21BSS21 – Algebra and Calculus
Unit - I
Matrices

Topic of Lecture : Characteristic Equation

Introduction :

The **characteristic equation** is the **equation** which is solved to find a matrix's eigenvalues, also called the **characteristic polynomial**.

Prerequisite knowledge for Complete understanding and learning of Topic :

Characteristic equation (calculus), **used** to solve linear differential **equations**. **Characteristic equation**, the **equation** obtained by equating to zero the **characteristic polynomial** of a matrix or of a linear mapping. **Characteristic equations**, auxiliary differential **equations**, **used** to solve a partial differential **equation**.

Detailed content of the Lecture:

Characteristic Equation

For 2 × 2 matrix

If A is a square matrix of order 2 , then its characteristic equation can be written as

$\lambda^2 - s_1\lambda + s_2 = 0$ where

S_1= sum of the main diagonal elements

S_2=Determinant value of A=$|A|$

For 3× 3 matrix

If A is a square matrix of order, then its characteristic equation can be written as

$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements

S_2= sum of minors of main diagonal elements =$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

S_3=Determinant value of A=$|A|$

Problems based on Characteristic Equation

1. **Find the Characteristic Equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$**

solution : Let A=$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The Characteristic Equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$ where

S_1= sum of the main diagonal elements =1+2=3

S_2=Determinant value of A=$|A|=\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$

=2-0=2

Hence the required Characteristic Equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the Characteristc Equation of the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{-3} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{3} \\ \mathbf{-5} & \mathbf{2} & \mathbf{-4} \end{pmatrix}$

solution : Let A=$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

The Characteristic Equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements =2+1+(-4) = -1

S_2= sum of minors of main diagonal elements =$\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

= (-4-6)+(-8+5)+(2+9)

= - 2

S_3=Determinant value of A=$|A|=\begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

=2(-4-6)-(-3)(-12+15)+1(6+5)

=0

Hence the required Characteristic Equation is $\lambda^3 + \lambda^2 - 2\lambda = 0$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	1.1-1.10

Topic of Lecture : Eigen Values and Eigen Vectors of Matrix

Introduction :

Eigen values:

Let A=a_{ij} be a square matrix. The Characteristic Equation of A is $|A - \lambda I| = 0$ .The roots of the Characteristic Equation are called Eigen values of A.

Eigenvectors:

Let A=a_{ij} be a square matrix of order n .If there exists a non zero vector X=$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

Such that AX=λ X, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ .

Prerequisite knowledge for Complete understanding and learning of Topic :

Working rule to find Eigen values and Eigen vectors

Step-1 Find the Characteristic Equation $|A-\lambda I|=0$

Step-2 Solving the Characteristic Equation, we get Characteristic roots called Eigen values.

Step-3 To find Eigen vectors, solve (A-λ I) X=0 for the different values of λ .

Detailed content of the Lecture:

Problems based on Non-symmetric matrices with non-repeated Eigen values

1. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

sol: Let A =$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements =1+2+3=6

S_2= sum of minors of main diagonal elements =$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$

=(6-2)+(3+2)+(2-0)=4+5+2=11

S_3=Determinant value of A=$|A|$=$\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

=1(6-2)-0(3-2)+(-1)(2-4)=6

Hence the required Characteristic Equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Step-2 **To find the roots of the Characteristic Equation**

by using calculator , the Eigen values are 1,2,3

Step-3 **To find the Eigen vectors**

To find the Eigen vectors, solve (A-λ I) X=0.

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{-------- (A)}$$

<u>Case –(i)</u> If λ =1,then the equation (A) becomes

$$\left[\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_3=0$ --------(1)

$x_1+x_2+x_3 =0$ --------(2)

$2x_1+2x_2+2x_3=0$ -------- (3)

We choose 1 and 2, since 2 and 3 are same.

Solving 1 and 2, we get $\frac{x_1}{0+1}=\frac{x_2}{-1+0}=\frac{x_3}{0-0}$

$$\frac{x_1}{1}=\frac{x_2}{-1}=\frac{x_3}{0}$$

Hence, a corresponding Eigen vector is $X_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

<u>Case –(ii)</u> If λ =2,then the equation (A) becomes

$$\left[\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1-x_3=0$ --------(4)

$x_1+x_3=0$ --------(5)

$2x_1+2x_2+x_3=0$ -------- (6)

We choose 5 and 6, since 4 and 5 are same.

Solving 5 and 6, we get $\frac{x_1}{0-2}=\frac{x_2}{2-1}=\frac{x_3}{2-0}$

$$\frac{x_1}{-2}=\frac{x_2}{1}=\frac{x_3}{2}$$

Hence, a corresponding Eigen vector is $X_2=\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

<u>Case –(iii)</u> If λ =3,then the equation (A) becomes

$$\left[\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1+0x_2-x_3=0$ ------- (7)

$x_1-x_2+x_3=0$ --------(8)

$2x_1+2x_2+0x_3=0$ ------- (9)

Solving 8 and 9, we get $\frac{x_1}{0-2}=\frac{x_2}{2-0}=\frac{x_3}{2+2}$

$$\frac{x_1}{-2}=\frac{x_2}{2}=\frac{x_3}{4}$$

Hence, a corresponding Eigen vector is $X_3=\begin{bmatrix}1\\-1\\-2\end{bmatrix}$

Problems based on Non-symmetric matrices with repeated Eigen values

1. Find the Eigen values and Eigen vectors of $\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

sol: Let A $=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3-s_1\lambda^2+s_2\lambda-s_3=0$ where

S_1= sum of the main diagonal elements = - 2+1+0 = -1

S_2= sum of minors of main diagonal elements $=\begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix}+\begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix}+\begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix}$

=(0-12)+(0-3)+(-2-4)= -12-3-6= - 21

S_3=Determinant value of A=$|A|=\begin{vmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{vmatrix}$

= - 2(0-12) – 2(0-6) + (-3) (-4+1)=45

Hence the required Characteristic Equation is $\lambda^3+\lambda^2-21\lambda-45=0$

Step-2 **To find the roots of the Characteristic Equation**

by using calculator , the Eigen values are -3 , - 3 ,5

Step-3 **To find the Eigen vectors**

To find the Eigen vectors, solve (A- λ I) X=0.

$$\left[\begin{pmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{pmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{pmatrix}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{-------- (A)}$$

<u>**Case –(i)**</u> If λ = - 3 ,then the equation (A) becomes

$$\left[\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+2x_2-3x_3=0$ --------(1)

$2x_1+4x_2-6x_3=0$ --------(2)

$-x_1-2x_2+3x_3=0$ -------- (3)

Here 1,2 and 3 are same. Consider $x_1+2x_2-3x_3=0$

put $x_1=0$, we get $2x_2=3x_3$

$$\frac{x_2}{3}=\frac{x_3}{2}$$

Hence, a corresponding Eigen vector is $X_1=\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

put $x_2=0$, we get $x_1=3x_3$

$$\frac{x_1}{3}=\frac{x_3}{1}$$

Hence, a corresponding Eigen vector is $X_2=\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

<u>Case –(ii)</u> If $\lambda=5$,then the equation (A) becomes

$$\left[\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1+2x_2-3x_3=0$ ------- (4)

$2x_1-4x_2-6x_3=0$ --------(5)

$-x_1-2x_2-5x_3=0$ ------- (6)

Solving 4 and 5, we get $\frac{x_1}{-12-12}=\frac{x_2}{-6-42}=\frac{x_3}{28-4}$

$$\frac{x_1}{-24}=\frac{x_2}{-48}=\frac{x_3}{24}$$

Hence, a corresponding Eigen vector is $X_3=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	1.20-1.35

Topic of Lecture : Eigen Values and Eigen Vectors of Matrix

Introduction :

Eigen values:

Let A=a_{ij} be a square matrix. The Characteristic Equation of A is $|A-\lambda I|=0$ .The roots of the Characteristic Equation are called Eigen values of A.

Eigenvectors:

Let A=a_{ij} be a square matrix of order n .If there exists a non zero vector X=$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

Such that AX=λ X, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ .

Prerequisite knowledge for Complete understanding and learning of Topic :

Working rule to find Eigen values and Eigen vectors

Step-1 Find the Characteristic Equation $|A-\lambda I|=0$

Step-2 Solving the Characteristic Equation, we get Characteristic roots called Eigen values.

Step-3 To find Eigen vectors, solve (A-λ I) X=0 for the different values of λ .

Detailed content of the Lecture:

Problems based on Symmetric matrices with non-repeated Eigen values

2. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

sol: Let A =$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements =7+6+5=18

S_2= sum of minors of main diagonal elements =$\begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$

=(30-4)+(35-0)+(42-4)=99

S_3=Determinant value of A=$|A|=\begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$=7(30-4)+2(-10-0)+0( )=162

Hence the required Characteristic Equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

Step-2 **To find the roots of the Characteristic Equation**

by using calculator , the Eigen values are 3,6,9

Step-3 **To find the Eigen vectors**

To find the Eigen vectors, solve (A-λ I) X=0.

$$\left[\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{-------- (A)}$$

<u>Case –(i)</u> If λ =3,then the equation (A) becomes

$$\left[\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1-2x_2+0x_3 = 0$ --------(1)

$-2x_1+3x_2-2x_3 = 0$ --------(2)

$0x_1-2x_2+2x_3 = 0$ -------- (3)

Solving 2 and 3, we get $\frac{x_1}{6-4} = \frac{x_2}{0+4} = \frac{x_3}{4-0}$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence, a corresponding Eigen vector is $X_1=\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

<u>Case –(ii)</u> If λ =6,then the equation (A) becomes

$$\left[\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1-2x_2+0x_3 = 0$ --------(4)

$-2x_1+0x_2-2x_3 = 0$ --------(5)

$0x_1-2x_2-x_3 = 0$ -------- (6)

Solving 5 and 6, we get $\frac{x_1}{0-4} = \frac{x_2}{0-2} = \frac{x_3}{4-0}$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, a corresponding Eigen vector is $X_2=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case –(iii) If λ =9,then the equation (A) becomes

$$\left[\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1-2x_2+0x_3=0$ ------- (7)

$-2x_1-3x_2-2x_3=0$ --------(8)

$0x_1-2x_2-4x_3=0$ ------- (9)

Solving 8 and 9, we get $\frac{x_1}{12-4}=\frac{x_2}{0-8}=\frac{x_3}{4-0}$

$$\frac{x_1}{2}=\frac{x_2}{-2}=1$$

Hence, a corresponding Eigen vector is $X_3=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Problems based on Symmetric matrices with repeated Eigen values

1. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

sol: Let A $=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements =0+0+0=0

S_2= sum of minors of main diagonal elements $=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

=(0-1)+(0-1)+(0-1)= - 3

S_3=Determinant value of A=$|A|=\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$=0(0-1) -2(0-1)+1(1-0)=2

Hence the required Characteristic Equation is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

Step-2 **To find the roots of the Characteristic Equation**

by using calculator , the Eigen values are -1 ,-1,2

Step-3 **To find the Eigen vectors**

To find the Eigen vectors, solve (A-λ I) X=0.

$$\left[\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} - \lambda \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\left[\begin{pmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{pmatrix}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \text{ -------- (A)}$$

Case –(i) If λ =2,then the equation (A) becomes

$$\left[\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$-2x_1+x_2+x_3 = 0$ --------(1)

$x_1-2x_2+x_3 = 0$ --------(2)

$x_1+x_2-2x_3 = 0$ -------- (3)

Solving 1 and 2, we get $\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, a corresponding Eigen vector is $X_1=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$

Case –(ii) If λ = -1 ,then the equation (A) becomes

$$\left[\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$x_1+x_2+x_3 = 0$ --------(4)

$x_1+x_2+x_3 = 0$ --------(5)

$x_1+x_2+x_3 = 0$ -------- (6)

Here Equation 4,5 and 6 are same. $x_1+x_2+x_3 = 0$

put $x_1=0$,we get $\frac{x_2}{1} = \frac{x_3}{-1}$

Hence, a corresponding Eigen vector is $X_2=\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$

Case –(iii) To find pairwise orthogonal vector

Let $X_3=\begin{bmatrix}l\\ m\\ n\end{bmatrix}$ as X_3 is orthogonal to x_1 and x_2

therefore , l+m+n=0 ……..(7)

0l+m-n=0 ...…..(8)

Solving 7 and 8, we get $\frac{l}{-1-1} = \frac{m}{0+1} = \frac{n}{1-0}$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$$

Hence, a corresponding Eigen vector is $X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	1.20-1.35

Topic of Lecture : Properties of Eigen values and Eigen vectors

Introduction : The eigen functions represent stationary states of the system i.e. the system can achieve that state under certain conditions and **eigenvalues** represent the value of that property of the system in that stationary state

Prerequisite knowledge for Complete understanding and learning of Topic :

The **eigenvalue problem** is related to the homogeneous system of linear equations, as we will see in the following discussion. To find the **eigenvalues** of n × n matrix A we rewrite (1) as. Ax=λIx E3. or by inserting an identity matrix I equivalently. (A−λI)x=0.

Detailed content of the Lecture:

1. **Find the sum and product of the Eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   Sol: Sum of the eigenvalues = sum of the diagonal elements

   $= -1 - 1 - 1 = -3$

   Product of the eigen values = Determinant value of A= $|A|$

   $$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

   $= -1\,(1-1) - 1(-1-1) + 1(1+1) = 4$

2. **The product of two eigen values of of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16.Find the third**

Eigen value.

Sol: Let the eigen values of the matrix $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2=16$. wkt , $\lambda_1 \lambda_2 \lambda_3=|A|$.

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$=6(9-1)+2(-6+2)+2(2-6)=32$

$16\ \lambda_3=32$

$\boldsymbol{\lambda_3=2}$

3. **If 2,2,3 are the eigen values of the matrix $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$.Find the eigenvalues of A^T**

Sol: A square matrix A and its transpose A^T have the same eigen values.

Hence the eigenvalues of A^T are 2,2,3

4. **Find the Eigen values of $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$**

Sol: Given A=$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Clearly given matrix A is lower triangular matrix

Hence by property the eigenvalues of are 2,3,4

5. **Two of the Eigen values of of the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6.Find the Eigen values of A^{-1}.**

Sol: Sum of the Eigen values = Sum of the main diagonal elements

= 3+5+3=11

Let k be the third Eigen value.

3+6+k=11

k = 2

The Eigen values of A are 2, 3, 6.

Hence by property, the Eigen values of $\mathbf{A^{-1}}$ are $\frac{1}{2},\frac{1}{3},\frac{1}{6}$.

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=yuE86XeGhEA&index=15&list=PL05FE76A34A3C0C71

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	1.36-1.50

Topic of Lecture : Cayley –Hamilton Theorem

Introduction :

The **Cayley Hamilton theorem** is one of the most powerful results in linear algebra. This **theorem** basically gives a relation between a square matrix and its characteristic polynomial. One important application of this **theorem** is to find inverse and higher powers of matrices.

Prerequisite knowledge for Complete understanding and learning of Topic :

In linear **algebra**, the Cayley–Hamilton theorem states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

Detailed content of the Lecture:

3. Verify Cayley –Hamilton theorem or the matrix $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1} & A^4

sol: Let A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ where

S_1= sum of the main diagonal elements =2+2+2=6

S_2= sum of minors of main diagonal elements = $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

=(4-1)+(4-2)+(4-1)=8

S_3=Determinant value of A= $|A|$ = $\begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$ =2(4-1)+1(-2+1)+2(1-2)=3

Hence the required Characteristic Equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

By Cayley –Hamoilton theorem,

$$A^3 - 6A^2 + 8A - 3I = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Verification:

$$\boldsymbol{A^2 = A \times A} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$\boldsymbol{A^3 = A \times A^2} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 56 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 2 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

Hence Cayley –Hamoilton theorem is verified.

To find A^{-1}:

(1) X A^{-1} , we get $A^2-6A+8I-3A^{-1}=0$

$$3A^{-1}=A^2-6A+8I$$

$$3A^{-1}=\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}+\begin{bmatrix}-12&6&-12\\6&-12&6\\-6&6&-12\end{bmatrix}+\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}$$

$$3A^{-1}=\begin{bmatrix}3&0&-3\\1&2&0\\-1&1&3\end{bmatrix}$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}3&0&-3\\1&2&0\\-1&1&3\end{bmatrix}$$

To find A^4:

(1) **X** A , we get $A^4=6A^3-8A^2+3A$

$= 6(6A^2-8A+3I)-8A^2+3A$

$= 36A^2 - 48A+ 18I-8A^2+3A$

$= 28A^2-45A+18I$

$$A^4=\begin{bmatrix}196&-168&252\\-140&168&-168\\140&-140&196\end{bmatrix}-\begin{bmatrix}90&-45&90\\-45&90&-45\\45&-45&90\end{bmatrix}+\begin{bmatrix}18&0&0\\0&18&0\\0&0&18\end{bmatrix}$$

$$A^4=\begin{bmatrix}124&-123&162\\-95&96&-123\\95&-95&124\end{bmatrix}$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}3&0&-3\\1&2&0\\-1&1&3\end{bmatrix}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=qLOaCyqSuVA

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi,2018	1.80-1.90

Topic of Lecture : Cayley –Hamilton Theorem

Introduction :

The **Cayley Hamilton theorem** is one of the most powerful results in linear algebra. This **theorem** basically gives a relation between a square matrix and its characteristic polynomial. One important application of this **theorem** is to find inverse and higher powers of matrices.

Prerequisite knowledge for Complete understanding and learning of Topic :

In linear **algebra**, the Cayley–Hamilton theorem states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

Detailed content of the Lecture:

Using Cayley-Hamilton theorem to find the value of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{bmatrix} \text{ given by } \mathbf{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 8 + A^2 - 2A + I}$$

Sol: The characteristic equation of A is $\boldsymbol{\lambda}^3 - S_1\boldsymbol{\lambda}^2 + S_2\boldsymbol{\lambda} - S_3 = 0$ where

$$S_1 = sum\ of\ the\ main\ diagonal\ elements = 2 + 1 + 2 = 5$$

S_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = (2-0)+(4-1)+(2-0\ 7$$

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2-0) - 1(0-0) + 1(0-1) = 3$$

Therefore , The characteristic equation is $\boldsymbol{\lambda}^3 - 5\boldsymbol{\lambda}^2 + 7\boldsymbol{\lambda} - \mathbf{3} = 0$

By **Cayley-Hamilton theorem , we get** $\mathbf{A}^3 - 5\mathbf{A}^2 + 7A - \mathbf{3}\boldsymbol{I} = 0 \quad \ldots (1)$

Let $f(A) = \mathrm{A}^8 + -5\mathrm{A}^7 7\mathrm{A}^6 - 3\mathrm{A}^5 + \mathrm{A}^4 - 5\mathrm{A}^3 + 8\mathrm{A}^2 - 2\mathrm{A} + \mathrm{I}$

$= (\mathrm{A}^3 - 5\mathrm{A}^2 + 7\mathrm{A} - 3\mathrm{I})(\mathrm{A}^5 + \mathrm{A}) + \mathrm{A}^2 + \mathrm{A} + \mathrm{I}$

$= 0 + \mathrm{A}^2 + \mathrm{A} + \mathrm{I}$

$= \mathrm{A}^2 + \mathrm{A} + \mathrm{I}$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$f(A)=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=qLOaCyqSuVA

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	1.80-1.90

Tutorial on Eigen values and Eigen vectors

Introduction: Let A=a_{ij} be a square matrix. The Characteristic Equation of A is $|A-\lambda I|=0$ .The roots of the Characteristic Equation are called Eigen values of A. Let A=a_{ij} be a square matrix of order n .If there exists a non zero vector X such that AX=λ X, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ .

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Characteristic Equation
2.Matrix Multiplication

Detailed content of the Lecture:

1. Find the Eigen values and Eigen vectors of $\begin{bmatrix}6&-6&5\\14&-13&10\\7&-6&4\end{bmatrix}$

Sol : Step-1 To find the Characteristic Equation

The Characteristic Equation of A is $\lambda^3-s_1\lambda^2+s_2\lambda-s_3=0$ where

S_1= sum of the main diagonal elements

S_2= sum of minors of main diagonal elements

S_3=Determinant value of A

Hence the required Characteristic Equation is $\lambda^3+3\lambda^2+3\lambda+1=0$

Step-2 **To find the roots of the Characteristic Equation**

by using calculator , the Eigen values are -1,-1,-1

Step-3 **To find the Eigen vectors**

To find the Eigen vectors, solve (A-λ I) X=0.

$$\left[\begin{bmatrix}6&-6&5\\14&-13&10\\7&-6&4\end{bmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, a corresponding Eigen vector is $X_1=\begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$ $X_2=\begin{bmatrix} -5 \\ 0 \\ 7 \end{bmatrix}$ & $X_3=\begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$

2. **For a given matrix A of order 3, $|A| = 32$ and two of its eigen values of 8 & 2. Find the sum of the eigenvalues.**

Solution:

Step 1: The eigenvalues are $\lambda_1, \lambda_2, \lambda_3$.

$|A|$= Product of the eigenvalues $=\lambda_1\ \lambda_2\ \lambda_3$

Step 2: $|A|$=32 & $\lambda_1 = 8$, $\lambda_2 = 2$

(ie) $\lambda_1\ \lambda_2\ \lambda_3$=32

$\lambda_3 = 2$

Sum of the Eigen values = $\lambda_1 + \lambda_2 + \lambda_3 = 8 + 2 + 2 = 12$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	1.29-1.31

Topic of Lecture : Diagonalization of Matrix

Introduction :

Matrix diagonalization is the process of taking a square **matrix** and converting it into a special type of **matrix**--a so-called diagonal **matrix**--that shares the same fundamental properties of the underlying **matrix**.

Prerequisite knowledge for Complete understanding and learning of Topic :

Diagonalizable A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, i.e. if A = PDP-1 where P is invertible and D is a diagonal matrix.

Detailed content of the Lecture:

1.$\boldsymbol{Diagonalise\ the\ matrix}$ $\mathbf{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$

Sol: The characteristic equation of A is $\boldsymbol{\lambda}^3 - S_1\boldsymbol{\lambda}^2 + S_2\boldsymbol{\lambda} - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 8 + 7 + 3 = 18$

S_2 = sum of the minors of the main diagonal elements

$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$ = (21-16)+(24-4)+(56-36)=45

$S_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 0$

Therefore , The characteristic equation is $\boldsymbol{\lambda}^3 - 18\boldsymbol{\lambda}^2 + 45\boldsymbol{\lambda} = 0$

To solve the characteristic equation

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$\lambda = 0, \lambda = 3 , \lambda = 15$ Hence the Eigen values are 0 , 3 , 15

To find the Eigen vectors :

To find the Eigen vectors , solve (A- $\lambda I) = 0$

ie, $\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ …… (A)

Case-(i) when $\lambda = 0$, equation (A) becomes

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$ ... (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ ... (2)

$2x_1 - 4x_2 + 3x_3 = 0$ ... (3)

Solving (1) and (2) , we get ,

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence the corresponding Eigenvector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case-(ii) when $\lambda = 3$, equation (A) becomes

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$ ... (4)

$-6x_1 + 4x_2 - 4x_3 = 0$ ... (5)

$2x_1 - 4x_2 + 0x_3 = 0$ ... (6)

Solving (4) and (5), we get ,

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the corresponding Eigenvector X $_1$ = $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case-(iii) when $\lambda = 15$, equation (A) becomes

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \qquad \text{... (7)}$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \qquad \text{... (8)}$$

$$2x_1 - 4x_2 - 12x_3 = 0 \qquad \text{... (9)}$$

Solving (7) and (8) , we get ,

$$\frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector X $_1$ = $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

To form Normalized matrix N :

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Find N^T:

$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Calculate N^{T}AN:

$$\mathbf{N^TAN} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$= D$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=ERB5GY1P-Ns

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Jain R.K. , Iyengar S.R.K.	Advanced Engineering Mathematics, 4th edition	Alpha Science International Ltd	1.91-1.100

Topic of Lecture : Canonical Form

Introduction :

This explains about how to reduce the *Quadratic form to Canonical form* through Orthogonal transformation.

Prerequisite knowledge for Complete understanding and learning of Topic :

A quadratic equation is an **equation** of the second degree, **meaning** it contains at least one term that is squared. The standard form is ax² + bx + c = 0 with a, b, and c being constants, or numerical coefficients, and x is an unknown variable.

Detailed content of the Lecture:

1. **Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ into canonical form by an orthogonal transformation**

Sol: The matrix of the quadratic form is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $\boldsymbol{\lambda}^3 - S_1\boldsymbol{\lambda}^2 + S_2\boldsymbol{\lambda} - S_3 = 0$ where

$$S_1 = sum\ of\ the\ main\ diagonal\ elements = 6 + 3 + 3 = 12$$

S_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = (9-1)+(18-4)+(18-4)=36$$

$$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(2-6) = 32$$

Therefore , The characteristic equation is $\boldsymbol{\lambda}^3 - 18\boldsymbol{\lambda}^2 + 45\boldsymbol{\lambda} = 0$

To solve the characteristic equation

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Hence the Eigen values are 2,2,8

To find the Eigen vectors :

To find the Eigen vectors , solve (A- $\lambda I)X = 0$

ie, $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ …… (A)

Case-(i) when $\lambda = 8$, equation (A) becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (1)}$$

$$-2x_1 - 5x_2 - x_3 = \quad \text{... (2)}$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \text{... (3)}$$

Solving (1) and (2) , we get ,

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case-(ii) when $\lambda = 2$, equation (A) becomes

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (4)}$$

$$-2x_1 + x_2 - x_3 = 0 \quad \text{... (5)}$$

$$2x_1 - x_2 + x_3 = 0 \quad \text{... (6)}$$

(4), (5) and (6) are same we get , $2x_1 - x_2 + x_3 = 0$

$$If\ x_1 = 0, we\ get - x_2 + x_3 = 0$$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

To find the third Eigen vector , let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$$2l + m + n = 0 \quad \text{…(7)}$$

$$l + m + n = 0 \quad \text{….(8)}$$

Solving 7 and 8 , we get

$$\frac{l}{-1-1} = \frac{m}{0-2} = \frac{n}{2-0}$$

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$$

Hence the corresponding Eigenvector $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

To form Normalized matrix N :

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Find N^T:

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Calculate N^TAN:

$$\mathbf{N^TAN} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{bmatrix} = \mathbf{D}$$

Canonical form :

$$(y_1 y_2 y_3) \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$$ is the required canonical form.

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=9UCIpfbJvzs

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Bali N. P Manish Goyal	A Text book of Engineering Mathematics, 9th edition	Laxmi Publications Pvt Ltd.	1.101-1.110

Tutorial on Properties & Cayley Hamilton theorem

Introduction : The **Cayley Hamilton theorem** is one of the most powerful results in linear algebra. This **theorem** basically gives a relation between a square matrix and its characteristic polynomial. One important application of this **theorem** is to find inverse and higher powers of matrices.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **Cayley Hamilton theorem** is one of the most powerful results in linear algebra. This **theorem** basically gives a relation between a square matrix and its characteristic polynomial. One important application of this **theorem** is to find inverse and higher powers of matrices.

Detailed content of the Lecture:

1. **For a given matrix A of order 3, $|A| = 32$ and two of its eigen values of 8 & 2. Find**

the sum of the eigenvalues.

Solution:

Step:1 Let the eigenvalues are $\lambda_1, \lambda_2, \lambda_3$.

Given: $|A|$=32 & $\lambda_1 = 8$, $\lambda_2 = 2$

Step:2 $\lambda_1\ \lambda_2\ \lambda_3$=32

(8)(2) λ_3=32

$\lambda_3 = 2$

Step:3 Sum of the eigenvalues = $\lambda_1 + \lambda_2 + \lambda_3 = 8 + 2 + 2 = 12$

2. **If the sum of two eigen values and trace of a 3X3 matrix A are equal , find the value of $|A|$**

Solution:

Step:1 Let λ_1 , λ_2 , λ_3 be the eigen values of the given 3X3 matrix A

Step:2 Sum of the eigen values = Trace of A

(ie) $\lambda_1 + \lambda_2 + \lambda_3$ = Trace A ……….(1)

By (1) $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2$

$\lambda_3 = 0$

Step:3 $|A|$= Product of the eigenvalues = $\lambda_1\ \lambda_2\ \lambda_3$ = 0

3. **Given $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Find the eigen values of A^2**

Solution:

Step:1 The given matrix "A" is a lower triangular matrix.

∴ The eigenvalues of "A" are -1 , -3 , 2

Step:2 The eigenvalues of A^2 are $(-1)^2$, $(-3)^2$, $(2)^2$

(ie) 1 , 9 , 4

4. **Use Cayley-Hamilton Theorem to find $(A^4 - 4A^3 - 5A^2 + A + 2I)$ when $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$**

Solution:

Step:1 Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

(ie) $\lambda^2 - S_1\lambda + S_2 = 0$

S_1= 1+3=4

S_2= $|A|$ = - 5

∴ The characteristic equation of A is

$\lambda^2 - 4\lambda - 5 = 0$

Step:2 By Cayley-Hamilton Theorem we get

$A^2 - 4A - 5I = 0$……..(1)

Step:3: $A^4 - 4A^3 - 5A^2 + A + 2I$

$\therefore A^4 - 4A^3 - 5A^2 + A + 2I = A^2(A^2 - 4A - 5I) + A + 2I$

$= A^2(0) + A + 2I$ (using (1))

= A+2I

$= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=qLOaCyqSuVA

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi,2018	1.80-1.90

Topic of Lecture : Nature of Quadratic from

Introduction :

Index: The number of positive square terms in the canonical form is called the index of the quadratic form

Signature: The difference of number of positive and negative square terms is called the signature of the quadratic form.

Prerequisite knowledge for Complete understanding and learning of Topic :

Positive definite : If all the Eigen values of A are positive numbers, then the quadratic form is said to be Positive definite

Negative definite: If all the Eigen values of A are negative numbers, then the quadratic form is said to be negative definite

Positive semi definite: If all the Eigen values of A are positive and at least one Eigen value is zero, then the quadratic form is said to be Positive semi-definite

Negative semi definite: If all the Eigen values of A are negative and at least one Eigen value is zero, then the quadratic form is said to be negative semi-definite

Indefinite: If A has both positive and negative Eigen values then the quadratic form is said to be Indefinite

Detailed content of the Lecture:

Problems based on nature of the quadratic form

1.Find the index and signature of the Q.F $x_1^2 + 2x_2^2 - 3x_3^2$

Solution: Let $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 3x_3^2$ it is already in the canonical form.

Index = Number of positive terms in the C.F = 2

Signature = Number of positive terms- Number of negative terms = 2-1=1.

2. **Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$**

Solution: Let $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$ it is already in the canonical form.

The C.F contains two positive and one zero term.

Hence QF is positive semi-definite

3. **Give the nature of a quadratic form whose matrix is** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Solution: The Eigen values of the given matrix are -1 , - 1 , - 2

All the Eigen values are negative numbers.

Hence the nature of the Q.F is negative definite.

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=9UCIpfbJvzs

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Bali N. P Manish Goyal	A Text book of Engineering Mathematics, 9th edition	Laxmi Publications Pvt Ltd.	1.110-1.115

Tutorial on Diagonalization and Quadratic form

Introduction :

The roots of the Characteristic Equation are called Eigen values of A.One important application of this **Cayley _Hamilton Theorem** is to find inverse and higher powers of matrices.

Prerequisite knowledge for Complete understanding and learning of Topic :
1. Charteristic Equation 2. Eigen values and Eigen vectors 3.Diagonalization

Detailed content of the Lecture:

1. $\boldsymbol{A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ **be diagonalized? why?**

Solution:

Step:1 The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1= Sum of the main diagonal elements = 1+1=2

S_2= $|A|$ = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$=1-0=1

Step:2 $\therefore$ $\lambda^2 - 2\lambda + 1 = 0$ => $(\lambda - 1)(\lambda - 1) = 0$

⇨ $\lambda = 1,1$

Since the eigen values are repeated , the matrix cannot be diagonalized

2.Write **down the quadratic form corresponding to the matrix** $A = \begin{bmatrix} \mathbf{0} & \mathbf{5} & \mathbf{-1} \\ \mathbf{5} & \mathbf{1} & \mathbf{6} \\ \mathbf{-1} & \mathbf{6} & \mathbf{2} \end{bmatrix}$

Solution:

Step:1 $$A = \begin{bmatrix} coeff\ x^2 & \frac{1}{2} coeff\ xy & \frac{1}{2} coeff\ xz \\ \frac{1}{2} coeff\ xy & coeff\ y^2 & \frac{1}{2} coeff\ yz \\ \frac{1}{2} coeff\ xz & \frac{1}{2} coeff\ yz & coeff\ z^2 \end{bmatrix}$$

Step:2 ∴ The quadratic form $y^2 + z^2 + 10\ xy - 2\ xz + 12yz$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	1.29-1.31

Unit – II

Geometrical Applications of Differential Calculus

Topic of Lecture : Representation of function

Introduction :

A **function** is a relation between two sets of variables such that one variable depends on another variable. We can represent different types of **functions** in different ways. Usually, **functions** are **represented** using formulas or graphs.

Prerequisite knowledge for Complete understanding and learning of Topic :

Graph, Table, Symbols, Words, & Picture/context. A recursive relationship represents the slope of the line in the equation.

Detailed content of the Lecture:

1. **Find the domain and range of the function $f(x) = 1 + x^2$**

Solution : Given $f(x) = 1 + x^2$

i.e., $y = 1 + x^2$

$y - 1 = x^2$

$Always\ x^2 \geq 0$

$hence\ y - 1 \geq 0 \quad \Rightarrow\ y \geq 1$

So the domain is $(-\infty, \infty)$ and the range is $[1, \infty)$

2. **Find the domain and range of the function $f(x) = \frac{4}{3-x}$**

Solution : Given $f(x) = \frac{4}{3-x}$

i.e., $y = \frac{4}{3-x}$ division by zero is not allowed

for x=3, we get 3 – x = 0

So the domain is $(-\infty, 3) \cup (3, \infty)$ and the range is $(-\infty, 0) \cup (0, \infty)$

3. **Find whether the function $f(x) = x^2 + 1$ is even or odd?**

Solution : Given $f(x) = x^2 + 1$

$$f(-x) = (-x)^2 + 1$$

$$f(-x) = x^2 + 1 = f(x)$$

Hence the given function is an even function.

4. **Find whether the function $f(x) = x^3 + x$ is even or odd?**

Solution : Given $f(x) = x^3 + x$

$$f(-x) = (-x)^3 + (-x)$$

$$f(-x) = -x^3 - x = -x^3 + x) = -f(x)$$

Hence the given function is an odd function.

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=L6_c6qvlB8I&list=PLzJaFd3A7DZuyLLbmVpb9e9VLf3Q9cYBL

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	2.1.1-2.1.5

Topic of Lecture : Limit of a function

Introduction :

The **limit of a function** at a point a in its domain (if it exists) is the value that the **function** approaches as its argument approaches. Informally, a **function** is said to have a **limit** L at a if it is possible to make the **function** arbitrarily close to L by choosing values closer and closer to a.

Prerequisite knowledge for Complete understanding and learning of Topic :

Let f(x) be a function **defined** on an interval that contains x=a , except possibly at x=a . Then we say that, limx→af(x)=L lim x → a if for every number ε>0 there is some number δ>0 such that. |f(x)−L|<εwhenever0<|x−a|<δ

Detailed content of the Lecture:

1. **Evaluate :** $\lim\limits_{x\to\infty} \frac{2x+4}{x+1}$

   Solution : $\lim\limits_{x\to\infty} \frac{2x+4}{x+1}$ $\left(\frac{\infty}{\infty} form\right)$

   By using L'hospitals rule

   $$\lim_{x\to\infty} \frac{2x+4}{x+1} = \lim_{n\to\infty} \frac{2}{1} = 2$$

2. **Evaluate :** $\lim\limits_{x\to 3} \frac{x^3-27}{x-3}$

   Solution : $\lim\limits_{x\to 3} \frac{x^3-27}{x-7} = \lim\limits_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3}$

   $= \lim\limits_{x\to 3} (x^2+3x+9)$

   $= 27.$

3. **Evaluate :** $\lim\limits_{x\to 3} \frac{x^2-9}{x-3}$

Solution : $\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$

$= \lim_{x\to 3}(x+3)$

$= 6.$

4. **Evaluate :** $\lim_{\theta\to 0}\frac{\sin n\theta}{\theta}$

Solution : $\lim_{\theta\to 0}\frac{\sin n\theta}{\theta} = \lim_{\theta\to 0} n\frac{\sin n\theta}{n\theta}$

$= n \lim_{n\theta\to 0}\frac{\sin n\theta}{n\theta}$

= n . 1 = n

5. **Prove that** $\lim_{x\to 0}|x| = 0$

Solution : Given $f(x) = |x| = \begin{cases} x \,, if\ x \geq 0 \\ -x\,, if\ x < 0 \end{cases}$

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0 \ for\ |x| = x\,, x > 0$$

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0 \ for\ |x| = -x\,, x < 0$$

therefore , $\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)$

Hence $\lim_{x\to 0}|x| = 0$

Video Content / Details of website for further learning (if any):
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https://www.youtube.com/watch?v=L6_c6qvlB8I&list=PLzJaFd3A7DZuyLLbmVpb9e9VLf3Q9cYBL

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	2.10-2.18

Topic of Lecture : Continuity

Introduction :

A **function** is said to be continuous on the interval [a,b] if it is continuous at each point in the interval. Note that this definition is also implicitly assuming that both f(a) and limx→af(x) lim x → a exist. If either of these do not exist the **function** will not be continuous at x=a

Prerequisite knowledge for Complete understanding and learning of Topic :

The limit must exist at that point. The function must be defined at that point, and. The limit and the function must have equal values at that point.

Detailed content of the Lecture:

1. **Explain why the function $f(x) = \frac{1}{x+2}$ $at\ a = -2$ is discontinuous at the given number 'a'**

Solution : Given $f(x) = \frac{1}{x+2}\ at\ a = -2$

$$f(-2) = \frac{1}{-2+2} = \frac{1}{0} = \infty = undefined$$

Hence f(x) is discontinuous at the given number 'a'.

2. **Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1 , 1]**

Solution : If -1 < a < 1 , then $\lim_{x\to a} f(x) = \lim_{x\to a} 1 - \sqrt{1-x^2}$

$$= 1 - \lim_{x\to a} \sqrt{1-x^2}$$

$$= 1 - \sqrt{1-a^2} = f(a)$$

Hence f is continuous at 'a' if -1 < a < 1

3. **Where is the following function $f(x) = sin(x^3)$ continuous?**

Solution : we have $f(x) = f(g(x))$,where

$g(x) = (x^3)\ \ and\ f(x) = sinx$

Now g is continuous on R, since it is polynomial and f is continuous everywhere .

Thus h = f . g is continuous on R.

4. **Show that $f(x) = 3x^2 + 2x - 1$ is continuous at x = 2.**

Solution : Given $f(x) = 3x^2 + 2x - 1$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x^2 + 2x - 1) = 3(2^2) + 2(2) - 1 = 15$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x^2 + 2x - 1) = 3(2^2) + 2(2) - 1 = 15$$

therefore , $\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$

Hence , $f(x) = 3x^2 + 2x - 1$ is continuous at x = 2

5. **Find the domain where the function f is continuous. Also find the numbers at which the**

function f is discontinuous, where $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$.

Solution : At x=0

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x^2) = 1 \qquad \text{---------------} (1)$$

$$f(0^-) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1 \qquad \text{---------------}(2)$$

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2 \qquad \text{--------------}(3)$$

From 1 , 2 and 3 , we get , $f(0) = f(0^-) \neq f(0^+)$

so , f is continuous on the left at x=0

f is discontinuous on the right at x=0

Hence , f is discontinuous at x=0

At x=2

$$f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 0} (2-x) = 0 \qquad \text{---------------} (1)$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 0 \qquad \text{---------------}(2)$$

$$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2 = 0 \qquad \text{--------------}(3)$$

From 1 , 2 and 3 , we get , $f(2) = f(2^-) = f(2^+)$

Hence , f is discontinuous at x=2.

Video Content / Details of website for further learning (if any): -----------
https://www.youtube.com/watch?v=lxksRct1QOY&list=PLVCBPCYGv7bC1JwOGH-X0FS-jTI4WRbI1

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	2.18-2.5

Tutorial on Limits and Continuity

Introduction :

A **function** is said to be continuous on the interval [a,b] if it is continuous at each point in the interval. Note that this definition is also implicitly assuming that both f(a) and limx→af(x) lim x

→ a exist. If either of these do not exist the **function** will not be continuous at x=a

Prerequisite knowledge for Complete understanding and learning of Topic :

The limit must exist at that point. The function must be defined at that point, and. The limit and the function must have equal values at that point.

Detailed content of the Lecture:

1. **Evaluate :** $\lim_{\theta \to 0} \frac{\tan n\theta}{\theta}$

Solution : $Step: 1$ $\lim_{\theta \to 0} \frac{\tan n\theta}{\theta} = \lim_{\theta \to 0} n \frac{\sin n\theta}{cosn\theta} \frac{1}{n\theta}$

S$tep: 2$ $= n\left(\lim_{n\theta \to 0} \frac{\sin n\theta}{n\theta}\right)\left(\lim_{n\theta \to 0} \frac{1}{cosn\theta}\right)$

= n . 1 = n

2. Find the domain where the function f is continuous. Also find the numbers at which the function f is discontinuous, where $f(x) = \begin{cases} 1 + x^2, & x \le 0 \\ 2 - x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$.

Solution : Step:1 At x=0

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x^2) = 1 \qquad \text{---------------} (1)$$

$$f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 + x^2) = 1 \qquad \text{---------------}(2)$$

$$f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x) = 2 \qquad \text{--------------}(3)$$

From 1 , 2 and 3 , we get , $f(0) = f(0^-) \neq f(0^+)$

so , f is continuous on the left at x=0

f is discontinuous on the right at x=0

Hence , f is discontinuous at x=0

Step:2 At x=2

$$f(2) = \lim_{x \to 2} f(x) = \lim_{x \to 0} (2 - x) = 0 \qquad \text{---------------} (1)$$

$$f(2^-) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0 \qquad \text{---------------}(2)$$

$$f(2^+) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2)^2 = 0 \qquad \text{--------------}(3)$$

From 1 , 2 and 3 , we get , $\boldsymbol{f(2) = f(2^-) = f(2^+)}$

Hence , f is discontinuous at x=2.

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=5yfh5cf4-0w

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	2.29-2.31

Topic of Lecture : DERIVATIVES

Introduction :

Differentiation is a process, in Maths, where we find the instantaneous rate of change in function based on one of its variables. The most common example is the rate change of displacement with respect to time, called velocity.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **Sum rule** says the derivative of a sum of functions is the sum of their derivatives.

The **Difference rule** says the derivative of a difference of functions is the difference of their derivatives.

Detailed content of the Lecture:

1. **Evaluate:** $\boldsymbol{\frac{d}{dx}(x^2 + x)}$

Solution: $\frac{d}{dx}(x^2+x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x)$

$$= 2x + 1$$

2. **If** $\boldsymbol{y = \frac{x^2+x+1}{2x}, find\ \frac{dy}{dx}}$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+x+1}{2x}\right)$

$$= \frac{d}{dx}\left(\frac{x^2}{2x} + \frac{x}{2x} + \frac{1}{2x}\right)$$

$$= \frac{d}{dx}\left(\frac{x^2}{2x}\right) + \frac{d}{dx}\left(\frac{x}{2x}\right) + \frac{d}{dx}\left(\frac{1}{2x}\right)$$

$$= \frac{d}{dx}\left(\frac{x}{2}\right) + \frac{d}{dx}\left(\frac{1}{2}\right) + \frac{d}{dx}\left(\frac{1}{2}\ x^{-1}\right)$$

$$= \frac{1}{2} + 0 + \frac{1}{2}(-1)\ x^{-2}$$

3. **Differentiate the following function : $y = e^x - x$**

Solution : Given $y = e^x - x$

$$y' = e^x - x$$

4. **Differentiate the following function : $y = a^x$**

Solution : Given **:** $y = a^x$

$$y = a^x = e^{\log a^x} = e^{x(\log a)} = e^{(\log a)x}$$

$$y' = e^{(\log a)x} = e^{(\log a)x}(\log a) = a^x \log a$$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=5yfh5cf4-0w

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Bali N. P Manish Goyal	A Text book of Engineering Mathematics, 9^{th} edition	Laxmi Publications Pvt Ltd.	2.-26-2.40

Topic of Lecture : Differentiation Rules

Introduction :

The **Sum rule** says the derivative of a sum of functions is the sum of their derivatives.

The **Difference rule** says the derivative of a difference of functions is the difference of their derivatives.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **derivative** of a constant is equal to zero. ...

The **derivative** of a constant multiplied by a function is equal to the constant multiplied by the **derivative** of the function. ...

The **derivative** of a sum is equal to the sum of the **derivatives**.

Detailed content of the Lecture:

1. **Differentiate the following function: $f(x) = xe^x$**

Solution: Given $f(x) = xe^x$

$$f'(x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$f'(x) = x.e^x + e^x.1$$

$f'(x) = (1 + x)e^x$

2. If $\boldsymbol{f(x) = \frac{e^x}{x}}$**, then find** $\boldsymbol{f'(x)}$**.**

Solution: Given $f(x) = \frac{e^x}{x}$

$$f'(x) = \frac{x\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x)}{x^2}$$

$$f'(x) = \frac{x(e^x) - e^x(1)}{x^2}$$

$$f'(x) = \frac{e^x(x-1)}{x^2}$$

3. **Find an equation of the tangent line to the curve** $\boldsymbol{y = 2x\, sinx\ \ at\ the\ point\ (\frac{\pi}{2}, \pi)}$

Solution: Given $y = 2x\, sinx$

$$y' = 2[x\, cosx + sinx\,.1\,] = 2x\, cosx + 2\, sinx$$

$$m = y'\ at\ \left(\frac{\pi}{2}, \pi\right) = 2\frac{\pi}{2}(0) + 2(1) = 2$$

Equation of the tangent line is $y - y_1 = m(x - x_1)$

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y = 2x$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=5yfh5cf4-0w

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Bali N. P Manish Goyal	A Text book of Engineering Mathematics, 9th edition	Laxmi Publications Pvt Ltd.	2.41-2.50

Topic of Lecture : Differentiation Rules

Introduction :

The **Sum rule** says the derivative of a sum of functions is the sum of their derivatives.

The **Difference rule** says the derivative of a difference of functions is the difference of their derivatives.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **derivative** of a constant is equal to zero. ...

The **derivative** of a constant multiplied by a function is equal to the constant multiplied by the **derivative** of the function. ...

The **derivative** of a sum is equal to the sum of the **derivatives**.

Detailed content of the Lecture:

1. **Find the derivatives of the following:** $\boldsymbol{y = (1 - x^2)^{10}}$

Solution: Given $y = (1 - x^2)^{10}$

put $u = (1 - x^2)$

$\frac{du}{dx} = 0 - 2x = -2x$

Therefore , $y = u^{10}$

$$\frac{dy}{du} = 10u^9$$

by the chain rule , $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\text{then } \frac{dy}{dx} = (10u^9)(-2x) = 10(1 - x^2)^9(-2x) = -20x(1 - x^2)^9$$

2. **Find the 50 th derivative of** $\boldsymbol{y = cos2x}$

Solution: Given : $y = cos2x$

$y' = -2\ sin2x$

$y'' = -4\ cos2x$

$y''' = 8\ sin2x$

....

....

$y^{49} = -2^{49}sin2x$

$y^{50} = -2^{50}cos2x$

3. **If** $\boldsymbol{x^3 + y^3 = 16.}$ $\boldsymbol{Find\ the\ value\ of\ \frac{d^2y}{dx^2}}$ **at (2,2)**

Solution: Given $x^3 + y^3 = 16$

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \quad \Rightarrow \quad x^2 + y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^2}{y^2} \qquad \therefore \ \frac{dy}{dx}\ at\ (2,2) = -1$$

$$2x + y^2\frac{d^2y}{dx^2} + \frac{dy}{dx}2y\frac{dy}{dx} = 0$$

at (2,2) $\quad 4+4\frac{d^2y}{dx^2}+4(-1)^2=0$

$$4\frac{d^2y}{dx^2}=-8 \quad .\, Hence\ \frac{d^2y}{dx^2}=-2$$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=5yfh5cf4-0w

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Bali N. P Manish Goyal	A Text book of Engineering Mathematics, 9th edition	Laxmi Publications Pvt Ltd.	2.41-2.50

Tutorial on Derivatives

Introduction :

The **Sum rule** says the derivative of a sum of functions is the sum of their derivatives.

The **Difference rule** says the derivative of a difference of functions is the difference of their derivatives.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **derivative** of a constant is equal to zero. ...

The **derivative** of a constant multiplied by a function is equal to the constant multiplied by the **derivative** of the function. ...

The **derivative** of a sum is equal to the sum of the **derivatives**.

Detailed content of the Lecture:

4. **Find an equation of the tangent line to the curve $\boldsymbol{y=2x\,sinx\ \ at\ the\ point\ (\frac{\pi}{2},\pi)}$**

Solution: Step:1 Given $y=2x\,sinx$

$$y'=2[x\,cosx+sinx\,.1\,]=2x\,cosx+2\,sinx$$

$\boldsymbol{Step:2}$ $\qquad m=y'\ at\ \left(\frac{\pi}{2},\pi\right)=2\frac{\pi}{2}(0)+2(1)=2$

Step:3 Equation of the tangent line is $y-y_1=m(x-x_1)$

$$y=2x$$

5. **If $\boldsymbol{x^3+y^3=16.\,Find\ the\ value\ of\ \frac{d^2y}{dx^2}}$ at (2,2)**

Solution: **Step:1** Given $x^3+y^3=16$

$$3x^2+3y^2\frac{dy}{dx}=0 \qquad \Rightarrow\!\!\!\!\gg \qquad x^2+y^2\frac{dy}{dx}=0$$

Step: 2 $\Rightarrow \frac{dy}{dx} = \frac{-x^2}{y^2}$ $\therefore \frac{dy}{dx}$ $at\ (2,2) = -1$

Step:3 at (2,2) $4 + 4\frac{d^2y}{dx^2} + 4(-1)^2 = 0$

$4\frac{d^2y}{dx^2} = -8$.

$Hence\ \frac{d^2y}{dx^2} = -2$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=5yfh5cf4-0w

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	2.50-2.65

Topic of Lecture : Maxima and Minima of functions of one variable

Introduction :

The **maxima** of a **function** f(x) are all the points on the graph of the **function** which are 'local maximums'. ... We can visualise this as our graph having the peak of a 'hill' at x=a. Similarly, the **minima** of f(x) are the points for which, when we move a small amount to the left or right, the value of f(x) increases.

Prerequisite knowledge for Complete understanding and learning of Topic :

Find the first derivative of a **function** f(x) and **find** the critical numbers. Then, **find** the second derivative of a **function** f(x) and put the critical numbers. If the value is negative, the **function** has relative **maxima** at that point, if the value is positive, the **function** has relative **maxima** at that point.

Detailed content of the Lecture:

1. **Find the critical values of the function :** $\boldsymbol{f(x) = 2x^3 - 3x^2 - 36x}$

Solution: Given $f(x) = 2x^3 - 3x^2 - 36x$

critical values of f occurs at $f'(x) = 0$

$6x^2 - 6x - 36 = 0$

$x = -2\ and\ x = 3$

Hence the critical values are -2 ,3

2. **Find the absolute maximum and minimum values of** $\boldsymbol{f(x) = 2\cos x + \sin 2x, [0, \frac{\pi}{2}]}$

Solution: Given $\boldsymbol{f(x) = 2\cos x + \sin 2x, [0, \frac{\pi}{2}]}$

critical values of f occurs at $f'(x) = 0$

$$f^{'}(x) = -2\, sinx + 2\, cos2x$$

$$-2\, sinx + 2\, cos2x = 0$$

$$cos2x = sinx$$

$$x = \frac{\pi}{6}$$

$$critical\ value = \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \frac{3}{2}\sqrt{3} \quad is\ the\ absolute\ maxmimum\ value\ of\ f.$$

$$f\left(\frac{\pi}{2}\right) = 0 \quad is\ the\ absolute\ minimum\ \ value\ of\ f.$$

3.Show that 5 is a critical number of the function $f(x) = 2 + (x-5)^3$ but does not have a local extreme value at 5.

Solution: Given $f(x) = 2 + (x-5)^3$

critical values of f occurs at $f'(x) = 0$

$$f^{'}(x) = 3(x-5)^2$$

$$0 = 3(x-5)^2$$

Hence x= 5.

Therefore 5 is the critical value of the given function.

x	0	5	6	7	…
f(x)	-123	2	3	10	…

Therefore f(5) = 2 is not a local extreme.

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=lxksRct1QOY&list=PLVCBPCYGv7bC1JwOGH-X0FS-jTI4WRbI1

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	2.55-2.70

Topic of Lecture : Maxima and Minima of functions of one variable

Introduction :

The **maxima** of a **function** f(x) are all the points on the graph of the **function** which are 'local maximums'. ... We can visualise this as our graph having the peak of a 'hill' at x=a. Similarly, the **minima** of f(x) are the points for which, when we move a small amount to the left or right, the value of f(x) increases

Prerequisite knowledge for Complete understanding and learning of Topic :

Find the first derivative of a **function** f(x) and **find** the critical numbers. Then, **find** the second derivative of a **function** f(x) and put the critical numbers. If the value is negative, the **function** has relative **maxima** at that point, if the value is positive, the **function** has relative **maxima** at that point.

Detailed content of the Lecture:

1. **Answer the following questions about the functions whose derivatives are given**

   i) What are the critical point of f ?

   ii) on what interval is f increasing or decreasing?

   iii)at what points ,if any, does f assume local maximum and minimum values?

   iv)Find intervals of concavity and the inflection points.

   $\boldsymbol{f(x) = x^4 - 2x^2 + 3}$

Solution: Given $f(x) = x^4 - 2x^2 + 3$

i) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

critical points are occur a $f'(x) = 0$ t

$$f'(x) = 4x(x-1)(x+1) = 0$$

This implies that $x = 0\,, x = 1 \; and \; x = -1$

Hence the critical points are $x = 0\,, x = 1 \; and \; x = -1$

ii)

Interval	Sign of f'	Behavior of f
$-\infty < x < -1$	-	Decreasing
$-1 < x < 0$	+	Increasing
$0 < x < 1$	-	Decreasing
$1 < x < \infty$	+	Increasing

iii) The first derivative test tells us that

a) there is a **local minimum** of x at $x = \pm 1$ Hence $f(\pm 1) = 2$

b) there is a **local maximum** of x at $x = 0$ Hence $f(0) = 3$

iv) $f''(x) = 12x^2 - 4$ $\quad f''(x) = 0 \quad Hence \quad 0 = 12x^2 - 4$

$3x^2 - 1 = 0 \quad \therefore x = \pm\frac{1}{\sqrt{3}}$

Interval	Sign of f'	Behavior of f
$-\infty < x < -\frac{1}{\sqrt{3}}$	+	Concave up
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	-	Concave down
$\frac{1}{\sqrt{3}} < x < \infty$	+	Concave up

v) Inflection points are $(\pm\frac{1}{\sqrt{3}}, \frac{22}{9})$ since $f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{22}{9}$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=lxksRct1QOY&list=PLVCBPCYGv7bC1JwOGH-X0FS-jTI4WRbI1

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	2.55-2.70

Tutorial on Maxima and Minima of functions of one variable

Introduction :

The **maxima** of a **function** f(x) are all the points on the graph of the **function** which are 'local maximums'. ... We can visualise this as our graph having the peak of a 'hill' at x=a. Similarly, the **minima** of f(x) are the points for which, when we move a small amount to the left or right, the value of f(x) increases.

Prerequisite knowledge for Complete understanding and learning of Topic :

Find the first derivative of a **function** f(x) and **find** the critical numbers. Then, **find** the second derivative of a **function** f(x) and put the critical numbers. If the value is negative, the **function** has relative **maxima** at that point, if the value is positive, the **function** has relative **maxima** at that point.

Detailed content of the Lecture:

1. **Find the absolute maximum and minimum values of $\boldsymbol{f(x) = 2\cos x + \sin 2x, [0, \frac{\pi}{2}]}$**

Solution: **Step:1** Given $\boldsymbol{f(x) = 2\cos x + \sin 2x, [0, \frac{\pi}{2}]}$

critical values of f occurs at $f'(x) = 0$

$$-2\sin x + 2\cos 2x = 0$$

$$critical\ value = \frac{\pi}{6}$$

Step:2 $f\left(\frac{\pi}{6}\right) = \frac{3}{2}\sqrt{3}$ $\quad$ *is the absolute maxmimum value of f.*

$f\left(\frac{\pi}{2}\right) = 0$ *is the absolute minimum value of* f.

2.Answer the following questions about the functions whose derivatives are given

i) What are the critical point of f ?

ii) on what interval is f increasing or decreasing?

iii)at what points ,if any, does f assume local maximum and minimum values?

iv)Find intervals of concavity and the inflection points.

$\boldsymbol{f(x) = x^4 - 2x^2 + 3}$

Solution: Given $f(x) = x^4 - 2x^2 + 3$

Step i) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

critical points are occur a $f'(x) = 0$

Hence the critical points are $x = 0, x = 1$ *and* $x = -1$

Step ii)

Interval	Sign of f'	Behavior of f
$-\infty < x < -1$	-	Decreasing
$-1 < x < 0$	+	Increasing
$0 < x < 1$	-	Decreasing
$1 < x < \infty$	+	Increasing

Step iii) The first derivative test tells us that

a) there is a **local minimum** of x at $x = \pm 1$ Hence $f(\pm 1) = 2$

b) there is a **local maximum** of x at $x = 0$ Hence $f(0) = 3$

Step iv) $f''(x) = 12x^2 - 4$ $f''(x) = 0$ $Hence$ $0 = 12x^2 - 4$

$3x^2 - 1 = 0$ $\therefore x = \pm \frac{1}{\sqrt{3}}$

Interval	Sign of f'	Behavior of f
$-\infty < x < -\frac{1}{\sqrt{3}}$	+	Concave up
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	-	Concave down
$\frac{1}{\sqrt{3}} < x < \infty$	+	Concave up

Inflection points are $\left(\pm \frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ since $f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{22}{9}$

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=lxksRct1QOY&list=PLVCBPCYGv7bC1JwOGH-X0FS-jTI4WRbI1

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd	Khanna Publications,	2.60-2.75

		Edition	Delhi	

Topic of Lecture : Mean Value Theroem

Introduction :

Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at x=c must be parallel. We can see this in the following sketch. Let's now take a look at a couple of examples using the *Mean Value Theorem*.

Prerequisite knowledge for Complete understanding and learning of Topic :

The **Mean Value Theorem** states that if a function f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there exists a point c in the interval (a,b) such that f'(c) is equal to the function's average rate of change over [a,b].

Detailed content of the Lecture:

1. **Verify mean value theorem for the function $\boldsymbol{f(x) = x^2 + 3x + 2 \, , 1 \le x \le 2}$**

Solution: Given $f(x) = x^2 + 3x + 2 \, , 1 \le x \le 2$

Here a= 1 and b = 2

$$f'(x) = 2x + 3 \;\; , f'(c) = 2c + 3$$

Also $f(b) = 2^2 + 3(2) + 2 = 12$

$$f(a) = 1^2 + 3(1) + 2 = 6$$

Now , $f(b) - f(a) = (b - a)f'(c)$

$$12 - 6 = (2 - 1)(2c + 3)$$

$$6 = (2c + 3)$$

c= $\frac{3}{2}$, ie., 1< $\frac{3}{2} < 2$

Hence mean value theorem is verified

2. **Verify mean value theorem for the function $\boldsymbol{f(x) = e^{-2x}, [0, 3]}$**

Solution: Given $\boldsymbol{f}(x) = e^{-2x}, [0,3]$ Here a=0 , b=3

$$f'(x) = (-2)\, e^{-2x}$$

Also $f(b) = f(3) = e^{-6}$, $f(a) = f(0) = e^{-0} = 1$

Now , $f(b) - f(a) = (b - a)f'(c)$

$$e^{-6} - 1 = (3 - 0)(-2)\, e^{-2c}$$

$$\frac{-1}{6} \quad (e^{-6} - 1) = e^{-2c}$$

$$\log(\frac{1}{6} \quad (1 - e^{-6}) = loge^{-2c}$$

$$c = -\frac{1}{2}\log(\frac{1}{6}(1 - e^{-6}))$$

ie., 0 < c < 3 , since c = 0.3896.

Hence mean value theorem is verified

Video Content / Details of website for further learning (if any):

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https://www.youtube.com/watch?v=6BikQbeW32k(mean value theorem)

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	2.71-2.85

Unit - III

Functions of Several Variables

Topic of Lecture : Functions of Two variables

Introduction :

A derivative is a contract between two parties which derives its value/price from an underlying asset. The most common types of derivatives are futures, options, forwards and swaps.

Prerequisite knowledge for Complete understanding and learning of Topic :

Description: It is a financial instrument which derives its value/price from the underlying assets.

Detailed content of the Lecture:

1. If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

$$u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$u(tx, ty, tz) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$= u(x, y, z) = t^0 u(x, y, z)$$

∴ $u(x, y, z)$ is a homogeneous function of degree n = 0

∴ By Eular's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

2. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, $find$ $\frac{\partial(u,v)}{\partial(x,y)}$

$$u = \frac{y^2}{2x}\ ,\ v = \frac{x^2+y^2}{2x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2+y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$$= \frac{-y^3}{2x^3} - \frac{y}{x}\left(\frac{x^2+y^2}{2x^2}\right)$$

$$= -\frac{y^3}{2x^3} - \frac{yx^2}{2x^3} - \frac{y^3}{2x^3}$$

$$= \frac{-2y^3 - yx^2}{2x^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-(2y^3+yx^2)}{2x^3}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=dKPZL3T4NMk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Jain R.K. , Iyengar S.R.K.	Advanced Engineering Mathematics, 4th edition	Alpha Science International Ltd	R-214-225

Topic of Lecture : Taylor's Series

Introduction:

In mathematics, the **Taylor series** of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. ...

Prerequisite knowledge for Complete understanding and learning of Topic :

The partial sum formed by the n first terms of a **Taylor series** is a polynomial of degree n that is called the nth **Taylor** polynomial of the function.

Detailed content of the Lecture:

1. ***Expand $e^x \log(1+y)$ in powers x and y upto terms of third degree.***

Solution:

Function	Value at (0,0)
$f(x,y) = e^x \log(1+y)$	$f(0,0) = 0$
$f_x(x,y) = e^x \log(1+y)$	$f_x(0,0) = 0$
$f_{xx}(x,y) = e^x \log(1+y)$	$f_{xx}(0,0) = 0$
$f_{xxx}(x,y) = e^x \log(1+y)$	$f_{xxx}(0,0) = 0$
$f_y(x,y) = e^x . \frac{1}{1+y}$	$f_y(0,0) = 1$
$f_{yy}(x,y) = \frac{-e^x}{(1+y)^2}$	$f_{yy}(0,0) = -1$
$f_{yyy}(x,y) = 2\frac{e^x}{(1+y)^3}$	$f_{yyy}(0,0) = 2$

$f_{xy}(x,y) = \dfrac{e^x}{1+y}$	$f_{xy}(0,0) = 1$
$f_{xxy}(x,y) = \dfrac{e^x}{1+y}$	$f_{xxy}(0,0) = 1$
$f_{xyy}(x,y) = \dfrac{-e^x}{(1+y)^2}$	$f_{xyy}(0,0) = -1$

…(A)

We know that by Taylor's series

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0)$$
$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots\ldots\ldots\ldots(B)$$

Substituting (A) in (B),

$$f(x,y) = 0 + x\times 0 + y\times 1 + \frac{1}{2!}[x^2\times 0 + 2xy\times 1 + y^2\times -1] + \frac{1}{3!}[x^3\times 0 + 3x^2 y\times 1 + 3xy^2\times -1 + y^3\times 2] + \ldots\ldots\ldots$$

$$\therefore e^x\log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}(x^2 y + xy^2) + \frac{1}{3}y^3 + \ldots\ldots\ldots$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=19x213y_uk4 (Taylor's series)

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	2.85-2.96

Tutorial on Function of Two Variables

Introduction: A derivative is a contract between two parties which derives its value/price from an underlying asset. The most common types of derivatives are futures, options, forwards and swaps.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Differentiation
2.Partial derivative

Detailed content of the Lecture:

1. **If $u = f(x-y,\ y-z,\ z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$**

Solution: **Step:1** $\frac{\partial u}{\partial x} = (y-z)[(x=y)(z-x)]$
$= (y-z)(z-x) - (y-z)(x-y)$

Step:2 $\frac{\partial u}{\partial y} = (z-x)[(x-y)(y-z)]$

$= (z-x)(x-y) - (z-x)(y-z)$

Step:3 $\frac{\partial u}{\partial z} = (x-y)[(y-z)(z-x)]$

$= (x-y)(y-z) - (y-z)(z-x)$

Hence $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

2. If $u = x^y$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution: **Step:2** $u = x^y$

$u = e^y . logx$ $[a^x = e^{xloga}]$

$\frac{\partial u}{\partial y} = e^{ylogx} . logx$

Step:2 $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial y})$

$= \frac{x^y}{x} + x^y \frac{y}{x} \; logx$ $[x^y = e^{ylogx}]$

$= \frac{x^y}{x}[1 + ylogx]$ ---(1)

$u = e^y . logx$

$\frac{\partial u}{\partial x} = e^{ylogx} \; \frac{y}{x}$

Step:3 $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial x})$

$= \frac{x^y}{x} + x^y \frac{y}{x} \; logx$

$= \frac{x^y}{x}[1 + ylogx]$ ---(2)

From (1)&(2) we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	3.20-3.35

Topic of Lecture : Taylor's Series

Introduction :

In mathematics, the **Taylor series** of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. ...

Prerequisite knowledge for Complete understanding and learning of Topic :

The partial sum formed by the n first terms of a **Taylor series** is a polynomial of degree n that is called the nth **Taylor** polynomial of the function.

Detailed content of the Lecture:

1. Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.

Solution:

Function	**Value at (0,0)**
$f(x,y) = e^x \sin y$	$f(0,0) = 0$
$f_x(x,y) = e^x \sin y$	$f_x(0,0) = 0$
$f_{xx}(x,y) = e^x \sin y$	$f_{xx}(0,0) = 0$
$f_{xxx}(x,y) = e^x \sin y$	$f_{xxx}(0,0) = 0$
$f_y(x,y) = e^x \cos y$	$f_y(0,0) = 1$
$f_{yy}(x,y) = -e^x \sin y$	$f_{yy}(0,0) = 0$
$f_{yyy}(x,y) = -e^x \cos y$	$f_{yyy}(0,0) = -1$
$f_{xy}(x,y) = e^x \cos y$	$f_{xy}(0,0) = 1$
$f_{xxy}(x,y) = e^x \cos y$	$f_{xxy}(0,0) = 1$
$f_{xyy}(x,y) = -e^x \sin y$	$f_{xyy}(0,0) = 0$

We know that by Taylor's series

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0)$$

$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \ldots\ldots\ldots\ldots \text{(B)}$$

Substituting (A) in (B),

$$f(x,y) = 0 + x \times 0 + y \times 1 + \frac{1}{2!}[x^2 \times 0 + 2xy \times 1 + y^2 \times 0] + \frac{1}{3!}[x^3 \times 0 + 3x^2 y \times 1 + 3xy^2 \times 0 + y^3 \times -1] + \ldots\ldots\ldots$$

$$\therefore e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots\ldots\ldots$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=19x213y_uk4 (Taylor's series)

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	2.85-2.96

Topic of Lecture : Partial Derivatives

Introduction :

A ***partial derivative*** of a function of several variables is its ***derivative*** with respect to one of those variables, with the others held constant (as opposed to the total ***derivative***, in which all variables are allowed to vary)

Prerequisite knowledge for Complete understanding and learning of Topic :

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.

$$\frac{\partial f}{\partial x_m} \equiv \lim_{h\to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}.$$

Detailed content of the Lecture:

1. ***If*** $\boldsymbol{u = cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)}$ ***then prove that*** $\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}cot\, u}$

Solution: $u = cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$cos\, u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

cos u is a homogeneous function of x and y of degree $\frac{1}{2}$.

By Euler's theorem $x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = \frac{1}{2}\cos u$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u$$

2. ***If*** $\boldsymbol{u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)}$ ***then prove that*** $\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin\, 2u}$

Solution:

$$u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\tan u = \left(\frac{x^3+y^3}{x-y}\right)$$

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}$$

$$= x^2 \; f \; \left(\frac{x}{y}\right)$$

$\therefore \tan u$ is homogeneous function of degree 2.

By Euler's theorem

$$x\frac{\partial(\tan\ u)}{\partial x} + y\frac{\partial(\tan\ u)}{\partial y} = 2\tan u$$

$$x.sec^2u.\frac{\partial u}{\partial x} + y.sec^2u.\frac{\partial u}{\partial y} = 2\tan u$$

$$sec^2u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan\ u}{sec^2\ u}$$

$$= 2\frac{\sin\ u}{\cos\ u} \times cos^2\ u$$

$$= 2\sin u\cos u$$

$$= \sin 2u$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=dKPZL3T4NMk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.10-3.35

Tutorial on *partial derivative*

Introduction: *A partial derivative* of a function of several variables is its *derivative* with respect to one of those variables, with the others held constant (as opposed to the total *derivative*, in which all variables are allowed to vary)

Prerequisite knowledge for Complete understanding and learning of Topic : Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.

$$\frac{\partial f}{\partial x_m} \equiv \lim_{h\to 0}\frac{f(x_1, \ldots, x_m+h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}.$$

Detailed content of the Lecture:

1. **Find $\frac{du}{dt}$ if $u = \sin\left(\frac{x}{y}\right)$ where $x = e^t, y = t^2$**

Solution:

Step:1 $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t, y = t^2$

Step:2 $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\frac{du}{dt} = e^t\left[\frac{t-2}{t^3}\right]cos\left[\frac{e^t}{t^2}\right]$$

2. **Using the definition of total derivative, find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax\ (or) u = x^2 + y^2, \quad x = at^2, y = 2at$**

Solution:

Step:1 $u = y^2 - 4ax,$ $\quad x = at^2,$ $\quad y = 2at$

Step:2 $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= -4a(2at) + 2y(2a)$$

$$= -8a^2 + 2(2at)(2a)$$

$$= -8a^2t + 8a^2t$$

$$\frac{du}{dt} = 0$$

3. **If $u = x^3y^2 + x^2y^3$ where $x = at^2\ and\ y = 2at$ then find $\frac{du}{dt}$?**

Solution:

Step:1 $$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$u = x^3y^2 + x^2y^3$$

Step:2 $\frac{\partial u}{\partial x} = 3x^2y^2 + 2xy^3$ $\qquad \frac{\partial u}{\partial y} = 2x^3y + 3x^2y^2$

$x = at^2$ $\qquad y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad \frac{dy}{dt} = 2a$

Step:3 $$\frac{du}{dt} = (3x^2y^2 + 2xy^3)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= [3(at^2)^2(2at)^2 + 2(at^2)(2at)^3](2at) + [2(at^2)^3 2at + 3(at^2)^2(2at)^2]2a$$

$$\frac{du}{dt} = 56a^5t^6(t + 1)$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	3.45-3.65

Topic of Lecture : Partial Derivatives

Introduction :

A ***partial derivative*** of a function of several variables is its ***derivative*** with respect to one of those variables, with the others held constant (as opposed to the total ***derivative***, in which all variables are allowed to vary)

Prerequisite knowledge for Complete understanding and learning of Topic :

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.

$$\frac{\partial f}{\partial x_m} \equiv \lim_{h\to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}.$$

Detailed content of the Lecture:

1. **If** $\boldsymbol{u = f(x,y), then\ \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}}$

Solution: we have $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

Put t=x , we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

2. **Find** $\boldsymbol{\frac{dy}{dx}\ when\ x^3 + y^3 = 3axy}$

Solution:

Let $\boldsymbol{f(x,y) = x^3 + y^3 - 3axy}$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=dKPZL3T4NMk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.10-3.35

Topic of Lecture : Jacobians

Introduction :

Prerequisite knowledge for Complete understanding and learning of Topic :

Detailed content of the Lecture:

1. ***If $y_1 = \frac{x_2\, x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$. Show that the Jacobian of $y_{1,} y_{2,} y_3$ with respect to $x_{1,} x_{2,} x_3$ is 4.***

Solution:

$y_1 = \frac{x_2\, x_3}{x_1}$	$y_2 = \frac{x_3 x_1}{x_2}$	$y_3 = \frac{x_1 x_2}{x_3}$
$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{{x_1}^2}$ $\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$ $\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$	$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$ $\frac{\partial y_2}{\partial x_2} = -\frac{x_3 x_1}{{x_2}^2}$ $\frac{\partial y_2}{\partial x_3} = \frac{\partial x_1}{\partial x_2}$	$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$ $\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$ $\frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{{x_3}^2}$

$$\frac{\partial (y_{1,} y_{2,} y_3)}{\partial (x_{1,} x_{2,} x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix} \qquad \text{[Using (A)}$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

= -1(1-1)-1(-1-1)+1(1+1)

= 0+2+2

= 4

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=UWiPnPITFfI (Jacobian & properties)

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.45-3.55

Topic of Lecture : Maxima and Minima

Introduction :

For a **function** of one variable, f(x), we find the local **maxima**/**minima** by differentiation. **Maxima/minima** occur when f (x) = 0. x = a is a maximum if f (a) = 0 and f (a) < 0; • x = a is a minimum if f (a) = 0 and f (a) > 0; A point where f (a) = 0 and f (a) = 0 is called a point of inflection.

Prerequisite knowledge for Complete understanding and learning of Topic :

Then the first step is to **find** the critical points x=a, where f′(a)=0. Just because f′(a)=0, it does not mean that f(x) has a local maximum or minimum at x=a. But, at all **extrema**, the derivative will be zero, so we know that the **extrema** must occur at critical points.

Detailed content of the Lecture:

1. ***Find the absolute maximum and minimum value of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on triangular plate in the first quadrant, bounded by the lines x=0, y=0 and y=9-x.***

Solution:

$f(x, y) = 2 + 2x + 2y - x^2 - y^2$	
$\frac{\partial f}{\partial x} = 2 - 2x$ $\frac{\partial f}{\partial x} = 0$ 2-2x=0 x=1	$\frac{\partial f}{\partial y} = 2 - 2y$ $\frac{\partial f}{\partial y} = 0$ 2-2y=0 y=1

Turning point is (1,1)

	At (1,1)
$r = \frac{\partial^2 f}{\partial x^2} = -2$	$-2 < 0$
$s = \frac{\partial^2 f}{\partial x \partial y} = 0$	0
$t = \frac{\partial^2 f}{\partial y^2} = -2$	$-2 < 0$
$rt - s^2$	$4 - 0 > 0$
Result:	$r < 0$ $rt - s^2 > 0$ ∴(1,1) – maximum point

Maximum value of '*f*' is = 2+2+2-1-1 = 4

2. ***Examine $f(x, y) = x^3 + y^3 - 3xy$ for maximum and minimum values.***

Solution:

$f(x, y) = x^3 + y^3 - 3xy$	
$\frac{\partial f}{\partial x} = 3x^2 - 3y$ $\frac{\partial f}{\partial x} = 0$ $3x^2 - 3y = 0$ $x^2 = y$ …(1)	$\frac{\partial f}{\partial y} = 3y^2 - 3x$ $\frac{\partial f}{\partial y} = 0$ $3y^2 - 3x = 0$ $y^2 = x$ …(2)

To find turning points:

(1) ⇨ $y = x^2$ …(3)

(2) ⇨ $y^2 = x$ …(4)

(3) ⇨ $x^4 = y^2$ …(5)

Substituting (4) in (5), we get

$$x^4 = x$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$
$$x = 0 \; or \; 1$$

Put x = 0 in (3), y = 0

x = 1 in (3), y = 1

Turning points are (0,0), (1,1)

	At (0,0)	At (1,1)
$r = \frac{\partial^2 f}{\partial x^2} = 6x$	0	6
$s = \frac{\partial^2 f}{\partial x \partial y} = -3$	-3	-3
$t = \frac{\partial^2 f}{\partial y^2} = 6y$	0	6
$rt - s^2$	$-9 > 0$	$36 - 9 = 27 > 0$
Result:	$r = 0$ $rt - s^2 > 0$ No extremum value	$r < 0$ $rt - s^2 > 0$ (1,1)–maximum point

Maximum value of '*f*' is = 1+1-3
= -1

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=SRfb-AjDCoc

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.45-3.55

Tutorial on Maxima and Minima

Introduction: To Introduce the jacobians to find the given functions u , v with respect to x,y

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Determinant
2. Differentiation

Detailed content of the Lecture:

1. Write the sufficient condition for $\boldsymbol{f(x, y)}$ to have a maximum value at $\boldsymbol{(a, b)}$

Solution:

Step:1 If $f_x(a, b) = 0, f_y(a, b) = 0 \; and \; f_{xx}(a, b) = A, f_{xy}(a, b) = B, f_{yy} = C$

Step:2 Then if $AC - B^2 > 0\ and\ A > 0\ \ (or) B > 0$.

2. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, $find\ \frac{\partial(u,v)}{\partial(x,y)}$ (May/June 2012, Jan/Feb 2010)

Solution:

Step:1 $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$

Step:2 $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2+y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$$= \frac{-y^3}{2x^3} - \frac{y}{x}\left(\frac{x^2+y^2}{2x^2}\right)$$

$$= -\frac{y^3}{2x^3} - \frac{yx^2}{2x^3} - \frac{y^3}{2x^3}$$

$$= \frac{-2y^3 - yx^2}{2x^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-(2y^3 + yx^2)}{2x^3}$$

3. If $x = rcos\theta$, $y = rsin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ $(or) \frac{\partial(r,\theta)}{\partial(x,y)}$

Solution:

Step:1 $x = rcos\theta$, $y = rsin\theta$

Step:2 $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= r(cos^2\theta + sin^2\theta)$$

$$= r$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	

Topic of Lecture : Lagrange's multipliers method

Introduction :

For a **function** of one variable, f(x), we find the local **maxima/minima** by differentiation. **Maxima/minima** occur when f (x) = 0. x = a is a maximum if f (a) = 0 and f (a) < 0; • x = a is a minimum if f (a) = 0 and f (a) > 0; A point where f (a) = 0 and f (a) = 0 is called a point of inflection.

Prerequisite knowledge for Complete understanding and learning of Topic :

Then the first step is to **find** the critical points x=a, where f′(a)=0. Just because f′(a)=0, it does not mean that f(x) has a local maximum or minimum at x=a. But, at all **extrema**, the derivative will be zero, so we know that the **extrema** must occur at critical points.

Detailed content of the Lecture:

1. ***A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box that requires the least material for its construction.***

Solution:

Let x,y,z be the length, breadth and height of the box respectively. When it requires least material, the surface area of the box should be least.

The surface area S = xy+2yz+2zx. Hence we have to minimize 'S' subject to the condition that the volume of the box xyz = 32.

F= (xy+2yz+2zx) + λ (xyz-32) ….(1)		
$\frac{\partial F}{\partial x} = 0$ $y + 2z + \lambda yz = 0$ $y + 2z = -\lambda yz$ $-\lambda = \frac{1}{z} + \frac{2}{y}$	$\frac{\partial F}{\partial y} = 0$ $x + 2z + \lambda zx = 0$ $x + 2z = -\lambda zx$ $-\lambda = \frac{1}{z} + \frac{2}{x}$	$\frac{\partial F}{\partial z} = 0$ $2x + 2z + \lambda xy = 0$ $2x + 2z = -\lambda xy$ $-\lambda = \frac{2}{y} + \frac{2}{x}$
(2)	(3)	(4)
$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz - 32 = p$ ….(5)		

From (2) and (3),

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

x=y …(6)

From (3) and (4),

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$\frac{1}{z} = \frac{2}{y}$$

y= 2z …(7)

∴ $x = y = 2z$ …(8)

Substituting (8) in (5),

(2z)(2z)(z) = 32

$4x^3$ = 32

$x^3 = \frac{32}{4} = 8$

x = 2

∴ y = 4, x = 4 [From (8)]

∴ The dimensions of the box are 4, 4, 2.

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=SRfb-AjDCoc

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.45-3.55

Topic of Lecture : Lagrange's Multipliers method

Introduction :

Prerequisite knowledge for Complete understanding and learning of Topic :

Detailed content of the Lecture:

1. ***Find the maximum value of $x^m y^n z^p$ when x+y+z = a.***

Solution:

Let $f = x^m y^n z^p$ and $g = x + y + z = a$

$$F = (x^m y^n z^p) + \lambda\,(x+y+z = a) \qquad \ldots(1)$$

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$mx^{m-1}y^n z^p + \lambda = 0$	$ny^{-1}x^m z^p + \lambda = 0$	$px^m y^n z^{p-1} + \lambda = 0$
$-\lambda = mx^{m-1}y^n z^p$	$-\lambda = ny^{-1}x^m z^p$	$-\lambda = px^m y^n z^{p-1}$
(2)	(3)	(4)

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x + y + z - a = \mathrm{p} \qquad \ldots(5)$$

From (2), (3) and (4), we get

$$mx^{m-1}y^n z^p = ny^{-1}x^m z^p = px^m y^n z^{p-1}$$

$\div$ by $x^m y^n z^p$, $\frac{mx^{m-1}y^n z^p}{x^m y^n z^p} = \frac{ny^{-1}x^m z^p}{x^m y^n z^p} = \frac{px^m y^n z^{p-1}}{x^m y^n z^p}$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$= \frac{m+n+p}{x+y+z}$$

$$= \frac{m+n+p}{a}$$

∴Hence maximum value of *f* occurs when

$x = \frac{am}{m+n+p}$ [Taking 1st and last] …(6)

$y = \frac{an}{m+n+p}$ [Taking 2nd and last] …(7)

$z = \frac{ap}{m+n+p}$ [Taking 3rd and last] …(8)

Substituting (6), (7), (8) in $f = x^m y^n z^p$, the maximum value of

$$f = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= a^{m+n+p} \frac{m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=N_CyeSqqYs4(lagrange's method of multipliers)

Important Books/Journals for further learning including the page nos.:

Sl.No	**Author(s)**	**Title of the Book**	**Publisher**	**Page.No**
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	3.71-3.85

Unit - IV
Integral Calculus

Topic of Lecture : Definite and Indefinite Integrals

Introduction: Integral Calculus is the study of finding a function based on the information about its rate of change.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Concept of integration
- Integral Mean Value Theorem
- Fundamental Theorem of Calculus

Detailed content of the Lecture:

Theorem 1:

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuous, then f is integrable on $[a, b]$

i.e., $\int_a^b f(x)\ dx$ exists.

Theorem 2:

If f is integrable on $[a, b]$, then $\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

1. **Evaluate $\int_0^3 (x^2 - 2x)dx$ by using Riemann sum by taking right end points as the sample points.**

   Solution:

   Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$$x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}, x_3 = \frac{9}{n}, \ldots, x_i = \frac{3i}{n}, \ldots x_n = \frac{3x}{n}$$

Since we are using right end points.

$$\int_0^3 (x^2 - 2x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^n \left[\frac{9i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^n i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^n i$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2}\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1 + \frac{1}{n}\right]\left[2 + \frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1 + \frac{1}{n}\right]$$

$$= \frac{27}{6}(1)(2) - 9(1) = 9 - 9 = 0$$

Video Content / Details of website for further learning (if any):
https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	4.15-4.25

Topic of Lecture : Definite and Indefinite Integrals

Introduction: The definite integral of f(x) is a NUMBER and represents the area under the curve f(x) from x=a to x=b. The indefinite integral of f(x) is a FUNCTION and answers the question, "What function when differentiated gives f(x)?"

Prerequisite knowledge for Complete understanding and learning of Topic :

- Concept of integration
- Integral Mean Value Theorem
- Fundamental Theorem of Calculus

Detailed content of the Lecture:

1. $\int x^{-4}dx$

Solution:

$$\int x^{-4}dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\text{C}$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{5}x^{5/2} + C$$

2. $\int \frac{x^3+2x+1}{x^4}dx$

Solution:

$$\int \frac{x^3+2x+1}{x^4}dx = \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right)dx$$

$$= \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right)dx$$

$$= \log x + 2\frac{x^{-2}}{(-2)} + \frac{x^{-3}}{(-3)}+\text{C}$$

$$= \log x - \frac{1}{x^2} - \frac{1}{3x^3} + C$$

3. $\int (u+4)(2u+1)du$

Solution:

$$\int(u+4)(2u+1)du = \int[2u^2+u+8u+4]du$$

$$= \int(2u^2+9u+4)\boldsymbol{du}$$

$$= \mathbf{2}\frac{u^3}{3} + \mathbf{9}\frac{u^2}{2}\ \mathbf{4u} + \boldsymbol{C}$$

4. $\int\left(\boldsymbol{ax}+\frac{\boldsymbol{b}}{x^2}\right)\boldsymbol{dx}$

Solution:

$$\int\left(\boldsymbol{ax}+\frac{\boldsymbol{b}}{x^2}\right)\boldsymbol{dx} = \int(\boldsymbol{ax}+\boldsymbol{bx}^{-2})\boldsymbol{dx}$$

$$= \boldsymbol{a}\frac{x^2}{2} - \boldsymbol{bx}^{-1} + \boldsymbol{C}$$

Video Content / Details of website for further learning (if any):
https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	4.26-4.35

Tutorial on Definite and Indefinite Integrals

Introduction: Integral Calculus is the study of finding a function based on the information about its rate of change.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Concept of integration
- Integral Mean Value Theorem
- Fundamental Theorem of Calculus

Detailed content of the Lecture:

2. **Evaluate $\int_0^3(x^2-2x)dx$ by using Riemann sum by taking right end points as the sample points.**

Solution $\boldsymbol{Step}$**: 1** $x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}, x_3 = \frac{9}{n}, \ldots, x_i = \frac{3i}{n}, \ldots x_n = \frac{3x}{n}$
Since we are using right end points.

$$\boldsymbol{Step}\mathbf{:2}\ \int_0^3(x^2-2x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} \left[\frac{9i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1+\frac{1}{n}\right]\left[2+\frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1+\frac{1}{n}\right] \qquad = \frac{27}{6}(1)(2) - 9(1) = 9 - 9 = 0$$

$$2.\ Evaluate \int \frac{x^3 + 2x + 1}{x^4} dx$$

Solution:

Step:1 $\int \frac{x^3+2x+1}{x^4} dx = \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right) dx$

$$= \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right) dx$$

Step:2 $= \log x + 2\frac{x^{-2}}{(-2)} + \frac{x^{-3}}{(-3)} + C$

$$= \log x - \frac{1}{x^2} - \frac{1}{3x^3} + C$$

Video Content / Details of website for further learning (if any):
https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	4.10-4.15

Topic of Lecture : Substitution Rule

Introduction: The method of substitution depends on finding a suitable substitution to convert the given integral into a standard form.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Integration
- Differentiation
- Properties of Integrals

Detailed content of the Lecture:

1. Let $I = \int \frac{1}{(ax+b)^4} dx$

Solution:

Let $I = \int \frac{1}{(ax+b)^4} dx$

Put $u = ax + b;$ $\quad du = a\,dx;$ $\quad \frac{du}{a} = dx$

$$I=\int \frac{1}{u^4}\frac{du}{a} = \frac{1}{a}\int u^{-4}\,du = \frac{1}{a}\left(\frac{u^{-3}}{-3}\right)+C$$

2. Let $I=\int (x+1)\sqrt{2x+x^2}\;dx$

Solution:

Let $I=\int (x+1)\sqrt{2x+x^2}\;dx$

Put $u=2x+x^2$; $\quad du=(2+2x)dx=2(1+x)dx$; $\quad \frac{du}{2}=(1+x)dx$

$$\therefore \boldsymbol{I}=\int \sqrt{\boldsymbol{u}}\frac{du}{2}=\frac{1}{2}\int u^{1/2}\,du=\frac{1}{2}\frac{u^{3/2}}{(3/2)}+C$$

$$=\frac{1}{3}\boldsymbol{u}^{3/2}+\boldsymbol{C} \;=\; \frac{1}{3}(\boldsymbol{2x}+\boldsymbol{x^2})^{3/2}+\boldsymbol{C}$$

3. Let $I=\int \frac{x^2}{\sqrt{x+5}}\,dx$

Solution:

Let $I=\int \frac{x^2}{\sqrt{x+5}}\,dx$

Put $u=\sqrt{x+5}$; $du=\frac{1}{2\sqrt{x+5}}\,dx$; $\quad 2du=\frac{1}{\sqrt{x+5}}\,dx$; $\quad dx=2u\;du$

$$\boldsymbol{u^2}=\boldsymbol{x}+\boldsymbol{5} \implies \boldsymbol{x}=\boldsymbol{u^2}-\boldsymbol{5} \Rightarrow \boldsymbol{x^2}=(\boldsymbol{u^2}-\boldsymbol{5})^2=\boldsymbol{u^4}-\boldsymbol{10u^2}+\boldsymbol{25}$$

$$\therefore \boldsymbol{I}=\int(\boldsymbol{u^4}-\boldsymbol{10u^2}+\boldsymbol{25})\boldsymbol{2}\,\boldsymbol{du}=\boldsymbol{2}\int(\boldsymbol{u^4}-\boldsymbol{10u^2}+\boldsymbol{25})\boldsymbol{du}$$

$$=\boldsymbol{2}\left[\frac{\boldsymbol{u^5}}{\boldsymbol{5}}-\boldsymbol{10}\frac{\boldsymbol{u^3}}{\boldsymbol{3}}+\boldsymbol{25u}\right]+\boldsymbol{C}$$

$$=\frac{\boldsymbol{2}}{\boldsymbol{5}}(\boldsymbol{x}+\boldsymbol{5})^{5/2}-\frac{\boldsymbol{20}}{\boldsymbol{3}}(\boldsymbol{x}+\boldsymbol{5})^{3/2}+\boldsymbol{50}(\boldsymbol{x}+\boldsymbol{5})^{1/2}+\boldsymbol{C}$$

Video Content / Details of website for further learning (if any):

https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	4.36-4.40

Topic of Lecture : Integration by parts

Introduction: Integration by parts is a technique for performing indefinite integration $\int udv$ or definite integration $\int_a^b udv$ by expanding the differential of a product of functions $d(uv)$ and expressing the original integral in terms of a known integral $\int v\,du$.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Product Rule for differentiation
- Definite Integrals
- Bernoulli's formula

Detailed content of the Lecture:

1. Evaluate $\int (\log x)^2 dx$

Solution:

$$\text{Let} \quad u = (\log x)^2 \qquad dv = dx$$
$$du = 2\log x\left(\frac{1}{x}\right)dx \qquad v = \int dx = x$$
$$\int udv = uv - \int vdu$$
$$\int (\log x)^2 dx = (\log x)^2 x - \int x\, 2\log x\left(\frac{1}{x}\right)dx$$
$$= (\log x)^2 x - 2\int \log x\, dx \text{--------------}(1)$$

Take $\int \log x\, dx$

$$\text{Let} \quad u = \log x \qquad dv = dx$$
$$du = \left(\frac{1}{x}\right)dx \qquad v = \int dx = x$$
$$\int udv = uv - \int vdu$$
$$\int \log x\, dx = (\log x)x - \int x\left(\frac{1}{x}\right)dx$$
$$= x(\log x) - \int dx$$
$$= x(\log x) - x + C$$
$$(1) \rightarrow \int (\log x)^2 dx = x(\log x)^2 - 2[x(\log x) - x] + C$$

2. Evaluate $\int e^x \cos x dx$

Solution:

$$I = \int e^x \cos x dx$$
$$\text{Let} \quad u = e^x \qquad dv = \cos x\, dx$$
$$du = e^x dx \qquad v = \int \cos x\, dx = \sin x$$
$$\int udv = uv - \int vdu$$
$$I = \int e^x \cos x dx = e^x \sin x - \int \sin x\, e^x dx \text{------------}(1)$$

Take $\int e^x \sin x\, dx$

$$\text{Let} \quad u = e^x \qquad dv = \sin x\, dx$$
$$du = e^x dx \qquad v = \int \sin x\, dx = -\cos x$$
$$\int udv = uv - \int vdu$$
$$\int e^x \sin x\, dx = e^x(-\cos x) - \int (-\cos x)\, e^x dx$$
$$= e^x(-\cos x) + \int \cos x\, e^x dx$$
$$= e^x(-\cos x) + I$$
$$(1) \rightarrow \qquad I = e^x \sin x - [e^x(-\cos x) + I] + C_1$$
$$I = e^x \sin x + e^x(\cos x) - I + C_1$$
$$2I = e^x \sin x + e^x(\cos x) + C_1$$
$$I = \frac{e^x}{2}[\sin x + \cos x] + C \quad \text{where } C = \frac{C_1}{2}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tGu-764KHCk

https://www.youtube.com/watch?v=cXgDjCO96Ug

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	

Topic of Lecture : Trigonometric Integrals

Introduction: In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called trigonometric integrals. They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Product Rule for differentiation
- Definite Integrals
- Bernoulli's formula
- Trigonometric Identities

Detailed content of the Lecture:

1. Evaluate $\int \tan^{-1}x\, dx$, Also find $\int_0^1 \tan^{-1}x\, dx$.

Solution:

Let $u = \tan^{-1}x \qquad dv = dx$

$$du = \frac{1}{1+x^2}dx \qquad v = \int dx = x$$

$$\int udv = uv - \int vdu$$

$$\int \tan^{-1}x\, dx = x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$$

$$= x\tan^{-1}x - \int \left(\frac{x}{1+x^2}\right)dx \qquad \text{--------------(1)}$$

Take $\int \left(\frac{x}{1+x^2}\right)dx$

Let $t = 1+x^2 \qquad dt = 2xdx$

$$\int \left(\frac{x}{1+x^2}\right)dx = \int \frac{1}{t}\frac{1}{2}dt = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\log t = \frac{1}{2}\log(1+x^2)$$

$$(1) \rightarrow \int \tan^{-1}x\, dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C \qquad \text{-------------(2)}$$

To find $\int_0^1 \tan^{-1}x\,dx$

$$(2) \rightarrow \int_0^1 \tan^{-1}x\,dx = [x\tan^{-1}x]_0^1 - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \tan^{-1}1 - 0 - \left[\frac{1}{2}\log 2 - \frac{1}{2}\log 1\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2 \qquad [\log 1=0]$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=lTqnlihOC4o

https://www.youtube.com/watch?v=flvhNBoOsiA

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	

Topic of Lecture : Trigonometric Integrals

Introduction: In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called trigonometric integrals. They are an important part of the integration technique called trigonometric substitution, which is featured in Trigonometric Substitution.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Product Rule for differentiation
- Definite Integrals
- Bernoulli's formula
- Trigonometric Identities

Detailed content of the Lecture:

1. Find the reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

Solution:

Let $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$---------(1)

$$(1) \rightarrow \qquad I_n = \left[\frac{-1}{n}\cos x \sin^{n-1}x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx$$

$$=(0\text{-}0)+ \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{n-2}=\frac{n-1}{n}I_{n-4}$$

$$I_n = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\ldots\ldots\frac{1}{2}I_0 & (\text{if } n \text{ is even}) \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\ldots\ldots\frac{2}{3}I_1 & (\text{if } n \text{ is odd}) \end{cases} \text{--------}(2)$$

From (2), put n=0,

$$I_0 = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = (-0) - (-1) = 1$$

$$(2) \to I_n = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\ldots\ldots\frac{1}{2}\frac{\pi}{2} & (\text{if } n \text{ is even}) \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\ldots\ldots\frac{2}{3} & (\text{if } n \text{ is odd}) \end{cases}.$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=lTqnlihOC4o

https://www.youtube.com/watch?v=flvhNBoOsiA

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1.	Grewal. B. S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	

Tutorial on Substitution Rule

Introduction: The method of substitution depends on finding a suitable substitution to convert the given integral into a standard form.

Prerequisite knowledge for Complete understanding and learning of Topic :

- Integration
- Differentiation
- Properties of Integrals

Detailed content of the Lecture:

1. Let $I = \int \frac{x^2}{\sqrt{x+5}}\, dx$

Solution:

Step:1 Let $I = \int \frac{x^2}{\sqrt{x+5}}\, dx$

Put $u = \sqrt{x+5}\,; du = \frac{1}{2\sqrt{x+5}}\, dx;\quad 2du = \frac{1}{\sqrt{x+5}}\, dx;\qquad dx = 2u\ du$

Step: 2 $\boldsymbol{u^2 = x+5 \implies x = u^2 - 5 \Rightarrow x^2 = (u^2-5)^2 = u^4 - 10u^2 + 25}$

Step:3 $\boldsymbol{\therefore I = \int (u^4 - 10u^2 + 25)2\, du = 2\int (u^4 - 10u^2 + 25) du}$

$$= 2\left[\frac{u^5}{5} - 10\frac{u^3}{3} + 25u\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + C$$

2. Evaluate $\int e^x \cos x dx$

Solution: **Step:1**

$$I = \int e^x \cos x dx$$

Let $u = e^x$ $\qquad dv = \cos x\, dx$

Step:2 $\quad du = e^x dx \qquad v = \int \cos x\, dx = \sin x$

$$\int u dv = uv - \int v du$$

$$I = \int e^x \cos x dx = e^x \sin x - \int \sin x\, e^x dx \text{------------}(1)$$

Take $\int e^x \sin x\, dx$

Step:3 $\quad$ Let $u = e^x \qquad dv = \sin x\, dx$

$$du = e^x dx \qquad v = \int \sin x\, dx = -\cos x$$

$$\int u dv = uv - \int v du$$

$$\int e^x \sin x\, dx = e^x(-\cos x) - \int(-\cos x)\, e^x dx$$

$$= e^x(-\cos x) + \int \cos x\, e^x dx$$

$$= e^x(-\cos x) + I$$

$(1) \rightarrow$

$$I = e^x \sin x - [e^x(-\cos x) + I] + C_1$$

$$I = e^x \sin x + e^x(\cos x) - I + C_1$$

$$2I = e^x \sin x + e^x(\cos x) + C_1$$

$$I = \frac{e^x}{2}[\sin x + \cos x] + C \quad \text{where } C = \frac{C_1}{2}$$

Video Content / Details of website for further learning (if any):
https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	4.30-4.45

Topic of Lecture : Integration of rational functions by partial fractions

Introduction: The integration of rational functions in one variable reduces, by the division algorithm, to that of proper fractions, which are then handled by expressing them as partial fractions.
Each proper fraction decomposes as a sum of simple proper fractions called partial fractions, each of which is easily integrated.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Partial fractions
2. Rational functions
3. Integration

4. Division algorithm

Detailed content of the Lecture:

1. Integration of rational functions by partial fraction $\int \frac{sec^2 x}{tan^2 x+3tanx+2}\, dx$

Solution:

Put $u = \tan x\,; du = \ sec^2 x\, dx$

$$\therefore I = \int \frac{1}{u^2 + 3u + 2}\, du$$

$$= \int \frac{1}{(u+1)(u+2)}\, du$$

$$= \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right)\, du$$

$$= \log(u+1) - \log(u+2) + c$$

$$= \log(\tan x + 1) - \log(\tan x + 2) + c$$

$$\int \frac{sec^2 x}{tan^2 x + 3tanx + 2}\, dx = \log\left(\frac{1+\tan x}{2+\tan x}\right) + c$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=daYVWmS9apI

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	

Topic of Lecture : Integration of rational functions by partial fractions

Introduction: The integration of rational functions in one variable reduces, by the division algorithm, to that of proper fractions, which are then handled by expressing them as partial fractions.
Each proper fraction decomposes as a sum of simple proper fractions called partial fractions, each of which is easily integrated.

Prerequisite knowledge for Complete understanding and learning of Topic :

5. Partial fractions
6. Rational functions
7. Integration

8. Division algorithm

Detailed content of the Lecture:

1. Integration of rational functions by partial fraction $\frac{1}{(x+1)(x+2)}$

Solution:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$1 = A(x+2) + B(x+1)$$

Put x = -1 , we get

$$1 = A(-1+2) + B(-1+1)$$

$$A = 1$$

Put x = -2 , we get

$$1 = A(-2+2) + B(-2+1)$$

$$B = -1$$

(1) $\Rightarrow$

$$(\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} + \frac{-1}{(x+2)}$$

$$\int \frac{1}{(x+1)(x+2)} dx = \int \frac{1}{(x+1)} dx + \int \frac{-1}{(x+2)} dx$$

$$= \log(x+1) - \log(x+2) + c$$

$$\frac{1}{(x+1)(x+2)} = \log\left(\frac{x+1}{x+2}\right) + c$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=daYVWmS9apI

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1.	Grewal. B. S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	

Tutorial on Integration of rational functions by partial fractions

Introduction: The integration of rational functions in one variable reduces, by the division algorithm, to that of proper fractions, which are then handled by expressing them as partial fractions.

Each proper fraction decomposes as a sum of simple proper fractions called partial fractions, each of which is easily integrated.

Prerequisite knowledge for Complete understanding and learning of Topic :

9. Partial fractions
10. Rational functions
11. Integration
12. Division algorithm

Detailed content of the Lecture:

2. Integration of rational functions by partial fraction $\frac{1}{(x+1)(x+2)}$

Solution:

Step:1 $\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$ ………………. (1)

$1 = A(x+2) + B(x+1)$

Put x = -1 , we get

$$1 = A(-1+2) + B(-1+1)$$

$$A = 1$$

Put x = -2 , we get

$$1 = A(-2+2) + B(-2+1)$$

$$B = -1$$

Step: 2 (1) $\Rightarrow$

$$(\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} + \frac{-1}{(x+2)}$$

$$\int \frac{1}{(x+1)(x+2)} dx = \int \frac{1}{(x+1)} dx + \int \frac{-1}{(x+2)} dx$$

$$= \log(x+1) - \log(x+2) + c$$

$$\frac{1}{(x+1)(x+2)} = \log\left(\frac{x+1}{x+2}\right) + c$$

2. Integration of rational functions by partial fraction $\int \frac{sec^2 x}{tan^2 x + 3tanx + 2} \, dx$

Solution: Step:1

Put $u = \tan x \, ; du = \, sec^2 x \, dx$

Step:2 $\therefore I = \int \frac{1}{u^2+3u+2} \, du$

$$= \int \frac{1}{(u+1)(u+2)} \, du$$

$$= \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du$$

$$= \log(u+1) - \log(u+2) + c$$

$$= \log(\tan x + 1) - \log(\tan x + 2) + c$$

$$\int \frac{sec^2 x}{tan^2 x + 3tanx + 2} \, dx = \log\left(\frac{1+\tan x}{2+\tan x}\right) + c$$

Video Content / Details of website for further learning (if any):
https://youtu.be/Zg4dJVvwRko

https://www.youtube.com/watch?v=bMnMzNKL9Ks

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	4.50-4.65

Topic of Lecture : Improper Integrals

Introduction: In a regular definite integral $\int_a^b f(x)dx$, it is assumed that the limit of integration are finite and that the integrand f(x) is continuous for every value of x in the interval $a \le x \le b$ .If atleast one of these conditions is violated, then the integral is known as an improper integral .

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Improper Integrals
2. Integration
3. Convergent & Divergent

Detailed content of the Lecture:

1. Evaluate $\int_1^\infty \frac{logx}{x}\ dx$

Solution:

Put $t = \log x\,; dt = \frac{1}{x}\ dx$

$$\therefore \int \frac{logx}{x} dx = \int t\ \ dt$$

$$= \frac{t^2}{2}$$

$$= \frac{(\log x)^2}{2}$$

$$\therefore \int_1^\infty \frac{logx}{x}\ dx = \left[\frac{(\log x)^2}{2}\right]_1^\infty$$

$$= \infty - 0$$

$$\therefore \int_1^\infty \frac{logx}{x}\ dx = \infty$$

$\therefore \int_1^\infty \frac{logx}{x}\ dx$ is divergent.

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=d_CnAKmQOKE

Important Books/Journals for further learning including the page nos.:

S.No	Author(s)	Title of the Book	Publisher	Page nos
1	James Stewart	Calculus with Early Transcendental Functions	Cengage Learning, New Delhi	

Unit - V
Multiple Integrals

Topic of Lecture : Double integrals in Cartesian coordinates

Introduction : When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Plane
2. Integration
3. Limit substitution

Detailed content of the Lecture:

1. **Evaluate** $\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dydx$

Solution :

$$\int_0^1 \int_0^{x^2} (x^2+y^2)\, dydx = \int_0^1 \left(\int_0^{x^2} (x^2+y^2)\, dy\right)dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{x^2}$$

$$= \int_0^1 (x^4 + \frac{x^6}{3})\, dx$$

$$= \left[\frac{x^5}{5} + \frac{1}{3}\frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{5} + \frac{1}{21}$$

$$= \frac{21+5}{105}$$

$$= \frac{26}{105}$$

2. **Evaluate** $\int_0^a \int_0^b (x + y)\, dxdy$

Solution :

$$\int_0^a \int_0^b (x+y)\, dxdy = \int_0^a \int_0^b (xdx + ydx)\mathrm{dy}$$

$$= \int_0^a \left[\frac{x^2}{2} + yx\right]_0^b dy$$

$$= \int_0^a [\frac{b^2}{2} + yb]\, dy$$

$$= \left[\frac{b^2 y}{2} + \frac{y^2 b}{2}\right]_0^a$$

$$= \frac{ab^2}{2} + \frac{a^2 b}{2}$$

$$= \frac{ab^2 + a^2b}{\mathbf{2}}$$

3. **Evaluate $\int_0^1 \int_0^{\sqrt{x}} xy(x+y)\, dxdy$.**

Solution :

$$\int_0^1 \int_0^{\sqrt{x}} xy(x+y)\, dxdy = \int_0^1 \int_0^{\sqrt{x}} (x^2y + xy^2)\, dydx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 [\frac{x^3}{2} + \frac{x^{\frac{5}{2}}}{3} - \frac{x^4}{2} - \frac{2x^4}{3}]\, dx$$

$$= \left[\frac{x^4}{8} + \frac{1}{3}\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{1}{2}\frac{x^5}{5} - \frac{1}{3}\frac{x^5}{5}\right]_0^1$$

$$= [\frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15}]$$

$$= \frac{3}{56}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=sQM-8Oj4Ecg

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9^{th} Edition	John Wiley and Sons, New Delhi	5.10-5.20

Topic of Lecture : Double integrals in Cartesian coordinates

Introduction : When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values.

Prerequisite knowledge for Complete understanding and learning of Topic :

4. Plane
5. Integration
6. Limit substitution

Detailed content of the Lecture:

1. **Evaluate $\int_0^a \int_0^b (x+y)\, dxdy$**

Solution :

$$\int_0^a \int_0^b (x+y)\, dxdy = \int_0^a \int_0^b (xdx + ydx)\mathrm{dy}$$

$$= \int_0^a \left[\frac{x^2}{2} + yx\right]_0^b dy$$

$$= \int_0^a [\frac{b^2}{2} + yb]\, dy$$

$$= \left[\frac{b^2 y}{2} + \frac{y^2 b}{2}\right]_0^a$$

$$= \frac{ab^2}{2} + \frac{a^2 b}{2}$$

$$= \frac{ab^2 + a^2 b}{\mathbf{2}}$$

2. **Evaluate $\int\int xy\, dxdy$ taken over the positive quadrant of the circle $x^2 + y^2 = a^2$**

Solution :

$$Given \quad x^2 + y^2 = a^2$$

$$x^2 = a^2 - y^2$$

$$x = \pm\sqrt{a^2 - y^2}$$

$$We\ need\ positive\ Quadrant \quad x = \sqrt{a^2 - y^2}$$

$$x \quad varies\ from\ 0\ to\ \sqrt{a^2 - y^2}, \quad y \quad varies\ from\ 0\ to\ a$$

$$Required\ area = \int_0^a \int_0^{\sqrt{a^2-y^2}} xydxdy$$

$$= \int_0^a \left[y\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{y(a^2 - y^2)}{2} dy$$

$$= \int_0^a \frac{(ya^2 - y^3)}{2} dy$$

$$= \frac{1}{2}\left[\frac{a^2 y^2}{2} - \frac{y^4}{2}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^4}{8} \quad sq.u$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=sQM-8Oj4Ecg

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Erwin Kreyszig	Advanced Engineering Mathematics, 9th Edition	John Wiley and Sons, New Delhi	5.21-5.30

Tutorial on Double integrals in Cartesian coordinates

Introduction: When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values.

Prerequisite knowledge for Complete understanding and learning of Topic :

7. Plane
8. Integration

3. Limit substitution

Detailed content of the Lecture:

1. **Evaluate** $\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dydx$

Solution:

Step:1 $\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dydx = \int_0^1 (\int_0^{x^2} (x^2 + y^2)\, dy)dx$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{x^2}$$

$$= \int_0^1 (x^4 + \frac{x^6}{3})\, dx$$

$$= \left[\frac{x^5}{5} + \frac{1}{3}\frac{x^7}{7}\right]_0^1$$

Step:2 $= \frac{1}{5} + \frac{1}{21} = \frac{21+5}{105} = \frac{26}{105}$

2. **Evaluate** $\int_0^a \int_0^b (x + y)\, dxdy$

Solution Step:1 $\int_0^a \int_0^b (x + y)\, dxdy = \int_0^a \int_0^b (xdx + ydx)\mathrm{dy}$

$$= \int_0^a \left[\frac{x^2}{2} + yx\right]_0^b dy$$

$$= \int_0^a [\frac{b^2}{2} + yb]\, dy$$

Step:2 $= \left[\frac{b^2 y}{2} + \frac{y^2 b}{2}\right]_0^a = \frac{ab^2}{2} + \frac{a^2 b}{2} = \frac{ab^2 + a^2 b}{\mathbf{2}}$

3. **Evaluate** $\int_0^1 \int_0^{\sqrt{x}} \mathrm{xy}(x + y)\, dxdy$

Solution: **Step: 1** $\int_0^1 \int_0^{\sqrt{x}} \mathrm{xy}(x + y)\, dxdy = \int_0^1 \int_0^{\sqrt{x}} (x^2\mathrm{y} + \mathrm{xy}^2)\, dydx$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 [\frac{x^3}{2} + \frac{x^{\frac{5}{2}}}{3} - \frac{x^4}{2} - \frac{2x^4}{3}]\, dx$$

$$= \left[\frac{x^4}{8} + \frac{1}{3}\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{1}{2}\frac{x^5}{5} - \frac{1}{3}\frac{x^5}{5}\right]_0^1$$

$$= [\frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15}]$$

Step:2 $= \frac{3}{56}$

4. **Evaluate** $\int_0^1 \int_0^2 e^{x+y} dxdy$

Solution: : **Step: 1** $\int_0^1 \int_0^2 e^{x+y}\, dxdy = \int_0^1 \int_0^2 e^x e^y\, dxdy$

$$= \int_{\mathbf{0}}^{\mathbf{1}} e^y dy \int_{\mathbf{0}}^{\mathbf{2}} e^x dx$$

$$= [e^y]_0^1 [e^x]_0^2$$

Step: 2 $= (e^1 - 1)\mathbf{(}\, e^2 - 1\mathbf{)}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.10-5.25

Topic of Lecture : Change of order of integration

Introduction :
To change order of integration, we need to write an integral with order dydx. This means that x is the variable of the outer integral. Its limits must be constant and correspond to the total range of x over the region D.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Integration
2. Path
3. Limit Substitution

Detailed content of the Lecture:

1. **Change the order of integration $\int_{\mathbf{0}}^{\boldsymbol{a}} \int_{\boldsymbol{a}-\sqrt{\boldsymbol{a}^2-\boldsymbol{y}^2}}^{\boldsymbol{a}+\sqrt{\boldsymbol{a}^2-\boldsymbol{y}^2}} \boldsymbol{dxdy}$ and hence evaluate it.**

Solution :

The region of integration is bounded by $y = 0, y = a, x = a - \sqrt{a^2 - y^2}\ and\ a + \sqrt{a^2 - y^2}$

Here y varies from $y = 0\ to\ y = a\ and\ x\ varies\ from\ x = a - \sqrt{a^2 - y^2}\ to\ x = a + \sqrt{a^2 - y^2}$

$$\text{Take , } x = a + \sqrt{a^2 - y^2}$$

$$x - a = \sqrt{a^2 - y^2}$$

$$(x - a)^2 = a^2 - y^2$$

$$(x - a)^2 + y^2 = a^2$$

This is a circle whose centre is $(a, 0)$ and radius is "a"

here, $y = 0\ to\ y = a$ represents horizontal path and $x = a - \sqrt{a^2 - y^2}$ to $x = a + \sqrt{a^2 - y^2}$ represents horizontal strip PQ sliding area.

Changing the order of integration is nothing but to change the Horizontal path into Vertical path and then to change the Horizontal strip PQ into Vertical strip RS.

Now, $x = 0\ to\ x = 2a$ represents Vertical path and $y = 0$ to $y = \sqrt{a^2 - (x - a)^2}$ represents Vertical strip RS sliding area.

Hence , by changing the order, we get

$$\int_{0}^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dxdy = \int_{0}^{2a}\int_{0}^{\sqrt{a^2-(x-a)^2}} dxdy$$

$$= \int_0^{2a} [y]_{y=0}^{y=\sqrt{a^2-(x-a)^2}}\ dx$$

$$= \int_0^{2a} \sqrt{a^2-(x-a)^2}\, dx$$

$$= \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^{2a}$$

$$= \left(0+\frac{\pi a^2}{4}\right) - \left(0-\frac{\pi a^2}{4}\right)$$

$$= \frac{\pi a^2}{4} + \frac{\pi a^2}{4}$$

$$= \frac{\pi a^2}{2}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=Xe1XaIUrfL4

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.35-5.40

Topic of Lecture : Change of order of integration

Introduction :

To change order of integration, we need to write an integral with order dydx. This means that x is the variable of the outer integral. Its limits must be constant and correspond to the total range of x over the region D.

Prerequisite knowledge for Complete understanding and learning of Topic :

4. Integration
5. Path
6. Limit Substitution

Detailed content of the Lecture:

1. **Change the order of integration in the integral $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}}$ ydxdy and then evaluate it**

Solution :

The region of integration is bounded by $y=-a, y=a, x=0\ and\ \sqrt{a^2-y^2}$

Here y varies from $y=-a\ to\ y=a\ and\ x\ varies\ from\ x=0\ to\ x=\sqrt{a^2-y^2}$

Take , $x=\sqrt{a^2-y^2}$

$$x = \sqrt{a^2 - y^2}$$

$$x^2 = a^2 - y^2$$

$$x^2 + y^2 = a^2$$

This is a circle whose centre is (0,0) and radius is "a"

here, $y = -a \ to \ y = a$ represents horizontal path and $x = 0$ to $x = \sqrt{a^2 - y^2}$ represents horizontal strip PQ sliding area.

Changing the order of integration is nothing but to change the Horizontal path into Vertical path and then to change the Horizontal strip PQ into Vertical strip RS.

Now, $x = 0 \ to \ x = a$ represents Vertical path and $y = -\sqrt{a^2 - y^2}$ to $y = \sqrt{a^2 - y^2}$ represents Vertical strip RS sliding area.

Hence , by changing the order, we get

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} x dx dy = \int_{0}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x dx dy$$

$$= \int_0^a [xy]_{y=\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \ dx$$

$$= \int_0^a 2x\sqrt{a^2 - x^2} \, dx$$

$$= 2\int_0^a x\sqrt{a^2 - x^2} \, dx$$

$$= 2\int_a^0 t(-t \, dt)$$

$$= 2\int_a^0 \boldsymbol{t^2 \, dt}$$

$$= -\frac{2}{3}[\boldsymbol{t^3}]_a^0$$

$$= -\frac{2}{3}[0 - \boldsymbol{a^3}]$$

$$= \frac{2}{3}a^3$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=Xe1XaIUrfL4

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.26-5.40

Topic of Lecture : Area between two curves

Introduction :

The area under a curve that exists between two points can be calculated by conducting a definite integral between the two points. To calculate the area under the curve y = f(x) between x = a & x = b, one must integrate y = f(x) between the limits of a and b.

Prerequisite knowledge for Complete understanding and learning of Topic :

1.Integration

2. Finding Limit

3. Limit substitution

Detailed content of the Lecture:

1. **Find the area enclosed by the curves $y = x^2$ and $x + y - 2 = 0$**

Solution :

$$\text{Given } y = x^2 ----------- (1)$$

$$x + y = 2 ----------------- (2)$$

∴ The required area $= \int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx$

$$= \int_{-2}^{1} [y]_{x^2}^{y=2-x} dx$$

$$= \int_{-2}^{1} [(2 - x) - x^2] dx$$

$$= \int_{-2}^{1} [2 - x - x^2] dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} - \frac{8}{3}\right)$$

$$= \frac{7}{6} - \left(-4 - 2 - \frac{8}{3}\right)$$

$$= \frac{7}{6} + 6 - \frac{8}{3}$$

$$= \frac{7 + 36 - 16}{6}$$

$$= \frac{27}{6} \quad square\ units$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/channel/UC_4NoVAkQzeSaxCgm-to25A

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.45-5.50

Tutorial on Change of order of integration

Introduction :
To change order of integration, we need to write an integral with order dydx. This means that x is the variable of the outer integral. Its limits must be constant and correspond to the total range of x over the region D.

Prerequisite knowledge for Complete understanding and learning of Topic :

7. Integration
8. Path
9. Limit Substitution

Detailed content of the Lecture:

1. **Change the order of integration $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)dxdy$**

Solution:

Step:1 In the given interval, y varies from $y = x^2$ to $y = 2 - x$ and x varies from 0 to

$y = x^2$ and $x + y = 2$

The region of integration is in the first quadrant bounded between $x^2 = y$ and the line

$x + y = 2$

Step:2 After dividing the region of integration by the line y=1

$$\int_0^1 \int_{x^2}^{2-x} f(x,y)dxdy = \int_0^1 \int_0^{\sqrt{y}} f(x,y)dxdy + \int_1^2 \int_0^{2-y} f(x,y)dxdy$$

2. **$\int\int xy\,dxdy$ taken over the positive quadrant of the circle $x^2+y^2 = a^2$**

Solution:

Step:1 Given $x^2+y^2 = a^2$

$x^2 = a^2 - y^2$

$x = \pm\sqrt{a^2 - y^2}$

We need positive Quadrant $x = \sqrt{a^2 - y^2}$

x varies from 0 to $\sqrt{a^2 - y^2}$, y varies from 0 to a

Step:2 Required area = $\int_0^a \int_0^{\sqrt{a^2-y^2}} xydxdy$

$= \int_0^a \left[y\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy = \int_0^a \frac{y(a^2-y^2)}{2} dy$

$= \int_0^a \frac{(ya^2-y^3)}{2} dy \quad = \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{2}\right]_0^a$

$= \frac{1}{2}[\frac{a^4}{2} - \frac{a^4}{4}] \quad = \frac{1}{2}[\frac{a^4}{4}]$

$= \frac{a^4}{8}$ sq. u

3. **Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solution:

Step:1 Area of the ellipse = 4x area of the Quadrant by considering the horizontal strip x varies 0 to $\frac{a}{b}\sqrt{b^2-y^2}$, y varies 0 to b

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$

$$x^2=\frac{a^2(b^2-y^2)}{b^2}$$

$$x=\frac{a}{b}\sqrt{b^2-y^2}$$

Step:2 Required area $= 4\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dxdy$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\, dy$$

Formula $\int\sqrt{a^2-x^2}\;dx = \frac{a^2}{2}\sin^{-1}(\frac{x}{a})+\frac{x}{2}\sqrt{a^2-x^2}$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\sin^{-1}\frac{y}{b}+\frac{y}{2}\sqrt{b^2-y^2}\right]_0^b$$

$$= \frac{4a}{b}\times\frac{b^2}{2}\times\frac{\Pi}{2}$$

Area $= \Pi ab\ sq.u$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.35-5.50

Topic of Lecture : Area of double integral

Introduction :

The area under a curve that exists between two points can be calculated by conducting a definite integral between the two points. To calculate the area under the curve y = f(x) between x = a & x = b, one must integrate y = f(x) between the limits of a and b.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Integration
2. Finding Limit
3. Limit substitution

Detailed content of the Lecture:

1. **Evaluate $\int\int xy\,dxdy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$**

Solution :

$$Given\quad x^2+y^2=a^2$$

$$x^2 = a^2-y^2$$

$$x = \pm\sqrt{a^2-y^2}$$

$$We\ need\ positive\ Quadrant\quad x=\sqrt{a^2-y^2}$$

$$x\quad varies\ from\ 0\ to\ \sqrt{a^2-y^2},\quad y\quad varies\ from\ 0\ to\ a$$

$$Required\ area\ = \int_0^a \int_0^{\sqrt{a^2-y^2}} xydxdy$$

$$= \int_0^a \left[y\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{y(a^2-y^2)}{2} dy$$

$$= \int_0^a \frac{(ya^2-y^3)}{2}\ dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{2}\right]_0^a$$

$$= \frac{1}{2}[\frac{a^4}{2} - \frac{a^4}{4}]$$

$$= \frac{1}{2}[\frac{a^4}{4}]$$

$$= \frac{a^4}{8}\ sq.u$$

2. **Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solution :

Area of the ellipse = 4x area of the Quadrant by considering the horizontal strip

x varies 0 to $\frac{a}{b}\sqrt{b^2-y^2}$, y varies 0 to b

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$x^2 = \frac{a^2(b^2-y^2)}{b^2}$$

$$x = \frac{a}{b}\sqrt{b^2-y^2}$$

Required area $= 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dxdy$

$$= \quad 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}}\ dy$$

$$= 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\ dy$$

$$Formula \int \sqrt{a^2-x^2}\ dx \quad = \frac{a^2}{2} sin^{-1}(\frac{x}{a}) + \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \frac{4a}{b}\left[\frac{b^2}{2} sin^{-1}\frac{y}{b} + \frac{y}{2}\sqrt{b^2 - y^2}\right]_0^b$$

$$= \frac{4a}{b} \, x \, \frac{b^2}{2} \, x \, \frac{\Pi}{2}$$

$$Area \;\; = \Pi ab \; sq.u$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/channel/UC_4NoVAkQzeSaxCgm-to25A

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.51-5.65

Topic of Lecture : Triple integration in Cartesian coordinates

Introduction :

Calculation of a triple integral in Cartesian coordinates can be reduced to the consequent calculation of three integrals of one variable.

Prerequisite knowledge for Complete understanding and learning of Topic :

1. Integration
2. Finding limit
3. Limit substitution
4. Volume integral

Detailed content of the Lecture:

1. **Evaluate $\int_0^1 \int_0^2 \int_0^3 xyzdzdydx$**

Solution:

$$Let\ I \;=\; \int_0^1\int_0^2\int_0^3 xyzdzdydx$$

$$= [\int_0^1 xdx]\,[\int_0^2 ydy]\,[\int_0^3 zdz]$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^3$$

$$= \left(\frac{1}{2}\right)\cdot\left(\frac{4}{2}\right)\cdot(\frac{9}{2})$$

$$= \frac{9}{2}$$

2. **Evaluate $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dxdydz$**

Solution :

Given : $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dxdydz$

$$= \int_0^a \int_0^b \int_0^c \boldsymbol{x^2\, dxdydz} + \int_0^a \int_0^b \int_0^c \boldsymbol{y^2\, dxdydz} + \int_0^a \int_0^b \int_0^c \boldsymbol{z^2\, dxdydz}$$

$$= I_1 + I_2 + I_3 ------------(1)$$

$$I_1 = \int_0^a \int_0^b \int_0^c \boldsymbol{x^2\, dxdydz}$$

$$= \left[\int_0^a dz\right]\left[\int_0^b dy\right]\left[\int_0^c \boldsymbol{x^2 dx}\right]$$

$$= [z]_0^a [y]_0^b \left[\frac{\boldsymbol{x^3}}{3}\right]_0^c$$

$$= (a-0)(b-0)\left(\frac{\boldsymbol{c^3}}{3} - 0\right)$$

$$= \frac{abc^3}{3}$$

$$I_2 = \int_0^a \int_0^b \int_0^c \boldsymbol{y^2\, dxdydz}$$

$$= \left[\int_0^a dz\right]\left[\int_0^b \boldsymbol{y^2} dy\right]\left[\int_0^c \boldsymbol{dx}\right]$$

$$= [z]_0^a \left[\frac{\boldsymbol{y^3}}{3}\right]_0^b [x]_0^c$$

$$= (a-0)\left(\frac{\boldsymbol{b^3}}{3} - 0\right)(c-0)$$

$$= \frac{ab^3c}{3}$$

$$I_3 = \int_0^a \int_0^b \int_0^c \boldsymbol{z^2\, dxdydz}$$

$$= \left[\int_0^a \boldsymbol{z^2} dz\right]\left[\int_0^b dy\right]\left[\int_0^c \boldsymbol{dx}\right]$$

$$= \left[\frac{\boldsymbol{z^3}}{3}\right]_0^a [y]_0^b [x]_0^c$$

$$= \left(\frac{\boldsymbol{a^3}}{3} - 0\right)(b-0)(c-0)$$

$$= \frac{a^3bc}{3}$$

$$\therefore (1) \implies \quad = \frac{abc^3}{3} + \frac{ab^3c}{3} + \frac{a^3bc}{3}$$

$$\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dxdydz = \frac{abc}{3}[\boldsymbol{a^2} + \boldsymbol{b^2} + \boldsymbol{c^2}]$$

Video Content / Details of website for further learning (if any):

https://youtu.be/Gt5ApX3oIN0

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.61-5.70

Tutorial on Triple integration in Cartesian coordinates

Introduction :

Calculation of a triple integral in Cartesian coordinates can be reduced to the consequent calculation of three integrals of one variable.

Prerequisite knowledge for Complete understanding and learning of Topic :

5. Integration
6. Finding limit
7. Limit substitution
8. Volume integral

Detailed content of the Lecture:

1. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dxdydz$

Solution:

Step:1 Let I = $\int_0^1 \int_0^y \int_0^{x+y} dzdxdy$

$= \int_0^1 \int_0^y [z]_0^{x+y}\, dxdy$

$= \int_0^1 \int_0^y (x+y)\, dxdy$

$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^y dy$

Step:2 $= \int_0^1 (\frac{y^2}{2} + y^2)\, dy$

$=\int_0^1 (\frac{3}{2}y^2)\, dy$

$= \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1$

$= \frac{1}{2}$

2. **Evaluate** $\int_0^1 \int_0^2 \int_0^3 xyzdzdydx$

Solution:

Step:1 Let $I = \int_0^1 \int_0^2 \int_0^3 xyzdzdydx$

$$= [\int_0^1 xdx]\,[\int_0^2 ydy]\,[\int_0^3 zdz]$$

Step:2

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^3$$

$$= \left(\frac{1}{2}\right) . \left(\frac{4}{2}\right) . \left(\frac{9}{2}\right) \quad = \frac{9}{2}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6Zacf25sXhk

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.65-5.70

Topic of Lecture : Triple integration in Cartesian coordinates

Introduction :
Calculation of a triple integral in Cartesian coordinates can be reduced to the consequent calculation of three integrals of one variable.

Prerequisite knowledge for Complete understanding and learning of Topic :

9. Integration
10. Finding limit
11. Limit substitution
12. Volume integral

Detailed content of the Lecture:

3. **Evaluate** $\int_0^1 \int_0^y \int_0^{x+y} dxdydz$

Solution :

$$Let\ I = \int_0^1 \int_0^y \int_0^{x+y} dzdxdy$$

$$= \int_0^1 \int_0^y [z]_0^{x+y}\, dxdy$$

$$= \int_0^1 \int_0^y (x+y)\, dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^y dy$$

$$= \int_0^1 (\frac{y^2}{2} + y^2)\, dy$$

$$= \int_0^1 (\frac{3}{2} y^2)\, dy$$

$$= \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}$$

4. Evaluate $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dxdydz$

Solution :

Given : $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dxdydz$

$$= \int_0^a \int_0^b \int_0^c x^2\, dxdydz + \int_0^a \int_0^b \int_0^c y^2\, dxdydz + \int_0^a \int_0^b \int_0^c z^2\, dxdydz$$

$$= I_1 + I_2 + I_3 ------------(1)$$

$$I_1 = \int_0^a \int_0^b \int_0^c x^2\, dxdydz$$

$$= \left[\int_0^a dz\right]\left[\int_0^b dy\right]\left[\int_0^c x^2 dx\right]$$

$$= [z]_0^a [y]_0^b \left[\frac{x^3}{3}\right]_0^c$$

$$= (a-0)(b-0)\left(\frac{c^3}{3} - 0\right)$$

$$= \frac{abc^3}{3}$$

$$I_2 = \int_0^a \int_0^b \int_0^c y^2\, dxdydz$$

$$= \left[\int_0^a dz\right]\left[\int_0^b y^2 dy\right]\left[\int_0^c dx\right]$$

$$= [z]_0^a \left[\frac{y^3}{3}\right]_0^b [x]_0^c$$

$$= (a-0)\left(\frac{b^3}{3} - 0\right)(c-0)$$

$$= \frac{ab^3c}{3}$$

$$I_3 = \int_0^a \int_0^b \int_0^c \boldsymbol{z^2\, dxdydz}$$

$$= \left[\int_0^a \boldsymbol{z^2} dz\right]\left[\int_0^b dy\right]\left[\int_0^c \boldsymbol{dx}\right]$$

$$= \left[\frac{\boldsymbol{z^3}}{3}\right]_0^a [y]_0^b [x]_0^c$$

$$= \left(\frac{\boldsymbol{a^3}}{3} - 0\right)(b-0)(c-0)$$

$$= \frac{a^3bc}{3}$$

$$\therefore (1) \Rightarrow \qquad = \frac{abc^3}{3} + \frac{ab^3c}{3} + \frac{a^3bc}{3}$$

$$\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\, dxdydz = \frac{abc}{3}[\boldsymbol{x^2+y^2+z^2}]$$

Video Content / Details of website for further learning (if any):

https://youtu.be/Gt5ApX3oIN0

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.70-5.80

Topic of Lecture : Volume as triple integrals

Introduction :

Triple iterated integrals If the solid W is a cube defined by a≤x≤b, c≤y≤d, and p≤z≤q, then we can easily write the triple integral as an iterated integral. We could first integrate x from a to b, then integrate y from c to d, and finally integrate z from p to q, ∭WfdV=∫qp(∫dc(∫baf(x,y,z)dx)dy)dz.

Prerequisite knowledge for Complete understanding and learning of Topic :

13. Integration
14. Finding limit
15. Limit substitution
16. Volume integral

Detailed content of the Lecture:

1. **Find the volume of the portion of the ellipsoid $\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1$ which has in first octant using triple integration.**

Solution :

Given $\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1$ ------------------(1)

Volume = $\iiint dz\, dy\, dx$

To find x limit put $y = 0$ and $z = 0$ we get (line integral)

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$x = \pm a$$

i.e., $x = 0 \; to \; x = a$

To find y limit put z $= 0$ we get (surface integral)

$$(1) \Longrightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

$i.e., y = 0 \; to \; y = b\sqrt{1 - \frac{x^2}{a^2}} \quad (\because first\ octant\ area)$

To find z limit (Volume integral)

$$(1) \Longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$z = \pm c\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}} \qquad (\because first\ octant\ area)$

$$i.e., \qquad z = 0 \;\; to \;\; z = c\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

$$\text{Volume } = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}+\frac{y^2}{b^2}}} dz \;\; dy \;\; dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}+\frac{y^2}{b^2}}} dy\, dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}} \;\; dy\, dx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2}{b^2}} \;\; dy\, dx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\right)^2 - y^2} \;\; dy\, dx$$

Video Content / Details of website for further learning (if any):

https://youtu.be/Gt5ApX3oIN0

Important Books/Journals for further learning including the page nos.:

Sl.No	Author(s)	Title of the Book	Publisher	Page.No
1.	Grewal. B.S	Higher Engineering Mathematics, 43rd Edition	Khanna Publications, Delhi	5.71-5.80