right)_0^1 dx$$

$$= \int_0^1 \frac{1}{2} dx = \frac{1}{2}(x)_0^1$$

$$= \frac{1}{2} \rightarrow (5)$$

Evalution of S_6:

$$\iint_{s6} \vec{F}.\hat{n} ds = \int_0^1 \int_0^1 yz \ dx \ dy$$

$$= \int_0^1 y \int_0^1 (0) \ dx \ dy$$

$$= 0 \rightarrow (6)$$

From,(1),(2),(3),(4),(5),(6)

$$\iint_s \vec{F}.\hat{n} ds = \iint_{s1} + \iint_{s2} + \iint_{s3} + \iint_{s4} + \iint_{s5} + \iint_{s6}$$

$$= 2+0-1+0+\frac{1}{2} + 0$$

$$= \frac{3}{2}$$

LHS = RHS

Hence,gauss divergence theorem is verify…..

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=_PNsPatRNVA (Gauss theorem and evaluation of integrals)

Important Books/Journals for further learning including the page nos.:

1. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 582-597**

Topic of Lecture : Green's theorem and evaluation of integrals

Introduction : Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

$$\int (udx + vdy) = \iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \ dx \ dy$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Harmonic function

2.Cauchy integral theorem

Detailed content of the Lecture:

1. Verify Green's theorem in the plane for $\int_C (x^2 - y^2)\boldsymbol{dx} + \boldsymbol{2xydy}$,where C is the curve in the XY plane given by x=0, y=0, x=a, y=b >0)

Solution:

The Green's Theorem is $\int (udx + vdy) = \iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \ dx \ dy$

RHS: $\iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \ dx \ dy$

Here , $u = x^2 - y^2$ v=2xy

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

$$\iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \int_o^b \int_0^a (2y + 2y) dx\, dy$$

$$= \int_o^b \int_0^a (4y) dx\, dy$$

$$= 4\int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= 4\int_0^a \left(\frac{b^2}{2}\right) dx$$

$$= 2b^2 \int_0^a dx$$

$$= 2b^2 (\ x\ )_0^a$$

$$= 2b^2 (a)$$

$$= 2ab^2 \text{………………(A)}$$

To evaluate,

Where C is the Curve in the xy plane place in 4 different segments,

1) Along OA
2) Along AC
3) Along BC
4) Along BO

1) Along OA : y=0;dy=0

$$\int_{OA} (x^2 - y^2)dx + 2xydy = \int_0^a x^2\, dx$$

$$= \left(\frac{x^2}{3}\right)_0^a$$

$$= \frac{a^3}{3} \text{………………………(1)}$$

2) Along AC : x=a; dx=0

$$\int_{AC} (x^2 - y^2)dx + 2xydy = \int_0^b 2ay\, dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b$$

$$= 2a\left(\frac{b^2}{2}\right)$$

$$= ab^2 \text{……………………(2)}$$

3) Along BC : y=b; dy=0

$$\int_{BC} (x^2 - y^2)dx + 2xydy = \int_a^0 (x^2 - b^2)\, dx$$

$$= \left(\frac{x^3}{3} - b^2 x\right)_a^0$$

$$= \left(b^2 a - \frac{a^3}{3}\right) \text{………………(3)}$$

5) Along CO : x=0;dx=0

$$\int_{OC} (x^2 - y^2)dx + 2xydy = \int_b^0 2xy\, dy$$

$$= \int_b^0 0(2y)dy$$

$$= 0 \text{…………………(4)}$$

From(1),(2),(3),(4)

$$\int (udx + vdy) = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2 \text{………………(B)}$$

A = B

LHS = RHS

The Green's therom is Verified .

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=a_zdFvYXX_c(Green's theorem and evaluation of integrals)

Important Books/Journals for further learning including the page nos.:

28. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 575-579**

Topic of Lecture : Green's theorem and evaluation of integrals

Introduction : Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

$$\int (udx + vdy) = \iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dx\, dy$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Harmonic function

2.Cauchy integral theorem

Detailed content of the Lecture:

2. Verify Green's theorem in the plane for $\int (x^2\, dx + xy\, dy)$,where C is the curve in the XY plane given by x=0, y=0, x=a, y=a (a>0)

Solution:

The Green's Theorem is $\int (udx + vdy) = \iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dx\, dy$

RHS: $\iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dx\, dy$

Here , u=x^2 v=xy

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial u} = y$

$$\iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dx\, dy = \int_o^a \int_0^a (y - 0) dx\, dy$$

$$= \int_0^a (\frac{y^2}{2})_0^a\, dx$$

$$= \int_0^a \left(\frac{a^2}{2}\right) dx$$

$$= \frac{a^2}{2} \int_0^a dx$$

$$=\frac{a^2}{2}(x)_0^a$$
$$=\frac{a^2}{2}(a)$$
$$=\frac{a^3}{2}\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

To evaluate,

Where C is the Curve in the xy plane place in 4 different segments,

6) Along OA
7) Along AC
8) Along BC
9) Along BO

1) Along OA : x=a, y=0;dy=0

$$\int(x^2\,dx + xy\,dy) = \int_0^a x^2\,dx$$
$$=\int_0^a a^2\,dx$$
$$=(x^2)_0^a$$
$$=\frac{a^3}{3}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

2) Along AC : x=a; dx=0

$$\int(x^2\,dx + xy\,dy) = \int x^2\,dx + xy\,dy$$
$$=\int_0^a ay\,dy$$
$$=a\left(\frac{y^2}{2}\right)_0^a$$
$$=a\left(\frac{a^2}{2}\right)$$
$$=\frac{a^3}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

3) Along BC : y=0; dy=0

$$\int(x^2\,dx + xy\,dy) = \int_a^0 x^2\,dx$$
$$=\left(\frac{x^3}{2}\right)_a^0$$
$$=\left(0-\frac{a^3}{3}\right)$$
$$=-\frac{a^3}{3}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

10) Along CO : x=0;dx:0

$$\int(x^2\,dx + xy\,dy) = \int_a^0 xy\,dy$$
$$=\int_a^0 0(y)dy$$
$$=0\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\int(x^2\,dx + xy\,dy) = \frac{a^3}{3} + \frac{a^2}{2} - -\frac{a^3}{3} - 0$$
$$=\frac{a^3}{2}\ldots\ldots\ldots\ldots\ldots\ldots(B)$$

A = B

LHS = RHS

The Green’s therom is Verified .

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=a_zdFvYXX_c(Green's theorem and evaluation of integrals)

Important Books/Journals for further learning including the page nos.:

29. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 575-579**

Topic of Lecture : Stoke's theorem and evaluation of integrals

Introduction : The surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the vector function around that surface.

$$\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}\,ds$$

Prerequisite knowledge for Complete understanding and learning of Topic :

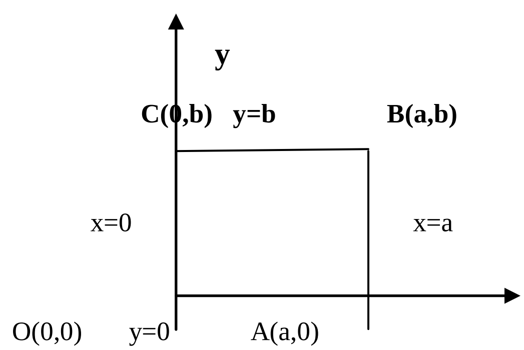
1.Surface integral
2.Line integral

Detailed content of the Lecture:

1. Verify stoke's therorem for the function $\vec{F} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$ integrated round the rectangular in the z=0 plane whose sides are along the lines $x = o, y = 0, x = a, y = b$.

Solution:

Stoke's theorem is $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}\,ds$



L.H.S

Evaluation of $\int_C \vec{F}.\overrightarrow{dr}$

Given $\vec{F} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$

$\overrightarrow{dr} = dx\vec{\imath} + dy\vec{\jmath}$

$\vec{F}.\overrightarrow{dr} = (x^2 - y^2)dx + 2xydy$

$\int_C \vec{F}.\overrightarrow{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Along OA (y=0)

$\int_{OA}\vec{F}.\overrightarrow{dr} = (x^2 - y^2)dx + 2xydy$

$= \int_0^a x^2dx\ [y = 0, dy = 0\ and\ x\ varies\ from\ 0\ to\ a]$

$= \left(\frac{x^3}{3}\right)_0^a$

$= \left(\frac{a^3}{3}\right)$--------------(1)

Along AB (x=a)

$\int_{AB}\vec{F}.\overrightarrow{dr} = \int_{AB}(x^2 - y^2)dx + 2xydy)$

$= \int_0^b \boldsymbol{2ay\, dy}\ [\boldsymbol{x = a, dx = 0\ and\ y\ varies\ from\ 0\ to\ a}]$

$= 2a\left(\frac{y^2}{2}\right)_0^b$

$= ab^2$--------------(2)

Along BC (y=a)

$\int_{BC} \vec{F}.\overrightarrow{dr} = \int_{BC}(\boldsymbol{x^2} - y^2)\boldsymbol{dx} + \boldsymbol{2xydy})$

$= \int_a^0(\boldsymbol{x^2} - b^2)\boldsymbol{dx}\ [\boldsymbol{y = b, dy = 0\ and\ x\ varies\ from\ a\ to\ 0}]$

$= \left(\frac{x^3}{3} - b^2x\right)_a^0$

$= \left(\frac{-a^3}{3}\right) + ab^2$--------------(3)

Along CO (x=0)

$\int_{CO} \vec{F}.\overrightarrow{dr} = \int_{CO}(\boldsymbol{x^2} - y^2)\boldsymbol{dx} + \boldsymbol{2xydy})$

$= \int_b^0 \boldsymbol{a(0)y\, dy}\ [\boldsymbol{x = 0, dx = 0\ and\ y\ varies\ from\ a\ to\ 0}]$

$= 0$--------------(4)

From (1),(2),(3),(4)

$\int_C \vec{F}.\overrightarrow{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} = \frac{a^3}{3} - \frac{a^3}{3} + ab^2 + ab^2$

$= 2ab^2$ ----------(A)

R.H.S

Evaluation of $\iint_S \boldsymbol{curl\vec{F}.\hat{n}\, ds}$

Given $\vec{F} = (\boldsymbol{x^2} - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & xy & 0 \end{vmatrix}$$

$=\vec{i}(0-0)$-$\vec{j}\,(0-0)$+$\vec{k}(y-0)$

$= \vec{k}4y$

Here the surface S the Square OABC and the unit outward normal vector is $\vec{k}$

$\boldsymbol{curl\vec{F}.\hat{n}} = \vec{k}4y.\vec{k} = 4y$

$\therefore \iint_S \boldsymbol{curl\vec{F}.\hat{n}\, ds} = \iint_S \boldsymbol{y\, ds}$

$= 4\iint_{00}^{ab} y\, dx\, dy$

$= 4\int_0^b y(x)_0^a dy$

$= \int_0^a 4ay\, dy$

$= 4a\left(\frac{y^2}{2}\right)_0^b$

$= 2ab^2$----------(B)

Hence Stoke's theorem is verified.

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=kFto6HjIp0Q (Method of variation of parameter)

Important Books/Journals for further learning including the page nos.:

30. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 581-582**

Topic of Lecture : Stoke's theorem and evaluation of integrals

Introduction : The surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the vector function around that surface.

$$\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}\, ds$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Surface integral

2.Line integral

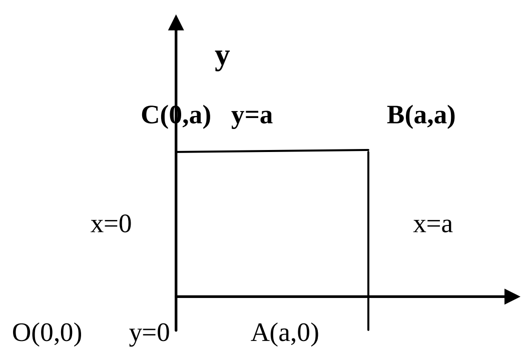
Detailed content of the Lecture:

2. Verify stoke's therorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ integrated round the square in the z=0 plane whose sides are along the lines $x = o, y = 0, x = a, y = a.$

Solution:

Stoke's theorem is $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}\, ds$

L.H.S



Evaluation of $\int_C \vec{F}.\overrightarrow{dr}$

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

$\overrightarrow{dr} = dx\vec{i} + dy\vec{j}$

$\vec{F}.\overrightarrow{dr} = x^2dx + xydy$

$\int_C \vec{F}.\overrightarrow{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Along OA (y=0)

$\int_{OA} \vec{F}.\overrightarrow{dr} = \int_{OA}(x^2dx + xydy)$

$= \int_0^a x^2dx\ [y = 0, dy = 0\ and\ x\ varies\ from\ 0\ to\ a]$

$= \left(\frac{x^3}{3}\right)_0^a$

$= \left(\frac{a^3}{3}\right)$--------------(1)

Along AB (x=a)

$\int_{AB} \vec{F}.\overrightarrow{dr} = \int_{AB}(x^2dx + xydy)$

$= \int_0^a ay\,dy\ [x = a, dx = 0\ and\ y\ varies\ from\ 0\ to\ a]$

$= a\left(\frac{y^2}{2}\right)_0^a$

$= \left(\frac{a^3}{2}\right)$--------------(2)

Along BC (y=a)

$\int_{BC} \vec{F}.\overrightarrow{dr} = \int_{BC}(x^2dx + xydy)$

$= \int_a^0 x^2dx\ [y = a, dy = 0\ and\ x\ varies\ from\ a\ to\ 0]$

$= \left(\frac{x^3}{3}\right)_a^0$

$= \left(\frac{-a^3}{3}\right)$--------------(3)

Along CO (x=0)

$\int_{CO} \vec{F}.\overrightarrow{dr} = \int_{CO}(x^2dx + xydy)$

$= \int_a^0 ay\,dy\ [x = 0, dx = 0\ and\ y\ varies\ from\ a\ to\ 0]$

$= 0$--------------(4)

From (1),(2),(3),(4)

$$\int_C \vec{F}.\overrightarrow{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3}$$

$= \frac{a^3}{2}$ ----------(A)

R.H.S

Evaluation of $\iint_S\ curl\vec{F}.\hat{n}\,ds$

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$=\vec{i}(0-0)-\vec{j}\,(0-0)+\vec{k}(y-0)$

$= \vec{k}y$

Here the surface S the Square OABC and the unit outward normal vector is $\vec{k}$

$curl\vec{F}.\hat{n} = \vec{k}y.\vec{k} = y$

$\therefore \iint_S\ curl\vec{F}.\hat{n}\,ds = \iint_S\ y\,ds$

$= \iint_{00}^{aa} y\,dx\,dy$

$= \int_0^a y(x)_0^a dy$

$= \int_0^a ay\, dy$

$= a\left(\frac{y^2}{2}\right)_0^a$

$= \frac{a^3}{2}$ ----------(B)

Hence Stoke's theorem is verified.

(A) = (B)

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=QS-zUSu-nxA(Stoke's theorem and evaluation of integrals)

Important Books/Journals for further learning including the page nos.:

31. **H.K. Dass, Er. Rajnish Verma, Higher Engineering Mathematics, S. Chand, New Delhi-2011. Page No: 581-582**

Unit-III

Analytic Function

Topic of Lecture : Analytic Functions

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Differentiation
- Integration
- Complex Analysis

Detailed content of the Lecture:

Analytic Function:

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire function:

A function which is analytic everywhere in the finite plane is called an entire function. An entire function is analytic everywhere except at $z = \infty$.

Example: e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$.

Cauchy-Riemann Equation:

Necessary Condition:

The necessary condition for $f(z)=u(x,y)+iv(x,y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Sufficient Condition:

If $f(z)$ to be analytic then the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D.

1. Show that the function $f(z)=\bar{z}$ is nowhere differentiable.

Given $f(z)=\bar{z}=x-iy$

$u=x, v=-y$

$$\frac{\partial u}{\partial x}=1, \frac{\partial v}{\partial x}=0 \qquad \frac{\partial u}{\partial y}=0, \frac{\partial v}{\partial y}=-1$$

$u_x \neq v_y$,C-R equations are not satisfied anywhere. Hence

$f(z)=\bar{z}$ is nowhere differentiable.

2. Find the constants a,b if $f(z)=x+2ay+i(3x+by)$ is analytic

$f(x)=x+2ay+i(3x+by)$

$$u=x+2ay \quad and \quad v=(3x+by)$$

Where $\frac{\partial u}{\partial x}=1, \frac{\partial u}{\partial y}=2a$

$$\frac{\partial v}{\partial x}=3, \quad \frac{\partial v}{\partial y}=b$$

We know that by CR equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$a=\frac{3}{2}, \quad b=1$$

3. If u+iv is analytic, show that v –iu &-v +iu are also analytic

Given u+iv is analytic

C –R equations are satisfied $u_x=v_y$ …….. (1)

$u_y= -v_x$ ……..(2)

Since the derivatives of u & v exist it is therefore continuous

Now to prove v –iu & -v+iu are also analytic, we should prove that

(i) $v_x = -u_y$ & $v_y = u_x$ &

(ii) $v_x = u_v$ & $v_y = u_x$

(iii) u_x, u_y, v_x, v_y are all continuous. Results (i) & (ii) follows from (1) & (2) . Since the derivatives of u &v exists from (1) and (2), the derivatives of u and v should be continuous . Hence the result

Topic of Lecture : Analytic Function--Properties

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

1. **If $f(z) = u + iv$ is a regular function of z in a domain D, then $\nabla^2|f(z)|^2 = 4|f'(z)|^2$**

Proof:

Let $f(z) = u + iv$ be an analytic function.

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$|f(z)|^2 = u^2 + v^2$$

$$\nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$$

$$= \nabla^2(u^2) + \nabla^2(v^2)$$

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = \left(\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}\right)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$$(2) \Rightarrow \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\because u \text{ is harmonic}]$$

Similarly,

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

$$\nabla^2(v^2) = 2v_x^2 + 2v_y^2$$

(1)$\Rightarrow$ $\nabla^2|f(z)|^2 = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$

$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2]$

$= 4[u_x^2 + v_x^2]$

i.e., $\nabla^2|f(z)|^2 = 4|f'(z)|^2$

Topic of Lecture : Analytic Function--Properties

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

2. **Prove that the real and imaginary parts of an analytic function are harmonic function.**

Proof:

Let $f(z) = u + iv$ **be an analytic function.**

$\Rightarrow u_x = v_y$**……(1)** **and** $u_y = -v_x$ **……….(2)** **by C-R Equation.**

Differentiate (1 & 2) p.w.r. to x**, we get**

$u_{xx} = v_{xy}$ **……………(3)** **and** $u_{xy} = -v_{xx}$ **…………(4)**

Differentiate (1 & 2) p.w.r. to y**, we get**

$u_{yx} = v_{yy}$ **……………(5)** **and** $u_{yy} = -v_{yx}$ **…………(6)**

(3)+(6) $\Rightarrow u_{xx} + u_{yy} = 0$ $[\because v_{xy} = v_{yx}]$

(5)-(4) $\Rightarrow v_{xx} + v_{yy} = 0$ $[\because u_{xy} = u_{yx}]$

$\therefore u$ **and** v **satisfy the Laplace equation.**

3. When the function $f(z) = u + iv$ is analytic, show that u =constant and v = constant are orthogonal.

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y$**......(1)** **and** $u_y = -v_x$ **...........(2)** **by C-R Equation.**

Given: $u = a$ **and** $v = b$

Differentiate p.w.r. to x, we get

$u_x + u_y \frac{dy}{dx} = 0$ **and** $v_x + v_y \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y}$ $\Rightarrow \frac{dy}{dx} = \frac{-v_x}{v_y}$

$\Rightarrow m_1 = \frac{-u_x}{u_y}$ $\Rightarrow m_2 = \frac{-v_x}{v_y}$

$m_1 . m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{v_x}{v_y}\right) = -1$ **by (1) and (2)**

Hence, the family of curves form an orthogonal system.

Topic of Lecture : Analytic Function

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Exact differential equation
- Milne-Thomson method
- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

Find the regular function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$.

Solution:

Given: $v = e^{-x}(x\cos y + y\sin y)$

$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$

$v_x(z,0) = e^{-z} + z(-e^{-z})$

$=(1-z)\, e^{-z}$

i.e., $$v_x(z,0) = (1-z)\, e^{-z}$$

$v_y = e^{-x}[-x\sin y + (y\cos y + \sin y(1))]$

$v_y(z,0) = e^{-z}[0+0+0] = 0$

i.e., $$v_y(z,0) = 0$$

Let $w = f(z) = u + iv$

$f'(z) = u_x + iv_x$

$=v_y + iv_x$ **[by C.R.,** $u_x = v_y,\ \ u_y = -v_x$**]**

$f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$

[by Milne-Thomson rule]

Where C is a complex constant.

$f(z) = \int 0\, dz + i\int (1-z)\, e^{-z}\, dz + C$

$= i\int (1-z)\, e^{-z}\, dz + C$

$=i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$

$=i[-(1-z)e^{-z} + e^{-z}] + C$

$=ize^{-z} + C$

Topic of Lecture : Analytic Function-Milne Thomson Method

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Exact differential equation
- Milne-Thomson method
- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y-\cos 2x}$ **.**

Solution:

Given: $u=\frac{\sin 2x}{\cosh 2y-\cos 2x}$ **………..(A)**

Differentiate (A) p.w.r. to x**, we get**

$$u_x=\frac{(\cosh 2y-\cos 2x)(2\cos 2x)-\sin 2x\,(2\sin 2x)}{(\cosh 2y-\cos 2x)^2}$$

$$u_x(z,0)=\frac{2\cos 2z(1-\cos 2z)-2sin^2 2z}{(1-\cos 2z)^2}$$

$$=\frac{2\cos 2z-2cos^2 2z-2sin^2 2z}{(1-\cos 2z)^2}$$

$$=\frac{2\cos 2z-2}{(1-\cos 2z)^2}=\frac{-2}{1-\cos 2z}$$

$$=\frac{-2}{2sin^2 z}=-cosec^2 z$$

$$u_x(z,0)=-cosec^2 z \text{ ……………………..(1)}$$

$$u_{y}=\frac{(\cosh 2y-\cos 2x)(0)-\sin 2x\,(2\sinh 2y)}{(\cosh 2y-\cos 2x)^2}$$

$$\boxed{u_y(z,0)=0}$$

Let $w=f(z)=u+iv$

$f^{'}(z)=u_x+iv_x$

$=u_x-iu_y$ **[by C.R.,** $u_x=v_y,\ \ u_y=-v_x$**]**

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$$

[by Milne-Thomson rule]

Where C is a complex constant.

$$f(z) = \int -cosec^2 z\, dz - i\int 0\, dz + C$$

$$= \cot z + C$$

Topic of Lecture : Construction of Analytic Function

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Exact differential equation
- Milne-Thomson method
- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Solution:

Given: $u - v = e^x(\cos y - \sin y)$ **………..(A)**

Differentiate (A) p.w.r. to x, we get

$u_x - v_x = e^x(\cos y - \sin y)$

$u_x(z,0) - v_x(z,0) = e^z$ **…………………..(1)**

Differentiate (A) p.w.r. to y, we get

$u_y - v_y = e^x(-\sin y - \cos y)$

$u_y(z,0) - v_y(z,0) = -e^z$

$-v_x(z,0) - u_x(z,0) = -e^z$ **……………………(2) [by C-R Equation]**

(1)+(2) $\Rightarrow -2v_x(z,0) = e^z - e^z = 0$

$\Rightarrow$ $\boxed{v_x(z,0) = 0}$

$\Longrightarrow$ $u_x(z,0) = e^z$

Let $w = f(z) = u + iv$

$f'(z) = u_x + iv_x$

$f(z) = \int u_x(z,0)dz + i\int v_x(z,0)dz + C$

[by Milne-Thomson rule]

Where C is a complex constant.

$$f(z) = \int e^z \, dz + i0 + C$$

$= e^z + C$

Topic of Lecture : Construction of Analytic Function

Introduction: A complex function is said to be analytic on a region R if it is complex differentiable at every point in R. The terms holomorphic function, differentiable function, and complex differentiable function are sometimes used interchangeably with "analytic function".

Prerequisite knowledge for Complete understanding and learning of Topic :

- Exact differential equation
- Milne-Thomson method
- Cauchy Riemann equation
- Partial derivative

Detailed content of the Lecture:

Find the analytic function for which $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ **is the real part. Hence determine the analytic function** $u + iv$ **for which** $u + v$ **is the above function.**

Solution:

Given: $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$ **………..(A)**

Differentiate (A) p.w.r. to x**, we get**

$$u_x + v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_x - u_y = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

[by C-R Equation]

$$u_x(z,0) - u_y(z,0) = \frac{2\cos 2z(1 - \cos 2x) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$=\frac{2\cos 2z-2cos^2 2z-2sin^2 2z}{(1-\cos 2z)^2}$$

$$=\frac{2\cos 2z-2}{(1-\cos 2z)^2}=\frac{-2}{1-\cos 2z}$$

$$=\frac{-2}{2sin^2 z}=-cosec^2 z$$

$$u_x(z,0)-u_y(z,0)=-cosec^2 z \text{ ....(1)}$$

Differentiate (A) p.w.r.to y, we get

$$u_y+v_y=\frac{0-\sin 2x(\sinh 2y)}{(\cosh 2y-\cos 2x)^2} \quad (2)$$

$$u_y+u_x=\frac{0-\sin 2x(\sinh 2y)}{(\cosh 2y-\cos 2x)^2} \quad (2)$$

[by C-R Equation]

$u_y(z,0)+u_x(z,0)=0$ **......................(2)**

(1)+(2)$\Longrightarrow 2u_x(z,0)=-cosec^2 z$

$$\boxed{u_x(z,0)=-\frac{1}{2}cosec^2 z}$$

(2)$\Longrightarrow$

$$\boxed{u_y(z,0)=\frac{1}{2}cosec^2 z}$$

Let $w=f(z)=u+iv$

$f'(z)=u_x+iv_x$

$=u_x-iu_y$ **[by C.R.,** $u_x=v_y,\ u_y=-v_x$**]**

$$f(z)=\int u_x(z,0)dz-i\int u_y(z,0)dz+C$$

[by Milne-Thomson rule]

Where C is a complex constant.

$$f(z)=\int -\frac{1}{2}cosec^2 z\,dz-i\int\frac{1}{2}cosec^2 z\,dz+C$$

$$=\frac{1}{2}\cot z+i\frac{1}{2}\cot z+C$$

$$=\frac{1+i}{2}\cot z+C$$

Topic of Lecture : Conformal Mapping

Introduction: In mathematics, a conformal map is a function that locally preserves angles, but not necessarily lengths. Conformal map preserves both angles and shape of infinitesimal small figures but not necessarily their size. More formally, a map $w = f(z)$ is called conformal (or angle-preserving) at z_0 if it preserves oriented angles between curves through z_0, as well as their orientation, i.e. direction.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Transformation
- Analytic Geomentry
- Complex function

Detailed content of the Lecture:

Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$

Solution: **Given $|z - 2i| = 2$…….(1) is a circle.**

Centre=$2i$ i.e., $(0, 2)$
Radius =2

Given: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$(1) \Rightarrow |\frac{1}{w} - 2i| = 2$$

$$\Rightarrow |1 - 2wi| = 2|w|$$

$$\Rightarrow |1 - 2ui + 2v| = 2|u + iv|$$

$$\Rightarrow |1 + 2v - 2ui| = 2|u + iv|$$

$$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

$$\Rightarrow (1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = -\frac{1}{4}$$

which is a straight line in w-plane.

Topic of Lecture : Conformal Mapping

Introduction: In mathematics, a conformal map is a function that locally preserves angles, but not necessarily lengths. Conformal map preserves both angles and shape of infinitesimal small figures but not necessarily their size. More formally, a map $w = f(z)$ is called conformal (or angle-preserving) at z_0 if it preserves oriented angles between curves through z_0, as well as their orientation, i.e. direction.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Transformation
- Analytic Geomentry
- Complex function

Detailed content of the Lecture:

Find the image of the infinite strips $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution: $w = \frac{1}{z}$ **(given)**

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

i.e., $x = \frac{u}{u^2+v^2}$…………(1) $\qquad y = \frac{-v}{u^2+v^2}$…………(2)

Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$\frac{1}{4} = \frac{u}{u^2+v^2}$ **by (2)**

$u^2 + v^2 = -4v$

$u^2 + v^2 + 4v = 0$

$u^2 + (v+2)^2 - 4 = 0$

$u^2 + (v+2)^2 = 4$ ……………………..(3)

which is a circle whose centre is at $(0, -2)$ in the w-plane and radius is 2.

when $y = \frac{1}{2}$

$\frac{1}{2} = \frac{-v}{u^2+v^2}$ **by (2)**

$u^2 + v^2 = -2v$

$u^2 + v^2 + 2v = 0$

$u^2 + (v+1)^2 - 1 = 0$

$u^2 + (v+1)^2 = 1$ ……………………..(3)

which is a circle whose centre is at $(0, -1)$ in the w-plane and unit radius.

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w-plane.

Topic of Lecture : Bilinear Transformation

Introduction: The bilinear transform is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice versa. The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Prerequisite knowledge for Complete understanding and learning of Topic :

- Complex analysis
- Analytic Geomentry

Detailed content of the Lecture:

Find the Bilinear transformation that maps the points z=$\mathbf{0}, \mathbf{-1}, \boldsymbol{i}$ onto the points w= $\boldsymbol{i}, \mathbf{0}, \infty$ respectively

Given: $z_1 = 0,\ z_2 = -1, z_3 = i \qquad w_1 = i,\ w_2 = 0,\ w_3 = \infty$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)}{(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\frac{(w-i)}{(-i)} = \frac{z}{(z-i)}(1+i)$$

$$w - i \ = \frac{z}{(z-i)}(-i+1)$$

$$w \ = \frac{z}{(z-i)}(-i+1) + i$$

$$w = \frac{-iz + z + iz + 1}{(z - i)}$$

$$w = \frac{z + 1}{(z - i)}$$

Topic of Lecture : Bilinear Transformation

Introduction: The bilinear transform is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice versa. The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Prerequisite knowledge for Complete understanding and learning of Topic :

- Complex analysis
- Analytic Geomentry

Detailed content of the Lecture:

Find the bilinear transformation which maps the points $1, i, 1$ onto the points $0, 1, \infty$, show that the transformation maps the interior unit circle of the z-plane onto the upper half of the w-plane.

Given: $z_1 = 1,\ z_2 = i, z_3 = -1 \qquad w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

Let the transformation be

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\frac{(w - w_1)}{(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w - 0)}{(1 - 0)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$w = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$= \frac{(z - 1)}{(z + 1)}[-i]$$

$$w = \frac{(-i)z + i}{(1)z + 1}$$

To find z

$$wz + w = -iz + i$$
$$wz + iz = -w + i$$
$$z[w + i] = -w + i$$
$$z = -\frac{(w - i)}{(w + i)}$$

To Prove: $|z| < 1$ maps $v > 0$

Proof: $|z| < 1$

$$\left|\frac{-(w - i)}{w + i}\right| < 1$$

$$\left|\frac{w - i}{w + i}\right| < 1$$
$$|w - i| < |w + i|$$
$$|u + iv - i| < |u + iv + i|$$
$$|u + i(v - 1)| < |u + i(v + 1)|$$
$$u^2 + (v - 1)^2 < u^2 + (v + 1)^2$$
$$(v - 1)^2 < (v + 1)^2$$
$$v^2 - 2v + 1 < v^2 + 2v + 1$$
$$-4v < 0$$
$$v > 0$$

Unit – IV

Complex integration

Topic of Lecture : Complex integration.

Introduction : Contour in the complete plane is just a curve,finite (or)infinite while has an arrow or orientation.It is a method of evaluating certain integrals alongpath (C) in the complete plane and is denoted by $\oint f(z)dz$.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Complex line integral
2. Real line integrals

Detailed content of the Lecture:

Simply connected region :

A region which has no holes is called simply connected region.

Multiple Connected Region :

A region which is not simply connected is called Multiple Connected Region.

Cauchy's Integral Theorem

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C.

$\oint f(z)dz = 0$

Cauchy's Integral Formula

$f(z)$ is analytic function, C be a closed curve, a is any point within C. $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a} dz$

Evaluate the integral $\oint \frac{z+4}{z^2+2z+5}\, dz$ where C is a circle $|z+1-i| = 2$ and $|z+1+i| = 2$

Solution

Given $\oint \frac{z+4}{z^2+2z+5}\, dz$

$z^2 + 2z + 5 = 0$, $\quad z = -2 \pm \sqrt{\frac{4-4(5)}{2}}$

$= -2 \pm \sqrt{\frac{-16}{2}}$

$= \frac{-2\pm 4i}{2} =$ -1$\pm 2i$, a= $-1+2i, -1-2i$

$|z+1-i| = 2$, $|x+iy+1-i| = 2$

$(x+1)^2 + (y-1)^2 = 2^2$

It is a circle of radius 2 and centre (-1,1)

a= $-1+2i$ lies inside the circle a= $-1-2i$ lies outside the circle

$\oint \frac{z+4}{z^2+2z+5}\, dz = \oint \frac{z+4}{[z-(-1+2i)]\,[z-(-1-2i)]} dz$

$f(z) = \frac{z+4}{z-(-1-2i)}$, $\quad f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

$\oint \frac{z+4}{z^2+2z+5}\, dz = 2\pi i f(-1+2i)$

$=2\pi i \left[\frac{3+2i}{4i}\right]$

$= \frac{\pi}{2}[3+2i]$

$\oint \frac{z+4}{z^2+2z+5}\, dz = \frac{\pi}{2}[3+2i]$

(ii) $|z+1+i| = 2$, $|x+iy+1-i| = 2$

$(x+1)^2 + (y+1)^2 = 2^2$

It is a circle of radius 2 and centre (-1,1)

a = $-1+2i$ lies inside the circle a= $-1-2i$ lies outside the circle

$\oint \frac{z+4}{z^2+2z+5}\, dz = \oint \frac{z+4}{[z-(-1+2i)]\,[z-(-1-2i)]} dz$

$f(z) = \frac{z+4}{z-(-1+2i)}$, $\quad f(-1-2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$

$\oint \frac{z+4}{z^2+2z+5}\, dz = 2\pi i f(-1-2i)$

$= 2\pi i \left[\frac{3-2i}{-4i}\right]$

$= \frac{-\pi}{2}[3-2i]$

$\oint \frac{z+4}{z^2+2z+5}\, dz =- \frac{\pi}{2}[3-2i]$

Topic of Lecture :Cauchy's integral formula

Introduction : If $f(z)$ is analytic in athe doubly connected region bounded by the curves c_1 and c_2 and then $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Complex line integral
2. Real line integrals

Detailed content of the Lecture:

Cauchy's Integral Formula

$f(z)$ is analytic function, C be a closed curve, a is any point within C. $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$

1. Evaluate $\int_C \frac{e^{-z}}{z^2}dz$ **, where C is a circle** $|z| = 1$.

Solution :

We know that $\int_c \frac{f(z)}{z-a}dz = 2\pi i f(a)$, $\int_c \frac{e^{-z}dz}{z^2} = \int_c \frac{e^{-z}}{(z-0)^2}$

Here $f(z) = e^{-z}$, a= 0 is lies inside $|z| = 1$

By cauchy's integral formula we get , $\int_C \frac{e^{-z}}{z^2}dz = 2\pi i f'(a) = 2\pi i(-1) = -2\pi i$

2. **Evaluate** $\int_C \frac{z^2+1}{z^2-1}dz$ **where C is a circle of unit radius and centre at z=i.**

Solution :

$|z-i| = 1$

The poles z=1,z=-1 lies outside the circle

$\therefore \frac{z^2+1}{z^2-1}$ is analytic inside $|z-i| = 1$

By Cauchy's theorem, $\int_C \frac{z^2+1}{z^2-1}dz$ **=0**

3. Evaluate $\int_z \sec z dz$ **where c is the unit circle** $|z| = 1$

Solution :

$$\int_z \sec z dz \quad = \int_z \frac{1}{\cos z} dz$$

The pole are given by the solution of cos z = 0

$$i.e., z = (2n+1)\frac{\pi}{2}, n = 0,1,2,.......$$

$$z = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2,}.........$$

Hence all the poles lies outside $|z| = 1$, secz is analytic with $|z| = 1$

By Cauchy's theorem $\int_z \sec z dz = 0$

Evaluate $\oint \frac{zdz}{(z-1)(z-2)^2}$ Where C is a $\left|z - 2 = \frac{1}{2}\right|$ by using Cauchy's Integral Formula

<u>Solution :</u>

$$f(z) = \frac{z}{(z-1)(z-2)^2}$$

Here z = 1 is a pole of order1 (outside $\left|z - 2 = \frac{1}{2}\right|$)

z = 2 is a pole of order1 (inside $\left|z - 2 = \frac{1}{2}\right|$)

$$\frac{z}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{c}{(z-2)^2}$$

$$z = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

$$put\ z = 1, \qquad A = 1$$

$$put\ z = 2, \qquad C = 2$$

Equatin the coefficient of z^2 , $A + B = 0$

$$B = -1$$

$$\oint \frac{zdz}{(z-1)(z-2)^2} = \oint \frac{1}{z-1} dz - \oint \frac{1}{z-2} dz + \oint \frac{2}{(z-2)^2} dz$$

$$= 0 - 2\pi i f(2) + 2 \times 2\pi i f'(2)$$

$$= -2\pi i + 0 \qquad [\ f'(2) = 0 \quad \therefore f(z) = 1]$$

$$\oint \frac{zdz}{(z-1)(z-2)^2} = -2\pi i$$

Topic of Lecture : Taylor Series

Introduction : It is necessary to expand a function $f(z)$ which fails to be analytic.In such cases it is not possible to apply Taylor's series.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Power series
2. Maclaurin's series
3. Laurent's series

Detailed content of the Lecture:

Power series :

A series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots$ is called power series of $(z-a)$, where z is complex variable and $a_0\ a_1\ a_2 \ldots\ldots a_n$ are complex constants and the constant a is called the centre of the series.

Taylor Series

A function $f(z)$ analytic inside a circle C with centre at a can be expanded in the series

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

Maclaurin's series

$f'(0)$ the first derivative of given function at $z = 0$.

$f''(0)$ the second derivative of given function at $z = 0$.

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

4. Find the Taylor series of the function f(z)=sinz about z=π/4

$$f(z) = \sin z$$
$$f'(z) = \cos z$$
$$f''(z) = -\sin z$$
$$f'''(z) = -\cos z$$

Here $a = \frac{\pi}{4}$, $f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}, \quad f''(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$f'''(\frac{\pi}{4}) = -\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Taylor's series is $f(z) = f(\frac{\pi}{4}) + \frac{z-\frac{\pi}{4}}{1!} f'(\frac{\pi}{4}) + \frac{(z-\frac{\pi}{4})^2}{2!} f''(\frac{\pi}{4}) + \ldots\ldots\ldots$

5. Obtain Taylor's series for $f(z) = \cos z$ about $z = \frac{\pi}{3}$,z=0

Solution ;

Given $f(z) = \cos z$

$f(z) = \cos z$ $\qquad f\left(\frac{\pi}{3}\right) = cos\frac{\pi}{3} = \frac{1}{2}$

$f'(z) = -sinz$ $\qquad f'\left(\frac{\pi}{3}\right) = -sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f''(z) = -cosz$ $\qquad f''\left(\frac{\pi}{3}\right) = -cos\frac{\pi}{3} = \frac{-1}{2}$

$f'''(z) = -cosz$ $\qquad f'''\left(\frac{\pi}{3}\right) = sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Using Taylor's series

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

At $z = \frac{\pi}{3}$

$$f(z) = f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(z-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(z-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(z-\frac{\pi}{3}\right)^3 + \cdots + \infty$$

$$cosz = \frac{1}{2} + \frac{\frac{-\sqrt{3}}{2}}{1!}\left(z-\frac{\pi}{3}\right) + \frac{\frac{-1}{2}}{2!}\left(z-\frac{\pi}{3}\right)^2 + \frac{\frac{-\sqrt{3}}{2}}{3!}\left(z-\frac{\pi}{3}\right)^3 + \cdots + \infty$$

$$cosz = \frac{1}{2}\left(1-\frac{\sqrt{3}}{1!}\right)\left(z-\frac{\pi}{3}\right) + \frac{-1}{2!}\left(z-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{3!}\left(z-\frac{\pi}{3}\right)^3 + \cdots + \infty$$

Topic of Lecture : Laurent's Series and singularities

Introduction : If c_1 *and* c_2 are two concentric circles with centre 'a' and radii r_1 *and* r_2 and if $f(z)$ is analytic on c_1 *and* c_2 and throughout the annular region R between them ,then at each point z in R between them,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-a)^n}$$

$$where\ a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{1+n}} dz\ , b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

An analytic function $f(z)$ is said to have a zero of order m if $f(z)$ is expressed as $f(z) = (z-a)^m\ \varphi(z)$ Where $\varphi(z)$ is analytic and $\varphi(a) \neq 0. When\ m = 1$,a is called simple zero.

Prerequisite knowledge for Complete understanding and learning of Topic :

3. Power series
4. Maclaurin's series
5. Isolated singularities
6. Removable singularities

Detailed content of the Lecture:

1. **Find the Laurent's series for the function f(z)=z^2 $e^{1/z}$ about z=0**

Solution :

$$z^2 e\frac{1}{z} = z^2\left[1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}.........\right]$$

$$= z^2 + z + \frac{1}{2} + ...$$

Removable singularity :

A point $z = a$ is called a removable singularity of $f(z) if$

(i) $z = a\ is\ singular$ point

(ii) $\lim_{z\to a} f(z)$ exists

1. Identify the types of singularities of the following function $f(z) = e^{\frac{1}{(z-1)}}$

Solution :

Here z=1 is a singular point

At z=1 , we get $f(z) = e^{\frac{1}{0}} = \infty$ which is not defined.

Also z=1 is not a pole or removable singularity

Z=1 is an essential singularity.

2. Discuss the nature of the singularities of the function f(z)= $\frac{\sin z}{z}$

Solution :

Poles of f(z) are obtained by equating the denominator to zero

i.e $f(z) = \frac{\sin z}{z}$

z=o is a pole of order 1

sinz=0

z=nπ where n= 0, $n = 0,\pm1,\pm2...$

Obtain the Laurent's series expansion of $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ **in (i) |z|>3 (ii)** $2 < |z| < 3$

Solution :

Given $$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$= \frac{z^2-1}{z^2+5z+6} = 1+\frac{-5z-7}{(z+2)(z+3)}$$

consider $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3} =$

$-5z - 3 = A(z+3) + B(z+2)$

Put $z = -3\ weget\ B = -8$

put $z = -2\ weget\ A = 3$

$\frac{-5z-7}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

(i) Given region $|z| > 3$, $|z| > 3 = |z| > 2$, $\left|\frac{2}{z}\right| < 1, \left|\frac{3}{z}\right| < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{z}\left[1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3.2^n}{z^{n+1}} - \frac{8.3^n}{z^{n+1}}\right]$$

$$f(z) = 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3.2^n - 8.3^n}{z^{n+1}}\right]$$

(ii) Given region $2 < |z| > 3$, $2 < |z|$ and $|z| < 3$, $\left|\frac{2}{z}\right| < 1, \left|\frac{3}{z}\right| < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$f(z) = 1 + \sum_{n=0}^{\infty}(-1)^n\left[\frac{3.2^n}{z^{n+1}} - \frac{8.(z)^n}{3^{n+1}}\right]$$

Topic of Lecture : Cauchy's residue theorem

Introduction : If $f(z)$ be analytic at all points inside and on a simple closed curve C .

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} R_i$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. **Derivatives**
2. **Analytic function**
3. **Harmonic function**

Detailed content of the Lecture:

Find the residue of $\frac{1-e^{2z}}{z^4}$ **at z=0**

Solution :

Given $f(z)=\frac{1-e^{2z}}{z^4}$

Here z = 0 is a pole of order 4

$$\operatorname{Re}s(z=0)=\frac{1}{3!}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[(z-0)^4\frac{1-e^{2z}}{z^4}\right]$$

$$=\frac{1}{6}\underset{z\to 0}{Lt}\frac{d^3}{dz^3}\left[1-e^{2z}\right]=-\frac{4}{3}$$

Find the resuidue of cot z at the pole z=0.

Solution :

f(z)=cotz= $\frac{\cos z}{\sin z}$ Poles of f(z) are sinz = 0 = sin n π

z = n π , where n = 0,±1, ±2,….

$[\text{Resf(z)}]_{z=n\pi}$ = $\lim_{z\to n\pi}(z-n\pi)\frac{\cos z}{\sin z}$ = $\lim_{z\to n\pi}\frac{-(z-n\pi)\sin z+\cos z(1)}{\cos z}$ (by L' Hospital rule)

$[\text{Resf(z)}]_{z=n\pi}$ = 1

Evaluate $\int_c \frac{e^z}{z-1}\,dz$ **if c is |z|=2**

Solution :

Z=1 is a pole of order 1 which lies inside |z|=2

$$\int_c \frac{e^z}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i e$$

Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ ***where c is the circle*** $|z-i|=2$.

Solution :

Given f(z) = $\frac{z-1}{(z+1)^2(z-2)}$

Res [f(z)]$_{z=2}$ = 0 [z= 2 $\rightarrow$ *outside*]

Res [f(z)]$_{z=-1}$ = $\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left[(z+1)^{2\,f(z)}\right]$

= $\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right]$

$= \lim_{z\to -1} \frac{d}{dz}[\frac{z-1}{z-2}]$
$= \lim_{z\to -1}[\frac{(z-1)-(z-1)}{(z-2)^2}]$
$= \lim_{z\to -1}[\frac{-2+1}{(z-2)^2}]$
$= \lim_{z\to -1}[\frac{-1}{(z-2)^2}]$
$= \frac{-1}{(-1-2)^2}$
$= \frac{-1}{9}$

By cauchy's residue theorem ,

$$\int_c \frac{z-1}{(z+1)^2(z-2)} dz = 2\pi i[sum\ of\ the\ residues]$$
$$= 2\pi i\ [0 - \frac{1}{9}]$$
$$= \frac{-2\pi i}{9}$$

Topic of Lecture : Cauchy's residue theorem

Introduction : If $f(z)$ be analytic at all points inside and on a simple closed curve C .

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} R_i$$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Derivatives
2. Analytic function
3. Harmonic function
4. Cauchy Riemann equation

Detailed content of the Lecture:

Evaluate $\int_c \frac{e^z}{z-1} dz$ **if c is |z|=2**

<u>Solution :</u>

Z=1 is a pole of order 1 which lies inside |z|=2

$$\int_c \frac{e^z}{z-1} dz = 2\pi i f(1)$$
$$= 2\pi i e$$

Evaluate $\int_c \frac{\sin \pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} dz$ **where C is |z|=3 Using Cauchy's Residue theorem**

<u>Solution :</u>

Given $\int_c \frac{\sin \pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} dz$

Given circle is $x^2 + y^2 = 9$

$z = 1$ *is a pole order* 2 *lies inside C*

$z = 1$ *is a pole order* 2 *lies inside C*

Residue for pole of order m at $z = a$

$$Res[f(z), a] = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Residue at z = 1

$$Res[f(z), 1] = \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d}{dz} (z-a)^2 \int_c \frac{\sin \pi z^2 + cos\pi\, z^2}{(z-1)^2(z-2)} dz$$

$$= \lim_{z\to 1} \frac{d}{dz} \int_c \frac{\sin \pi z^2 + cos\pi z^2}{(z-2)} dz$$

$$= \lim_{z\to 1} \frac{(z-2)\left(2\pi z cos\pi z^2 - 2\pi z sin\pi z^2\right) - (sin\pi z^2 - cos\pi z^2)}{(z-2)^2}$$

$= (-1)(2\pi cos\pi - 2\pi sin\pi) - (sin\pi + cos\pi)$

$= (-1)(-2\pi) - (-1) = 2\pi + 1$

$Res[f(z), 1] = 2\pi + 1$

Residue for the simple pole at $z = a$

$$Res[f(z), a] = \lim_{z\to a}(z-a)f(z)$$

Residue at z = 2

$$Res[f(z), 2] = \lim_{z\to 2}(z-2)\frac{\sin \pi z^2 + cos\pi\, z^2}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 2}\frac{\sin \pi z^2 + cos\pi\, z^2}{(z-1)^2}$$

$$= \frac{sin\, 4\pi + cos 4\pi}{1}$$

$Res[f(z), 2] = 1$

Using Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} R_i = 2\pi i[2\pi + 1 + 1]$$

$$\int_c \frac{\sin \pi z^2 + cos\pi\, z^2}{(z-1)^2(z-2)} dz = 2\pi i[2\pi + 1]$$

Topic of Lecture : Contour integration

Introduction : In this topic ,we shall use Caushy 's residue theorem to evaluate real definite integrals of complex functions over suitable contours or closed paths.This process of evaluation of definite integrals is called contour integrals.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Integral around the unit circle
2. Integral around the semi circular contour

Detailed content of the Lecture:

Cauchy's lemma : 1

If f(z) be a uniformly continuous function such that $|(z-a)f(z)| \to 0$ as $|(z-a))| \to 0$ then $\int_c f(z)dz \to 0$ where c is a circle $|(z-a)| = r$

Using contour integration on unit circle , evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

Solution :

Given $\int_0^{2\pi} \frac{d\theta}{5+4cos\theta}$

Let $z = e^{i\theta}$ $\quad dz = ie^{i\theta}\,d\theta,\ d\theta = \frac{dz}{iz}$

$cos\theta = \frac{1}{2}\left[\frac{z^2+1}{z}\right] \quad sin\theta = \frac{1}{2i}\left[\frac{z^2-1}{z}\right]$

$$\int_0^{2\pi} \frac{d\theta}{5+4cos\theta} = \frac{1}{2}\oint \frac{1}{5+\frac{4}{2}\left[\frac{z^2+1}{z}\right]}\frac{dz}{iz}$$

$$= \oint \frac{z}{5z+2z^2+2}\frac{dz}{iz} \qquad where\ C\ is\ |z=1|$$

$$= \frac{1}{i}\oint \frac{1}{2z^2+5z+2}dz = \frac{1}{i}\oint f(z)dz \quad \to (1)$$

$$f(z) = \frac{1}{2z^2+5z+2} = \frac{1}{2z(z+2)+(z+2)}$$

$$= \frac{1}{(z+2)(2z+1)}$$

$z = -2\ is\ a$ simple pole lies tside the circle

$z = -\frac{1}{2}\ is\ a$ simple pole lies outside the circle

Residue for the simple pole at $z = a$

$$Res[f(z), a] = \lim_{z\to a}(z-a)f(z)$$

$$Res\left[f(z), -\frac{1}{2}\right] = \lim_{z\to\left(-\frac{1}{2}\right)}\left(z+\frac{1}{2}\right)\left[\frac{1}{(z+2)(2z+1)}\right]$$

$$= \lim_{z\to\left(-\frac{1}{2}\right)}\left(\frac{2z+1}{2}\right)\left[\frac{1}{(z+2)(2z+1)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{3}{2}}\right] \quad = \frac{1}{2}\left[\frac{2}{3}\right]$$

$$Res\left[f(z), -\frac{1}{2}\right] = \frac{1}{3}$$

Using Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i\sum_{i=1}^{n} R_i = 2\pi i\left(\frac{1}{3}\right)$$

$$= \frac{2}{3}\pi i \qquad \to (2)$$

Using $(2)\ in\ (1)$

$$\int_0^{2\pi} \frac{d\theta}{5+4cos\theta} = \frac{1}{i}\left(\frac{2}{3}\pi i\right)$$

$$\int_0^{2\pi} \frac{d\theta}{5+4cos\theta} = \frac{2}{3}\pi$$

Topic of Lecture : Contour integration

Introduction : If $f(x) = \frac{P(x)}{Q(x)}$.Consider $I = \int_{-\infty}^{\infty} f(x)dx \quad where\ f(x) = \frac{p(x)}{q(x)}$ Where $P(x), Q(x)$ are polynomial,degree of $Q(x) > 1 + P(x)$

Then $\oint f(z)dz = \int_{-\infty}^{\infty} f(x)dx$ *where C is the closed contour integrals* consisting of semi circle γ *of radius R*,large enough to include all the polynomial $f(z)$ and the real axis from –R to R

$\therefore \int_{-\infty}^{\infty} f(x)dx = \oint f(z)dz$

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Integral around the semi circular contour
2. Cauchy's residue theorem

Detailed content of the Lecture:

Cauchy'slemma : II

If f(z) be a uniformly continuous function such that $|(z-a)f(z)| \to 0$ as $|(z-a))| \to \infty$ then $\int_c f(z)dz \to 0 as\ R \to \infty$ where c is a circle $|(z-a)| = R$

Jorden lemma :

If $f(z)$ be a uniformly continuous functions suchthat $|f(z)| \to 0$ as $|(z)| \to \infty\ then \int_c e^{imz} \int_c f(z)dz \to 0\ as\ R \to \infty$ where c is the semicircle $|(z)| = R$ above the real axis and m>0

Evaluate $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$

Solution:

Given $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$,

$f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)}$

Consider $\int_c f(z)dz = \int_c \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$ where C is upper half semi circle r bounded by diameter [-R,R]

$$\int_{-R}^{R} f(x)dx + \int_r f(z)dz = \int_c f(z)dz \to (1)$$

$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \frac{z^2}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

The poles z = ai , z = bi are simple poles which lies inside r

The poles z = -ai ,z = -bi simple poles which lies outside r

Residue of simple pole is Res [f(z) ,a] = $\lim_{z\to a}(z-a)f(z)$

Res[f(z), ai] $= \lim_{z\to ai}(z-ai)f(z) = \lim_{z\to ai}(z-ai)\frac{z^2}{(z-ai)(z+ai)(z^2+b^2)}$

$= \lim_{z\to ai}\frac{z^2}{(z+ai)(z^2+b^2)} = \frac{(ai)^2}{(ai+ai)[(ai)^2+b^2]} = \frac{-a^2}{(2ai)(-a^2+b^2)}$

Res[f(z),ai] $= \frac{a}{2i(a^2-b^2)}$

Res[f(z), bi] $= \lim_{z\to bi}(z-bi)f(z) = \lim_{z\to bi}(z-bi)\frac{z^2}{(z^2+a^2)(z+bi)(z-bi)}$

$= \lim_{z\to bi}\frac{z^2}{(z^2+a^2)(z+bi)} = \frac{(bi)^2}{[(bi)^2+a^2](z+bi)} = \frac{-b^2}{(a^2-b^2)(2bi)}$

Res[f(z),bi] $= \frac{-b}{2i(a^2-b^2)}$

By using Cauchy's residue theorem,

$\int_c f(z)dz = 2\pi i[sum\ of\ the\ residues] = 2\pi i[\frac{a}{2i(a^2-b^2)} + \frac{b}{2i(a^2-b^2)}]$

$= \frac{2\pi i}{2i(a^2-b^2)}[a-b] = \frac{\pi}{(a-b)(a+b)}[a-b]$

$\int_c f(z)dz = \frac{\pi}{a+b} \to (2)$

Using (2**) in (1)**

$$\int_R^R f(x)dx + \int_r f(z)dz = \frac{\pi}{(a+b)}$$

If R→ ∞ $\int_r f(z)dz = 0$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{ab(a+b)}$$

Since f(x) = $\frac{x^2}{(x^2+a^2)(x^2+b^2)}$

$$\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Topic of Lecture : Contour integration

Introduction : Evalution of the integrals of the form $\int_{-\infty}^{\infty} f(x)cosmx\ dx\ (or)\ \int_{-\infty}^{\infty} f(x)sinmx\ dx$ Where f(x) → 0 as x → ∞ given $f(x) = \frac{P(x)}{Q(x)}$ is a rational function in x.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Cauchy' s lemma II
2. Jorden lemma

Detailed content of the Lecture:

Cauchy'slemma : II

If f(z) be a uniformly continuous function such that $|(z-a)f(z)| \to 0$ as $|(z-a))| \to \infty$ then $\int_c f(z)dz \to 0 as\ R \to \infty$ where c is a circle $|(z-a)| = R$

Jorden lemma :

If $f(z)$ be a uniformly continuous functions suchthat $|f(z)| \to 0$ as $|(z)| \to \infty\ then \int_c e^{imz} \int_c f(z)dz \to 0\ as\ R \to \infty$ where c is the semicircle $|(z)| = R$ above the real axis and m>0

Evaluate $\int_{-\infty}^{\infty} \frac{\cos axdx}{(x^2+1)}$

Solution :

Let $e^{ix} = \cos x + i\sin x$ R.P$e^{ix} = \cos x$ I.P$e^{ix} = \sin x$

$\int_{-\infty}^{\infty} \frac{\cos\ axdx}{(x^2+1)} = R.P\ of\ \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)} dx$, $\int_{-\infty}^{\infty} f(x)dx = R.P\ of\ \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)} dx$

$\int_{-\infty}^{\infty} \frac{\sin\ axdx}{(x^2+1)} = I.P\ of\ \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)} dx$, $\int_{-\infty}^{\infty} f(x)dx = I.P\ of\ \int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+1)} dx$

Consider $\int_{-\infty}^{\infty} f(z)dz = \int_c \frac{e^{iaz}}{(z^2+1)} dz$ where C is the half of the semi circle r with bounding diameter (-R,R)

$$\int_c f(z)dz = \int_r f(z)dz + \int_{-R}^{R} f(x)dx \to (1)$$

f(z) = $\frac{e^{iaz}}{(z^2+1)} = \frac{e^{iaz}}{(z+i)(z-i)}$

z = -i is a double pole lies outside r

z = i is a double pole lies inside r

Residue of simple pole is

Res [f(z),a] = $\lim_{z\to a}(z-a)f(z)$

Res [f(z),i] = $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z+i)(z-i)} = \lim_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{i^2a}}{(i+i)}$

Res [f(z) , i] = $\frac{e^{-a}}{2i}$

Hence by Cauchy's residue theorem,

$$\int_c f(z)dz = 2\pi i(sum\ of\ the\ residues) = 2\pi i(\frac{e^{-a}}{2i})$$

$\int_c f(z)dz = \pi e^{-a} \to (2)$

Using (2) in (1)

$$\int_{-R}^{R} f(x)dx + \int_r f(z)dz = \pi e^{-a}$$

If R→ ∞, $\int_r f(z)dz = 0$

$\int_{-\infty}^{\infty} f(x)dx = \pi e^{-a}$

$\int_{-\infty}^{\infty} \frac{e^{-iax}}{(x^2+1)} dx = \pi e^{-a}$ $\int_{-\infty}^{\infty} \frac{\cos\ ax + i\sin\ ax}{(x^2+1)} dx = \pi e^{-a}$

$$\int_{-\infty}^{\infty} \frac{cosax}{(x^2+1)} dx + i\int_{-\infty}^{\infty} \frac{sinax}{(x^2+1)} dx = \pi e^{-a}$$

Equating real and imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{cosax}{(x^2+1)} dx = \pi e^{-a}$$

UNIT – V

Laplace Transform and Inverse Laplace Transform

Topic of Lecture : Laplace Transform - Basic properties

Introduction : The Laplace Transform method is a technique for solving linear differential equations with initial conditions. It is commonly used to solve electrical circuit and systems problems.

Prerequisite knowledge for Complete understanding and learning of Topic :

7. Laplace Transform
8. Integration
9. Differentiation

Detailed content of the Lecture:

1. **Find the Laplace transform of** $f(t) = t\cosh t$

$$L(t\cosh t) = \frac{-d}{ds}(L(\cosh t)) = \frac{-d}{ds}\left[\frac{s}{s^2-1}\right]$$

$$= -\left[\frac{(s^2-1)(1)-s(2s)}{(s^2-1)^2}\right] = -\left[\frac{-1-s^2}{(s^2-1)^2}\right] = \frac{s^2+1}{(s^2-1)^2}$$

2. **State and prove change of scale property in Laplace transform.**

If L(f(t)) = F(s), then L(f(at)) = $\frac{1}{a}F\left(\frac{s}{a}\right)$.

L(f(at)) = $\int_0^\infty e^{-st} f(at)dt$

at=u t=0 u=0

adt=du t=∞ u=∞

L(f(at)) = $\int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a}$

$= \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} f(u)du$

L(f(at)) $= \frac{1}{a}F\left(\frac{s}{a}\right)$

3. **State and prove the first shifting theorem on Laplace transforms.**

If L(f(t)) = F(s)

L[e^{-at}f(t)] = $\int_0^\infty e^{-st}e^{-at} f(t)dt$

$= \int_0^\infty e^{-(s+a)t} f(t)dt$

$= F(s+a)$

$= [F(s)]_{s\to s+a}$

$\therefore\ L[e^{-at}f(t)] = [F(s)]_{s\to s+a}$

4. **Find the Laplace transform of te^{-t} sint.**

$$L(te^{-t}\sin t) = -\frac{d}{ds}\ L(e^{-t}\sin t)$$

$$= -\frac{d}{ds}[L(\sin t)]_{s\to s+1}$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1^2}\right]_{s\to s+1}$$

$$= -\frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$= -\left[\frac{-1}{(s^2+2s+2)^2}(2s+2)\right]$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}$$

5. **Find $L\left[\cos^2 3t\right]$**

$$L\left[\cos^2 3t\right] = L\left[\frac{1+\cos 6t}{2}\right]$$

$$= \frac{1}{2}L\left[1+\cos 6t\right]$$

$$= \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+36}\right]$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=W8pYeITi9bQ

Important Books/Journals for further learning including the page nos:

1. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics'', S. Chand Private Ltd., (2011) Page No: 9.3-9.12**
2. **B.S.Grewel, higher Engineering Mathematics, Khanna Publishers, New Delhi, 2015 Page No: 726-735**

Topic of Lecture : Basic properties, Transforms of derivatives integrals Periodic functions

Introduction : Let a function f(t) be continuous and defined for positive values of t. The Laplace transformation of f(t) associates a function s defined by the equation

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Here, F(s) is said to be the Laplace transform of f(t) and it is written as $L[f(t)]\ or\ L(f)$.

Thus, F(s) = L[f(t)]

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt, t > 0$$

Prerequisite knowledge for Complete understanding and learning of Topic :

10. **Periodic function**
11. **Integration**

Detailed content of the Lecture:

Find the Laplace Transform of triangular wave function

$$f(t)=\begin{cases} t & , 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \text{ with } f(t+2a) = f(t)$$

Solution:

$$L[L(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} \; f(t)\; dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \; t\; dt + \int_a^{2a} e^{-st} \; (2a-t)\; dt\right] \quad \text{--------------------(A)}$$

$$\int_0^a e^{-st} \; t\; dt = [uv_1]_0^a - [u'v_2]_0^a + \; ....$$

	$v = e^{-st}$	
$u = t$	$v_1 = \frac{e^{-st}}{(-s)}$	$[uv_1]_0^a = \frac{ae^{-as}}{(-s)} - 0$ $= \frac{ae^{-as}}{(-s)}$
$u' = 1$	$v_2 = \frac{e^{-st}}{(-s)^2}$	$[u'v_2]_0^a = \frac{e^{-as}}{s^2} - \frac{1}{s^2}$

(1) $\Rightarrow$ $\int_0^a e^{-st} \; t\; dt = \frac{ae^{-as}}{(-s)} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$ --------------------(2)

$\int_0^{2a} e^{-st} (2a-t)\; dt = [uv_1]_a^{2a} - [u'v_2]_a^{2a} + \; ...$ ------------------(3)

	$v = e^{-st}$	
$u = 2a - t$	$v_1 = \frac{e^{-st}}{(-s)}$	$[uv_1]_a^{2a} = 0 - \frac{ae^{-as}}{(-s)}$ $= \frac{ae^{-as}}{s}$
$u' = -1$	$v_2 = \frac{e^{-st}}{(-s)^2}$	$[u'v_2]_a^{2a} = \frac{-e^{-2as}}{s^2} + \frac{e^{-as}}{s^2}$

(3) $\Rightarrow$ $\int_0^{2a} e^{-st} (2a-t)\; dt = \frac{ae^{-st}}{-s} + \frac{e^{-2st}}{(-s)^2} - \frac{e^{-sa}}{(-s)^2}$ ... -----------------(4)

(A) $\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}} \left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{ae^{-as}}{s} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right]$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 - 2e^{-as} + e^{-2as}}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})^2}{(1-e^{-as}\ )(1+e^{-as})}$$

$$= \frac{1}{s^2}\frac{1-e^{-as}}{1+e^{-as}}$$

We Know that, $\tan h\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}$

Here, $2\theta = as \Rightarrow \theta = \frac{as}{2}$

$$L[f(t)] = \frac{1}{s^2}\tan\frac{as}{2}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=W8pYeITi9bQ

Important Books/Journals for further learning including the page nos:

3. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.3-9.12**
4. **B.S.Grewel, higher Engineering Mathematics, Khanna Publishers, New Delhi, 2015 Page No: 726-735**

Topic of Lecture : Periodic functions

Introduction : A function f(t) is said to be periodic function with period 'p' if for all t,
F(t+p) = f(t) , where 'p' is a positive constant. The least value of p>0 is called the period of f(t).
Example:

1. tan t is periodic function with period 2π
2. sin 2t is periodic function with period π

Prerequisite knowledge for Complete understanding and learning of Topic :

12. Integration
13. Laplace Transform

Detailed content of the Lecture:

Find the Laplace transform of the Half-sine wave rectifier function $(t) = \begin{cases} \sin\omega t, 0 < t < \frac{\pi}{\omega} \\ 0\ , \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

Solution:

The given function is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\, f(t)dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin\omega t\ dt + 0\right]$$

We know that the formula $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

Here we apply this formula , We get

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}[-s \sin \omega t - \omega \cos \omega t]_0^{\frac{\pi}{\omega}}\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left[1+e^{\frac{-s\pi}{\omega}}\right]}{\left[1-e^{\frac{-s\pi}{\omega}}\right]\left[1+e^{\frac{-s\pi}{\omega}}\right][s^2+\omega^2]}$$

$$L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1-e^{\frac{-s\pi}{\omega}})}$$

Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Solution:

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty} L[\cos at - \cos bt] \, ds$$

$$= \int_S^{\infty}\left[\frac{s}{s^2+a^2} - \frac{s}{S^2+b^2}\right] ds$$

$$= \frac{1}{2}[\log(s^2+a^2) - \log(s^2+b^2)]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=MLSfh33ZCwE

Important Books/Journals for further learning including the page nos:

5. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.63-9.73**
6. **B.S.Grewel, higher Engineering Mathematics, Khanna Publishers, New Delhi, 2015 Page No: 736-748**

Topic of Lecture : Periodic functions

Introduction : A function f(t) is said to be periodic function with period 'p' if for all t,
F(t+p) = f(t) , where 'p' is a positive constant. The least value of p>0 is called the period of f(t).
Example:

3. tan t is periodic function with period 2π
4. sin 2t is periodic function with period π

Prerequisite knowledge for Complete understanding and learning of Topic :

14. Integration
15. Laplace Transform

Detailed content of the Lecture:

Find the Laplace transform of the Half-sine wave rectifier function $(t) = \begin{cases} \sin \omega t, 0 < t < \frac{\pi}{\omega} \\ 0 \ , \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

Solution:

The given function is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} [\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t \, dt + 0]$$

We know that the formula $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

Here we apply this formula , We get

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} [\frac{e^{-st}}{s^2+\omega^2} [-s \sin \omega t - \omega \cos \omega t]_0^{\frac{\pi}{\omega}}]$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} [\frac{e^{\frac{-s\pi}{\omega}} \omega + \omega}{s^2+\omega^2}]$$

$$= \frac{\omega [1 + e^{\frac{-s\pi}{\omega}}]}{\left[1 - e^{\frac{-s\pi}{\omega}}\right]\left[1 + e^{\frac{-s\pi}{\omega}}\right][s^2+\omega^2]}$$

$$L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1 - e^{\frac{-s\pi}{\omega}})}$$

Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Solution:

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty} L[\cos at - \cos bt] \, ds$$

$$= \int_S^{\infty} \left[\frac{s}{s^2 + a^2} - \frac{s}{S^2 + b^2}\right] ds$$

$$= \frac{1}{2} [\log(s^2 + a^2) - \log(s^2 + b^2)]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0-\log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$L\left[\frac{\cos at-\cos bt}{t}\right] = \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=MLSfh33ZCwE

Important Books/Journals for further learning including the page nos:

7. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.63-9.73**
8. **B.S.Grewel, higher Engineering Mathematics, Khanna Publishers, New Delhi, 2015 Page No: 736-748**

Topic of Lecture: Laplace Transform - Partial Fraction Method

Introduction : The Laplace Transform method is a technique for solving linear differential equations with initial conditions. It is commonly used to solve electrical circuit and systems problems.

Prerequisite knowledge for Complete understanding and learning of Topic: (Max. Four important topics)

1. Partial Fraction
2. Trigonometry
3. Integration

Detailed content of the Lecture:

1. **Find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$**

Solution:

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1}+\frac{Bs+c}{s^2+4s+13}$$

$$1-s=A(s^2+4s+13)+(Bs+c)(s+1)$$

$$2=10A$$

$$A=\frac{1}{5}$$

Equating coefficient of s^2,

$$A+B=0$$
$$B=-A$$
$$B=\frac{-1}{5}$$

Equating contant coefficient,

$$13A+C=1$$
$$C=1-\frac{13}{5}=\frac{-8}{5}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)}=\frac{\frac{1}{5}}{s+1}+\frac{-1/_5\,s-8/_5}{s^2+4s+13}$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]=\frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{5}L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right]-\frac{1}{5}L^{-1}\left[\frac{6}{(s+2)^2+3^2}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right]-\frac{6}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t-\frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t-\frac{2}{5}e^{-2t}\sin 3t$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=XIfaRY-DDuk

Important Books/Journals for further learning including the page nos.:

1. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.81-9.83**

Topic of Lecture:

CONVOLUTION THEOREM

Introduction : (Maximum 5 sentences)

"Reverse, shift, multiply and integrate" technique is called convolution.

The convolution of two function f (t) and g (t) is defined as

$$f(t)*g(t)=\int_0^t f(t)g(t-u)\,du$$

Prerequisite knowledge for Complete understanding and learning of Topic: (Max. Four important topics)

4. Partial Fraction
5. Trigonometry
6. **Integration**

Detailed content of the Lecture:

2**. Using convolution theorems find** $\boldsymbol{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]}$.

Solution:

$$L^{-1}[F(s)\,G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * \frac{1}{a} L^{-1}\left[\frac{a}{(s^2+a^2)}\right]$$

$$= \cos at * \frac{1}{a}\sin at$$

$$= \frac{1}{a}[\cos at * \sin at]$$

$$= \frac{1}{a}\int_0^t \cos au\ \sin(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \sin(at-au)\cos au\,du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\,\mathrm{du}$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin \mathrm{a}(t-2u)]\,\mathrm{du}$$

$$= \frac{1}{2a}\left[(\sin at)u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]$$

$$= \frac{1}{2a}\left[u(\sin at) + \left(\frac{\cos a(t-2u)}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[t \ \sin at + \left(\frac{\cos at}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[t \ \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a} \ t \ \sin at$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=6XIX5Z3ZMHQ
https://www.youtube.com/watch?v=bfWu14Ny-tc

Important Books/Journals for further learning including the page nos.:

2. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics'', S. Chand Private Ltd., (2011) Page No: 9.81-9.83**

Topic of Lecture:

CONVOLUTION THEOREM

Introduction :

In mathematics, the convolution theorem states that under suitable conditions the Fourier transform of a convolution of two signals is the pointwise product of their Fourier transforms. In other words, convolution in one domain (e.g., time domain) equals point-wise multiplication in the other domain (e.g., frequency domain). Versions of the convolution theorem are true for various Fourier-related transforms. Let f and g be two functions with convolution $f * g$.

Prerequisite knowledge for Complete understanding and learning of Topic:

7. Partial Fraction
8. Trigonometry
9. Integration

Detailed content of the Lecture:

1. Find the Laplace inverse of $\frac{s^2}{(s^2+a^2)^2}$ using convolution theorem.

Solution: $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \ \cos a(t-u) \ du = \int_0^t \cos au \ \cos(at-au) \ du$$

$$= \frac{1}{2}\int_0^t [\cos(au+at-au) + \cos(au-at+au)]\text{du}$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)]\ du$$

$$= \frac{1}{2}\left[(\cos at)u + \frac{\sin(2au - at)}{2a}\right]$$

$$= \frac{1}{2}\left[\left(t\ \cos at + \frac{\sin\ at}{2a}\right) - \left(0 - \frac{\sin\ at}{2a}\right)\right]$$

$$= \frac{1}{2}\left[\left(t\ \cos at + \frac{\sin\ at}{a}\right)\right] \quad = \frac{1}{2a}[\sin at + at\cos at]$$

2. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+b^2}\right)\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au\ \cos b(t-u)\ du$$

$$= \int_0^t \cos au\ \cos(bt - bu)\ du$$

$$= \frac{1}{2}\int_0^t [\cos(au + bt - bu) + \cos(au - bt + bu)]\, du$$

$$\left[\therefore\ \cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\right]$$

$$= \frac{1}{2}\int_0^t [\cos[(a-b)u + bt] + \cos[(a+b)u - bt]]\, du$$

$$= \frac{1}{2}\left[\left[\frac{\sin[(a-b)u + bt]}{a-b}\right] + \left[\frac{\sin[(a+b)u - bt]}{a+b}\right]\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(at - bt + bt)}{a-b} + \frac{\sin(at + bt - bt)}{a+b}\right) - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$

$$[\therefore \sin(-\theta) = -\sin(\theta)]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} + \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{2a}{a^2-b^2}\right) + \sin bt\left(\frac{-2b}{a^2-b^2}\right)\right]$$

$$= \frac{a\ \sin at - b\sin bt}{a^2-b^2}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=6XIX5Z3ZMHQ
https://www.youtube.com/watch?v=bfWu14Ny-tc

Important Books/Journals for further learning including the page nos.:

32. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.81-9.83**

Topic of Lecture:

CONVOLUTION THEOREM

Introduction :

"Reverse, shift, multiply and integrate" technique is called convolution.

The convolution of two function f (t) and g (t) is defined as

$$f(t)*g(t)=\int_0^t f(t)g(t-u)\,du$$

Prerequisite knowledge for Complete understanding and learning of Topic:

10. Partial Fraction
11. Trigonometry
12. Integration

Detailed content of the Lecture:

CONVOLUTION THEOREM

Convolution theorem:

Statement: If f (t) and g (t) are functions defined for t≥0,
then L^{-1} [f (t) * g (t)] = L^{-1} [f (t)]. L^{-1} [g (t)]

1.Using convolution theorem find $\boldsymbol{L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right]}$.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \cos at * \frac{1}{b}\sin bt$$

$$=\frac{1}{b}\int_0^t \cos au \sin b(t-u)\,du$$

$$=\frac{1}{b}\int_0^t \sin(bt-bu)\cos au\,du$$

$$=\frac{1}{2b}\int_0^t [\sin(bt-bu+au)+\sin(bt-bu-au)]\,du$$

$$= \frac{1}{2b}\int_0^t [\sin[bt + (a-b)u + \sin[bt - (a+b)u]\, du$$

$$= \frac{1}{2b}\left[\frac{-\cos[bt+(a-b)u]}{a-b} - \frac{\cos[bt-(a+b)u]}{-(a+b)}\right]$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos at}{a+b} + \frac{\cos at}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left[\cos at\left[\frac{1}{a+b} - \frac{1}{a-b}\right] - \cos bt\left[\frac{1}{a+b} - \frac{1}{a-b}\right]\right]$$

$$= \frac{1}{2b}\left[\frac{1}{a+b} - \frac{1}{a-b}\right][\cos at - \cos bt]$$

$$= \frac{1}{2b}\left[\frac{a-b-a-b}{a^2-b^2}\right][\cos at - \cos bt]$$

$$= \frac{\cos at - \cos bt}{b^2-a^2}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=6XIX5Z3ZMHQ
https://www.youtube.com/watch?v=bfWu14Ny-tc

Important Books/Journals for further learning including the page nos.:

1. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.81-9.83**

Topic of Lecture:
APPLICATION TO SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT CO-EFFICIENTS

Introduction : (Maximum 5 sentences)
A solution of a differential equation is a function that satisfies the equation. The solutions of a homogeneous linear differential equation form a vector space. In the ordinary case, this vector space has a finite dimension, equal to the order of the equation.

Prerequisite knowledge for Complete understanding and learning of Topic:

13. Laplace Transform
14. Inverse Laplace Transform
15. Partial Fraction Method

Detailed content of the Lecture:

APPLICATION TO SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT CO-EFFICIENTS

Formula:

$L[y'(t)] = s\,L[y(t)] - y(0)$

$L[y''(t)] = s^2\,L[y(t)] - sy(0) - y'(0)$

$L[y'''(t)] = s^3\,L[y(t)] - s^2y(0) - sy'(0) - y''(0)$

1. Solve by using L.T. $(D^2+9)\,y = \cos 2t$, given that if y (0) = 1, y (z) = -1.

Solution:

$(D^2+9)\,y = \cos 2t$

$y''(t) + 9y(t) = \cos 2t$

$L[y''(t)] + 9L[y(t)] = L[\cos 2t]$

$[s^2L[y(t)] - s\,y(0) - y'(0)] + 9L[y(t)] = \left[\frac{s}{s^2+4}\right]$

$[s^2L[y(t)-s-k] + 9L[y(t)] = \left[\frac{s}{s^2+4}\right]$

[given: y (0) =1→y'(0)=k]

$(s^2+9)\,L[y(t)] - s - k = \left[\frac{s}{s^2+4}\right]$

$(s^2+9)L[y(t)] = \left[\frac{s}{s^2+4}\right] + s + k$

$L[y(t)] = \left[\frac{s}{(s^2+4)(s^2+9)}\right]$ ------------------------------- (1)

Hint: $\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right]$

$\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}$

(1)------- $L[y(t)] = \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$

$= \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{k}{s^2+9}$

$= \frac{1}{5}\,L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}\,L^{-1}\left[\frac{s}{s^2+9}\right] + k\,L^{-1}\left[\frac{s}{s^2+4}\right]$

$= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{k}{3}\sin 3t$ -------------------- (2)

Given: $y\left(\frac{\pi}{2}\right) = -1$

$y\left(\frac{\pi}{2}\right) = \frac{1}{5}(-1) + \frac{4}{5}(0) + \frac{k}{3}(-1)$

$= - -\frac{1}{5} - \frac{k}{3}$

$= -1$

$$k = \frac{12}{5}$$

(2) ------- y (t) $= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{3}\sin 3t$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=3KtUt78p9a4
https://www.youtube.com/watch?v=iQoAbO_xlFs

Important Books/Journals for further learning including the page nos.:

1. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics", S. Chand Private Ltd., (2011) Page No: 9.96-9.108**

Topic of Lecture: APPLICATION TO SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT CO-EFFICIENTS

Introduction :

A solution of a differential equation is a function that satisfies the equation. The solutions of a homogeneous linear differential equation form a vector space. In the ordinary case, this vector space has a finite dimension, equal to the order of the equation.

Prerequisite knowledge for Complete understanding and learning of Topic:

16. Partial Fraction
17. Laplace transform
18. Inverse Laplace transform

Detailed content of the Lecture:

1. Using Laplace transform solve $y'' - 3y' + 2y = e^{-t}$ given $y(0) = 1$, $y'(0) = 0$.

Solution:

Given: $y'' - 3y' + 2y = e^{-t}$

(i.e.,) $L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$

$[s^2L[y(t)] - s\,y(0) - y'(0)] - 3[s\,L[y(t)] + 3[y(0)] + 2L[y(t)] = \left[\frac{1}{s+1}\right]$-------(1)

Given:

$y(0) = 1$

$y'(0) = 0$

(1) -------→ $s^2 L[y(t)] - 3s\ L[y(t)] + 3 + 2L[y(t)] = \left[\frac{1}{s+1}\right]$

$$(s^2 - 3s + 2)\,L[y(t)] = \left[\frac{1}{s+1}\right] + s - 3$$

$$(s-1)(s-2)\,L[y(t)] = \frac{1+s^2+s-3s-3}{s+1}$$

$$= \frac{s^2-2s-2}{s+1}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s-1)(s+1)(s-2)}$$

$$\frac{s^2-2s-2}{(s-1)(s+1)(s-2)} = \frac{A}{S-1} + \frac{B}{S+1} + \frac{C}{S-2}$$

$(s^2 - 2s - 2) = A (s-2)(s+1) + B (s-1)(s-2) + c (s-1)(s+1)$

Put s = 1, we get | put s = -1, we get | put s = 2, we get

$1 – 2 - 2 = A (-1)(2)$ | $1+2-2 = B (-2)(-3)$ | $4 – 4 – 2 = C(1)(3)$

$-3 = -2A$ | $1 = 6B$ | $-2 = 3C$

$A = 3/2$ | $B = 1/6$ | $C = -2/3$

(2) -------→ $L[y(t)] = \frac{(3/2)}{S-1} + \frac{(1/6)}{S+1} + \frac{(-2/3)}{S-2}$

$y(t) = \frac{3}{2} L^{-1} \left[\frac{1}{S+4}\right] + \frac{1}{6} L^{-1} \left[\frac{1}{S+1}\right] + \frac{2}{3} L^{-1} \left[\frac{1}{S-2}\right]$

$y(t) = \frac{3}{2} e^{t} + \frac{1}{6} e^{-t} + \frac{2}{3} e^{2t}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=3KtUt78p9a4
https://www.youtube.com/watch?v=iQoAbO_xlFs

Important Books/Journals for further learning including the page nos.:

1. **Dass, H. K., and Er. Rajinish Verma, "Higher Engineering Mathematics'', S. Chand Private Ltd., (2011) Page No: 9.96-9.108**