MUTHAYAMMAL ENGINEERING COLLEGE

(An Autonomous Institution)

(Approved by AICTE, New Delhi, Accredited by NAAC & Affiliated to Anna University)
Rasipuram - 637 408, Namakkal Dist., Tamil Nadu

Estd. 2000

LECTURE HANDOUTS

L

AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture:**

Topic of Lecture: Introduction to Fourier transforms.
Fourier transforms pair.

Introduction :

For every time domain waveform there is a corresponding frequency domain waveform, and vice versa. For example, a rectangular pulse in the time domain coincides with a sine function [i.e., sin(x)/x] in the frequency domain. ...

Prerequisite knowledge for Complete understanding and learning of Topic:

Waveforms that correspond to each other in this manner are called Fourier transform pairs.

Detailed content of the Lecture:

Find Fourier Transform $F(X) = \begin{cases} X & \text{if } |x| \leq a \\ 0 & \text{if } |x| \geq a \end{cases}$

Solution:

Given that,

$$F(x)=\begin{cases} X & \text{if } |x| \leq a \\ 0 & \text{if } |x| \geq a \end{cases}$$

Forier Transform,

$$F(s)=1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$=1/\sqrt{2\pi}\int_{-a}^{a} x\, e^{isx}$ dx

[$e^{i\theta}$=cosθ+i sinθ]

[e^{isx}=sx($\theta = sx$)]

$=1/\sqrt{2\pi}\int_{-a}^{a} x(\cos sx + i\sin sx)dx$

$=1/\sqrt{2\pi}\int_{-a}^{a}\{(x\cos sx) + i\sin sx)\}dx$

$=1/\sqrt{2\pi}\int_{-a}^{a}\{(0 + i\, x\sin sx)dx$

$=2i/\sqrt{2\pi}\int_{0}^{a} x\sin sx\, dx$

[$\int udv = uv - u'v1 + u"v2 \ldots\ldots$.]

u =x dv=sin sx

u' =1 v=-cossx/s

u" =0 v1=-sin sx/s^2

$=\frac{\sqrt{2\sqrt{2i}}}{\sqrt{2\sqrt{\pi}}}\left[(x)\left(-\frac{\cos sx}{s}\right)-(1)\left(-\frac{\sin sx}{s^2}\right)\right]_0^a$

$=i\sqrt{2}/\pi\left[\left(-\frac{a\cos sa}{s}\right)+\frac{\sin sa}{s^2}-(0+0)\right]$

F(S)=I $\boldsymbol{\sqrt{2}/\pi\left[\frac{\sin sa - a\cos s\, a}{s^2}\right]}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/channel/UC_4NoVAkQzeSaxCgm-to25A

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2rd Edition, 1986, Page.No : 5-7

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LECTURE HANDOUTS | **L**

AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Statement of Fourier integral theorem.

Introduction :

he shift **theorem**: If f(x) has the **Fourier** transform F(u), then f(x − a) has the **Fourier** transform F(u)$e^{-2i\pi au}$. ... The convolution **theorem**: If the convolution between two functions f(x) and g(x) is defined by the **integral**

Prerequisite knowledge for Complete understanding and learning of Topic :

A mathematical **theorem** stating that a PERIODIC function f(x) which is reasonably continuous may be expressed as the sum of a series of sine or cosine terms.

Detailed content of the Lecture:

State Fourier integral theorem.

Solution:

If f(x) is piece wise continuous and absolutely integrable in $(-\infty,\infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(x-t)}dtds$$

This is known as Fourier integral theorem or Fourier integral formula

2. Write down the Fourier transform pair.

Solution:

If f(x) is a given function, then $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx = \mathrm{F(s)}$

$and\ f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s).e^{-isx}ds$ are called Fourier transform pair.

Video Content / Details of website for further learning (if any): https://www.youtube.com/watch?v=ZZfRuPpRX-o
Important Books/Journals for further learning including the page nos.: 1. Ronald N.Bracewell – The Fourier transform and its application , 2nd Edition, 1986, Page.No : 5-7

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Fourier transforms pair.
Fourier Cosine Transform.

Introduction :

Fourier transform (FT) is a mathematical **transform** that decomposes functions depending on space or time into functions depending on spatial or temporal frequency, such as the expression of a musical chord in terms of the volumes and frequencies of its constituent notes.

Prerequisite knowledge for Complete understanding and learning of Topic :

Volumes and frequencies of its constituent.

Detailed content of the Lecture:

Find The Fourier Cosine Transform Of Function f(x) =cos x if $\begin{cases} 0 < x < a \\ 0 \text{ if } a < x < 0 \end{cases}$

Solution:

Given that,

$$F(x) = \begin{cases} \text{Cos } x \text{ if } 0 < x < a \\ 0 \quad \text{if } a < x < 0 \end{cases}$$

Fourier cosine transfourm :

$$F_c(s)=\sqrt{2}/\pi \int_0^\infty f(x) \cos x$$

$$= \frac{\sqrt{2}}{\pi} \int_0^a \cos x \cos x \, dx$$

[cosAcosB=1/2(cos(A+B)+cos(A-B)]

$= \frac{\sqrt{2}}{\pi} \int_0^a$ 1/2(cos(x+sx)+cos(x-sx)

$= 1/2 \frac{\sqrt{2}}{\pi} \int_0^a$ [cos(1+s)x+cos(1-s)x]dx

$= \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}\sqrt{\pi}} \left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right]_0^a$

$=1/\sqrt{2\pi} \left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right]$

Fc(s) $=1/\sqrt{2\pi} \left[\frac{(1-s)\sin(1+s)a+(1+s)\sin(1-s)a}{1+s^2}\right]$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=bKTzqIdk-Hg

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2rd Edition, 1986, Page.No : 5-7

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AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Fourier sine transforms.

Introduction :

In mathematics, the **Fourier sine and cosine transforms** are forms of the **Fourier** integral **transform** that do not use complex numbers. They are the forms originally used by Joseph **Fourier** and are still preferred in some applications, such as signal processing or statistics.

Prerequisite knowledge for Complete understanding and learning of Topic :

They are the forms originally used by Joseph **Fourier** and are still preferred in some applications, such as signal processing or statistics.

Detailed content of the Lecture:

Find Fourier sine transform of

$$F(x)=\left\{\begin{array}{ll} \text{Sin } X & \text{If } 0<x<a \\ 0 & \text{If } 0>a \\ & a<x<0 \end{array}\right\}$$

Fourier sine transform:

$$F_s(S)=\sqrt{2/\pi}\int_0^\infty f(x)\text{sinsxdx}$$

$$=\sqrt{2/\pi}\int_0^\infty \text{sinx sinsx dx}$$

sinAsin B= $[½\cos(A-B)-\cos(A+B)]$

$$=\sqrt{2/\pi}\int_0^a [½(x-sx)-\text{cox}(x+sx) \quad dx]dx$$

$=1/2\sqrt{2}/\pi \quad [\sin(x - sx) - \sin(1+s)a\,]_0^a$

$=1/\sqrt{}_{2\pi}\left[\frac{\sin(1-s)a}{1-s} - \frac{\sin(1+s)a}{1+s}\right]$

$F_s(s)=1/\sqrt{}_{2\pi}\left[\frac{(1+s)\sin(1-s)a}{1-s} - \frac{(1-s)\sin(1=s)a}{1-s}\right]$

Using Fourier Sine Transform of e^{-ax} , a>0 and deduce that

$\int_0^\infty \frac{s}{a^2+s^2}\sin sx\, dx = \frac{\pi}{2}e^{-ax}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=0USI-48ovJI

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2rd Edition, 1986, Page.No : 17-20

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LECTURE HANDOUTS | **L**

AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Fourier cosine transforms.

Introduction :

Fourier Transform of the Sine and **Cosine** Functions
Equation [2] states that the **fourier transform** of the **cosine** function of frequency A is an impulse at f=A and f=-A.

Prerequisite knowledge for Complete understanding and learning of Topic :

That is, all the energy of a sinusoidal function of frequency A is entirely localized at the frequencies given by |f|=A.

Detailed content of the Lecture:

Using Fourier Sine Transform of e^{-ax} , a>0 and deduce that

$\int_0^\infty \frac{s}{a^2+s^2}$ sinsx dx = $\frac{\pi}{2}$ e^{-ax}

Given that

F(x)= e^{-ax} , a>0

Fourier sine transform.

$F_s(s)= \sqrt{\frac{2}{\pi}} \int_0^e e - ax$ Sinsx dx

$=\sqrt{2}/\pi \int_0^\infty e{-}ax$sinsx dx

Fs(s) $=\sqrt{2}/\pi \left[\frac{s}{a^2+s^2}\right]$

Inverse fourier sine transform:

$F(x)= \sqrt{2/\pi}\int_0^\infty F_s(s) \sin sx \, ds$

$= {}^{2}/_{\pi} \int_0^\infty \sqrt{2/\pi}\left(\frac{s}{a^2+s^2}\right) \sin sx \, ds$

${}^{2}/_{\pi} \int_0^\infty \left(\frac{s}{a^2+s^2}\right) \sin sx \, ds = fx$

$\int_0^\infty \left(\frac{s}{a^2+s^2}\right) \sin sx \, ds = \frac{\pi}{2} e^{-ax}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/channel/UC_4NoVAkQzeSaxCgm-to25A

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2nd Edition, 1986, Page.No : 17-20

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AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Properties.

Introduction :

Fourier transform of the **Fourier transform** is proportional to the original signal re- versed in time. ... The time-shifting

Prerequisite knowledge for Complete understanding and learning of Topic :

Property identifies the fact that a linear displacement in time corresponds to a linear phase factor in the frequency domain.

Detailed content of the Lecture:

1.Without finding the value of a_0 , a_n&b_n for the function $f(x) = x^2$ in $(0, \pi)$, find the value of $\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)$

Solution: Given $f(x) = x^2$ in $(0, \pi)$

By Parseval's Theorem

$$\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2) = \frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[x^2]^2 dx = \frac{\pi^4}{5}.$$

2. If $f(x) = 2x$ in the interval $(0,4)$, find the value of a_2.

Solution:

Given $f(x) = 2x$ in $(0,4)$

$$\therefore a_2 = \frac{1}{2}\int_0^4 2x \cos\frac{2\pi x}{2}\, dx$$

$$= \frac{1}{2}\int_0^4 2x \cos \pi x\, dx = \int_0^4 x \cos \pi x\, dx$$

$$= \left[x\left[\frac{\sin \pi x}{\pi}\right] - (1)\left[\frac{-cos\pi x}{\pi^2}\right]\right]_0^4 = 0.$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=wAMxETEtRos

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2^{rd} Edition, 2011,
Page.No : 2.36-2.43

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AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Transforms of simple functions.

Introduction :

Fourier transform of the **Fourier transform** is proportional to the original signal re- versed in time. ... The time-shifting.

Prerequisite knowledge for Complete understanding and learning of Topic :

property identifies the fact that a linear displacement in time corresponds to a linear phase factor in the frequency domain.

Detailed content of the Lecture:

1.Write the Dirichlet's conditions on the existence of Fourier series

Solution:

Any function $f(x)$ can be developed as a Fourier series in any one period, provided

i) It is periodic, single valued, finite.

ii) The number of discontinuities if any is finite.

iii) The number of maxima and minima if any is finite.

2.Obtain the first term of the Fourier series for the function $f(x) = x^2, (-\pi, \pi)$.

Solution:

Given $f(x) = x^2 , -\pi < x < \pi$ is an even function

Hence $b_n = 0$ and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$..............(1)

First term of the Fourier series is $\frac{a_0}{2}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}(\frac{x^3}{3})_0^{\pi} = \frac{2}{\pi}[\frac{\pi^3}{3} - 0]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^2.$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=bKTzqIdk-Hg

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011,
Page.No : 2.36-2.43

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Convolution theorem.

Introduction :

Convolution is a mathematical way of combining two signals to form a third signal. It **is** the single most important technique in Digital Signal Processing.

Prerequisite knowledge for Complete understanding and learning of Topic :

Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

Detailed content of the Lecture:

1.Find the Fourier Cosine transform to evaluate $\int_0^\infty dx/(a^2+x^2)(b^2+x^2)$

Solution:

Given that:

$\int_0^\infty dx/(a^2+x^2)(b^2+x^2)$

$f(x)=e^{-ax}$ $g(x)=e^{-bx}$

Fourier Cosine Transforms:

$F_c(s) = \sqrt{\left(\frac{2}{\pi}\right)}\, f(x) \cos sx\, dx$

$F_C(S) = \sqrt{(\frac{2}{\pi})}(a/(a^2+s^2))$

$G_c(s) = \int_0^\infty \sqrt{(\frac{2}{\pi})} \int_0^\infty e^{-bx} \text{cossx dx}$

$G_c(s) = \sqrt{(\frac{2}{\pi})}(b/(b^2+s^2))$

Parseval's identity of Fourier Cosine Transform:

$$= \int_0^\infty F_C[S]\, G_C[S]ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$= \int_0^\infty [\sqrt{(\tfrac{2}{\pi})}(a/(a^2+s^2)).\sqrt{(\tfrac{2}{\pi})}(b/(b^2+s^2))]ds = \int_0^\infty e^{-ax}e^{-bx}\, dx$$

$$= (2/\pi)\int_0^\infty [ab\,/(a^2+s^2)(b^2+s^2)]ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$= (2/\pi)\int_0^\infty [ab\,/(a^2+s^2)(b^2+s^2)]ds = [e^{\infty}/-(a+b)]-[-e^{0}/-(a+b)]$$

$$= (2/\pi)\int_0^\infty [ab\,/(a^2+s^2)(b^2+s^2)]ds = 0+ 1/a+b$$

REPLACE 'S' BY 'X',

$$\int_0^\infty dx/(a^2+x^2)(b^2+x^2) = \pi/2ab(a+b)$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=wAMxETEtRos

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2nd Edition, 1986, Page.No : 108-112

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **I-Fourier Transforms** **Date of Lecture :**

Topic of Lecture : Parseval's identity (Problems)

Introduction :

Convolution is a mathematical way of combining two signals to form a third signal. It **is** the single most important technique in Digital Signal Processing.

Prerequisite knowledge for Complete understanding and learning of Topic :

Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

Detailed content of the Lecture:
Find the Fourier Sine transforms to evaluate

$\int_0^\infty dx/(a^2+x^2)(b^2+x^2)$

Solution:

Given that:

$\int_0^\infty dx/(a^2+x^2)(b^2+x^2)$

$f(x)=e^{-ax}$ $g(x)=e^{-bx}$

Fourier sine transform:

$Fs(s) = \sqrt{\left(\frac{2}{\pi}\right)}\, f(x) \sin sx\, dx$

$F_s(S) = \sqrt{(\frac{2}{\pi})}(s/(a^2+s^2))$

$G_s(s) = \int_0^\infty \sqrt{(\frac{2}{\pi})} \int_0^\infty e^{-bx} \sin sx\, dx$

$G_s(s) = \sqrt{(\frac{2}{\pi})}(s/(b^2+s^2))$

Parseval's identity of Fourier Sine transform:

$\int_0^{\infty} F_s[S]\, G_s[S] ds = \int_0^{\infty} f(x)\, g(x)\, dx$

$\int_0^{\infty} [\sqrt{(\frac{2}{\pi})}(s/(a^2+s^2)).\sqrt{(\frac{2}{\pi})}(s/(b^2+s^2))]ds = \int_0^{\infty} e^{-ax}\, e^{-bx}\, dx$

$(2/\pi)\int_0^{\infty} [s^2/(a^2+s^2)(b^2+s^2)]\, ds = [e^{\infty}/-(a+b)]-[-e^0/-(a+b)]$

$(2/\pi)\int_0^{\infty} [s^2/(a^2+s^2)(b^2+s^2)]\, ds = 0+ 1/a+b$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=wAMxETEtRos

Important Books/Journals for further learning including the page nos.:

1.Ronald N.Bracewell – The Fourier transform and its application , 2rd Edition, 1986,
Page.No : 108-112

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Z- transforms and Elementary properties

Introduction :The z-transform plays a similar role for discrete systems, i.e. ones where sequences are involved, to that played by the Laplace transform for systems where the basic variable t is continuous. Specifically:

1. The z-transform definition involves a summation
2. The z-transform converts certain difference equations to algebraic equations
3. Use of the z-transform gives rise to the concept of the transfer function of discrete (or digital) systems.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Summation
2. Z-Transform Formula
3. Properties

Detailed content of the Lecture:

1. **Find Z – Transform of a^n**

Solution :

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1}$$

$$Z[a^n] = \frac{z}{z-a}$$

2. **Find the Z Transform of *n*.**

Solution :

W.K.T $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[n] = \sum_{0}^{\infty} n \left(\frac{1}{z}\right)^n$$

$$= 0+\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \ldots \quad =\left(\frac{1}{z}\right)\left(\frac{z-1}{z}\right)^{-2} = \frac{z}{(z-1)^2}$$

3. **Find Z –Transform of na^n .**

Solution : $$Z[a^n n] = Z[n]_{z\to\frac{z}{a}} = \left[\frac{z}{(z-1)^2}\right]_{z\to\frac{z}{a}} = \left[\frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}\right] = \left[\frac{z}{a}\,\frac{a^2}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$$

4. **Find Z – Transform of $\cos\frac{n\pi}{2}$ and $\sin\frac{n\pi}{2}$**

Solution :

$$i)\ W.K.T\,Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

Put $\theta = \frac{\pi}{2}$, we get $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z(z-\cos\frac{\pi}{2})}{z^2-2z\cos\frac{\pi}{2}+1} = \frac{z^2}{z^2+1}$.

$$ii)\ \ Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Put $\theta = \frac{\pi}{2}$, we get $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1} = \frac{z}{z^2+1}$

5. **Find Z – Transform of $\frac{1}{n}$.**

Solution :

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left[\frac{1}{n}\right] = \sum_0^\infty \left(\frac{1}{n}\right)\left(\frac{1}{z}\right)^n = \frac{1}{1}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 = -\log\left(1-\frac{1}{z}\right) = \log\left(\frac{z}{z-1}\right).$$

6. **Prove that $Z\left[\frac{1}{n+1}\right] = z.\log\left(\frac{z}{z-1}\right)$**

Solution : $Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{z}\right)^n = 1+\frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots = z\left[\frac{\left(\frac{1}{z}\right)}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right]$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right] = z.\log\left(\frac{z}{z-1}\right)$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=RYXlHkqqdh8

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 5.1-5.33

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LECTURE HANDOUTS

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Initial and final value theorem

Introduction :The z-transform plays a similar role for discrete systems, i.e. ones where sequences are involved, to that played by the Laplace transform for systems where the basic variable t is continuous. Specifically:
1. The z-transform definition involves a summation
2. The z-transform converts certain difference equations to algebraic equations
3. Use of the z-transform gives rise to the concept of the transfer function of discrete (or digital) systems.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Theorem Statement
2. First Shifting Theorem
3. Second Shifting Theorem

Detailed content of the Lecture:

1. **Initial value theorem**

Statement : $\mathbf{Z[f\,(n)] = F\,(Z)}$ **then** $\boldsymbol{f(0) = \lim_{z\to\infty} F(Z)}$

Proof

$$\mathrm{Z[f\,(n)] = F\,(Z)} = \sum_{n=0}^{\infty} f(0)\, z^{-n}$$

$$F(Z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots$$

$$\lim_{z\to\infty} F(Z) = \lim_{z\to\infty}\left(f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\right)$$

$$\therefore \boldsymbol{f(0) = \lim_{z\to\infty} F(Z)}$$

2. **Final value theorem**

Statement : $\boldsymbol{Z[f(0)] = F(Z)\,then\ \lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(Z)}$

Proof

$$Z[f(n+1) - f(n)] = \sum_{n=0}^{\infty}[f(n+1) - f(n)]\, z^{-n}$$

$$Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$zF(Z) - zf(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$(z-1)F(Z) - zf(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]\, z^{-n}$$

$$\lim_{x\to 1}(z-1)F(Z) - \lim_{Z\to 1} zf(0) = \lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$\lim_{z\to 1}(z-1)F(Z) - f(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]$$

$$\lim_{z\to 1}(z-1)F(Z) - f(0) = \lim_{n\to\infty}\{[f(1) - f(0)] + \cdots [f(n+1) - f(n)]\}$$

$$\lim_{z\to 1}(z-1)F(Z) - f(0) = \lim_{n\to\infty}[f(n)]$$

$$\therefore \boldsymbol{\lim_{z\to 1}(z-1)F(Z) = \lim_{n\to\infty} f(n)}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=5nQ03xrxkVw

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 5.15-5.33

Course Faculty

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Course Name with Code : Transforms and Partial Differential Equations / 19BSS23

Course Faculty : M.Nazreen Banu

Unit : II-Z-Transforms And Difference Equations

Date of Lecture:

Topic of Lecture: Inverse Z - transforms – Partial fraction method

Introduction:

$If F(Z) is a rational function in which the denominator is factorisable , F(Z) is resolve$
$into partial fractions and then z^{-1}[F(Z)] is divided as the sum of the inverse Z transforms$
Of $the partial fractions$.

i) $\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$ ii) $\frac{1}{(z-a)(z-b)^2} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{(z-b)^2}$

iii) $\frac{1}{(z-b)(z^2+b)} = \frac{A}{z-a} + \frac{Bz+c}{z^2+b}$

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Z-Inverse Property
2. Partial Fraction Method
3. Factorize

Detailed content of the Lecture:

1. **Find** $Z^{-1}\left[\frac{z}{(z^2+7z+10)}\right]$

Solution :

$$Y(Z) = \frac{Z}{(z^2+7z+10)} = \frac{z}{(z+5)(z+2)}$$

$$\frac{Y(Z)}{Z} = \frac{1}{(z+5)(z+2)}$$

$$\frac{1}{(z+5)(z+2)} = \frac{A}{(z+2)} + \frac{B}{(z+5)}$$

$$1 = \mathrm{A}(z+5) + \mathrm{B}(z+2)$$

Put z = -2 ⇒ $A(-2+5) + B(-2+2)$ $1 \Rightarrow 3A \Rightarrow A = \frac{1}{3}$

Put z = -5 ⇒ $A(-5+5) + B(-5+2)$ $1 \Rightarrow -3B \Rightarrow B = -\frac{1}{3}$

$$\frac{Y(Z)}{Z} = \frac{A}{(z+2)} + \frac{B}{(z+5)} = \frac{\frac{1}{3}}{(z+2)} + \frac{-\frac{1}{3}}{(z+5)}$$

Y(Z)= $\frac{\mathbf{1}}{\mathbf{3}}\frac{z}{(z+2)} - \frac{\mathbf{1}}{\mathbf{3}}\frac{z}{(z+5)}$

$$Z^{-1}[y(z)] = \frac{1}{3}Z^{-1}\left[\frac{z}{(z+2)}\right] - \frac{1}{3}Z^{-1}\left[\frac{z}{(z+5)}\right] \therefore \boldsymbol{z^{-1}}\left[\frac{\boldsymbol{z}}{\boldsymbol{z-a}}\right] = \boldsymbol{a^2}$$

$\mathrm{y}(n) = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$ $\mathbf{y}(\boldsymbol{n}) = \frac{\mathbf{1}}{\mathbf{3}}\left[(-\mathbf{2})^{\boldsymbol{n}} - \frac{\mathbf{1}}{\mathbf{3}}(-\mathbf{5})^{\boldsymbol{n}}\right]$

2. **Find the inverse Z transform of** $\frac{\boldsymbol{z(z^2-z+2)}}{\boldsymbol{(z+1)(z+2)^2}}\,\boldsymbol{by\ partial\ fractions}$

Solution :

$$Y(Z) = \frac{z(z^2-z+2)}{(z+1)(z-1)^2}$$

$$\frac{Y(Z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2}$$

$$\frac{z^2-z+2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

$$z^2-z+2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

Put z = -1 ⇒ $(-1)^2 - (-1) + 2 = A(-1-1)^2 + B(-1+1) + C(-1+1)$

4=4$A \Rightarrow A = 1$

Put z = 1 ⇒ $(1)^2 - (1) + 2 = A(1-1)^2 + B(1+1) + C(1+1)$

2=2$C \Rightarrow C = 1$

Equating coefficient of z^2 1=A+B

B=$1 - A \Rightarrow B = 0$

$$\frac{Y(Z)}{\boldsymbol{Z}} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} = \frac{1}{(z+1)} + \frac{0}{(z-1)} + \frac{1}{(z-1)^2}$$

$$\frac{Y(Z)}{\boldsymbol{Z}} = \frac{z}{(z+1)} + \frac{z}{(z-1)^2}$$

$Z^{-1}[Y(Z)] = Z^{-1}\left[\frac{z}{(z+1)}\right] + z^{-1}\left[\frac{z}{(z-1)^2}\right]$ $\therefore Z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n$

$$y(n) = (-1)^n + n \ \therefore Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=XRWaKZoJVZY

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 5.34-5.36

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Residue method

Introduction : The residue theorem, sometimes called Cauchy's residue theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals and infinite series as well.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Residue method
2. Poles and Order
3. Formula

Detailed content of the Lecture:

1. **Find** $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$ ***by residues***

Solution :

$$Y(Z) = \frac{z(z+1)}{(z-1)^3}$$

Z=1 is a pole of order 3

$$Y(Z)z^{n-1} = \frac{z(z+1)}{(z-1)^3}z^{n-1}$$

$$Y(Z)z^{n-1} = \frac{z^n(z+1)}{(z-1)^3}$$

Residue for the pole z=1

$$Res_{z=1}Y(Z)Z^{n-1} = Lt_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}(z-1)^3\frac{z^n(z+1)}{(z-1)^3}$$

$$=Lt_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}(z^{n+1}+z^n)$$

$$=\frac{1}{2}Lt_{z\to 1}\frac{d}{dz}[(n+1)z^n+(n)z^{n-1}]$$

$$=\frac{1}{2}Lt_{z\to 1}[(n+1)(n)z^{n-1}+(n)(n+1)z^{n-2}]$$

$$=\frac{1}{2}\left[(n+1)(n)(1)^{n-1}+(n)(n+1)(1)^{n-2}\right]$$

$$=\frac{1}{2}\left[n^2+n+n^2-n\right]$$

$$=\frac{1}{2}\left[2n^2\right]$$

$$Res_{z=1}Y(Z)Z^{n-1}=n^2$$

$$y(n)=sum\ of\ the\ residues \qquad y(n)=n^2$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=9BJBMooYeDE

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011,
Page.No : 5.60-5.67

Course Faculty

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Convolution theorem

Introduction :Convolution is a mathematical way of combining two signals to form a third signal. It is the single most important technique in Digital Signal Processing. Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Convolution method
2. Convolution Property
3. Geometric Progression

Detailed content of the Lecture:

1. **Using convolution theorem,find $z^{-1}[\frac{z^2}{(z-4)(z-3)}]$**

   Solution :

$$z^{-1}[\frac{z^2}{(z-4)(z-3)}] = z^{-1}(\frac{z}{z-4})\ z^{-1}(\frac{z}{z-3})$$

$$= 4^n * 3^n$$

$$= \sum_{r=0}^{n} 4^r\, 3^{n-r}$$

$$= 3^n \sum_{r=0}^{n} 4^r\, 3^{-r}$$

$$= 3^n \sum_{r=0}^{n} (\frac{4}{3})^r$$

$$= 3^n(1 + \frac{4}{3} + (\frac{4}{3})^2 + \ldots\ldots\ldots\ldots\ldots + (\frac{4}{3})^n)$$

Geometric series: a+ar+ar^2+……..+ar^n

If r>1 --------------------$\frac{a(r^{n+1}-1)}{r-1}$

If r<1 ---------------------a$(\frac{1-r^{n+1}}{1-r})$

$$= 3^n\left[\frac{\left(\frac{4}{3}\right)^{n+1}-1)}{\frac{4}{3}-1}\right]$$

$$= 3^n\left[\frac{\frac{4^{n+1}-3^{n+1}}{3^{n+1}}}{\frac{4-3}{3}}\right]$$

$$= 3^n\left[\frac{4^{n+1}-3^{n+1}}{3^{n+1}}*\frac{3}{1}\right]$$

$$=3^n\left[\frac{4^{n+1}-3^{n+1}}{3^n*3}*\frac{3}{1}\right]$$

$$=4^{n+1}-3^{n+1}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6XIX5Z3ZMHQ

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 5.36-5.50

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Convolution theorem

Introduction :Convolution is a mathematical way of combining two signals to form a third signal. It is the single most important technique in Digital Signal Processing. Using the strategy of impulse decomposition, systems are described by a signal called the impulse response.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Convolution method
2. Convolution Property
3. Geometric Progression

Detailed content of the Lecture:

1. **Using convolution theorem,** $z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=(-1)^n\left[\frac{b^{n+1}-a^{n+1}}{b-a}\right]$

Solution :

Z[f(n)*g(n)]=F[Z].G[Z]

f(n)*g(n)= $z^{-1}[F(Z).G(Z)]$

$=z^{-1}F(Z).z^{-1}G(Z)$

$z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=z^{-1}\left[\frac{z}{z+a}\right]*z^{-1}\left[\frac{z}{z+b}\right]$

$\therefore z^{-1}\left[\frac{z}{z+a}\right]=a^n$

$=(-a^n)*(-b)^n$

=f(n)*g(n)

By convolution definition,

f(n)*g(n)= $\sum_{r=0}^{n} f(r)g(n-r)$

$(-a)^n*(-b)^n=\sum_{r=0}^{n}(-a)^r(-b)^{n-r}$

$$=\sum_{r=0}^{n}(-a)^r\frac{(-b)^n}{(-b)^r}$$

$$=(-b)^n\sum_{r=0}^{n}(\frac{-a}{-b})^r$$

$$=(-b)^n\sum_{r=0}^{n}(\frac{a}{b})^r$$

$$\therefore\sum_{r=0}^{n}(\frac{a}{b})^r=\frac{1}{b^n}[\frac{a^{n+1}-b^{n+1}}{a-b}]$$

$$=(-1)^n b^n\frac{1}{b^n}[\frac{a^{n+1}-b^{n+1}}{a-b}]$$

$$=(-1)^n[\frac{a^{n+1}-b^{n+1}}{a-b}]$$

$$=(-1)^n[\frac{b^{n+1}-a^{n+1}}{b-a}]$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=6XIX5Z3ZMHQ

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2[rd] Edition, 2011,
Page.No : 5.36-5.50

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Formation of difference equations

Introduction:

A difference equation is formed by eliminating the arbitrary constants from a given relation. The order of the difference equation is equal to the number of arbitrary constants in the given relation. Following examples illustrate the formation of difference equations.

Prerequisite knowledge for Complete understanding and learning of Topic:

A difference equation is relation between the differences of an unknown function at one or more general values of the argument.

Ex: Δy(n+2)+y(n)=2

Detailed content of the Lecture:

Form the difference equation from $\boldsymbol{y_n = a + b3^n}$

Solution:

Given: $y_n = a + b3^n$

$y_{n+1} = a + b3^{n+1} = a + 3b3^n$ ---(1)

$y_{n+2} = a + b3^{n+2} = a + 9b3^n$ ---(2)

Eliminating a and b from (1) and(2)

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n([9-3]-(1)[9y_{n+1} - 3y_{n+2}] + (1)[y_{n+1} - y_{n+2}] = 0$$

$6y_n - 9y_{n+1} + 3y_{n+2} + y_{n+1} - y_{n+2} = 0$

$2y_{n+2} - 8y_{n+1} + 6y_n = 0$

2. Form the difference equation from $\boldsymbol{y_n = (A + Bn)2^n}$

Solution:

Given: $y_n = (A + Bn)2^n = A2^n + Bn2^n$

$y_{n+1} = A2^{n+1} + B(n+1)2^{n+1}$

$y_{n+1} = 2A2^n + 2B(n+1)2^n$ ---(1)

$y_{n+2} = A2^{n+2} + B(n+2)2^{n+2}$

$y_{n+2} = 4A2^n + 4B(n+2)2^n$ ---(2)

Eliminating a and b from (1) and(2)

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix}=0$$

y_n [8(n+2)-8(n+1)]-(1)[4(n+2)$y_{n+1} - 2(n+1)\ y_{n+2}$] + (n)[4y_{n+1} -2y_{n+2}]=0

y_n [8n+16-8n-8]-(4n+8)y_{n+1} +(2n+2)y_{n+2} $+ 4ny_{n+1}$ -2ny_{n+2} $= 0$

(2n+2-2n)y_{n+2} $+ y_{n+1}$ (-4n-8+4n)+y_n[8]=0

$2y_{n+2} - 8y_{n+1}$ +8y_n =0

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=_A-ozcPiFvg

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 5.71-5.77

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Solution of difference equations using Z - transforms

Introduction : Using the initial conditions, we get an algebraic equation of the form F(z) = f(z). By taking the inverse Z-transform, we get the required solution f_n of the given difference equation. Solve the difference equation $y_{n+1} + y_n = 1$, $y_0 = 0$, by Z - transform method. Let Y(z) be the Z -transform of $\{y_n\}$

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Formation of Difference Equations
2. Poles and order
3. Residue Method

Detailed content of the Lecture:

1. **solve$(y_{n+2}) + 4(y_{n+1}) + 3y_n = 2^n$ with $y_0 = 0$ &$y_1 = 1$ using Z-transform.**

Solution:

$$Z(y_{n+2}) + 4Z(y_{n+2}) + 3Z(y_n) = Z(2^n)$$

$$z^2y(z) - z^2y(0) - Zy(1) + 4(Z(y(z) - zy(0)) + 3y(z) = \frac{z}{z-2}$$

$$z^2y(z) - 0 - Z + 4(Z(y(z) - 0) + 3y(z) = \frac{z}{z-2}$$

$$y(z)(z^2 + 4z + 3) = \frac{z}{z-2} + z$$

$$(z)(z+1)(z+3 = \frac{z}{z-2} + z$$

$$y(z) = \frac{z}{(z-2)(z+1)(z+3)} + \frac{z}{(z+1)(z+3)}$$

$$y(z) = \frac{z^n}{(z-2)(z+1)(z+3)} + \frac{z^n}{(z+1)(z+3)}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=2} = \lim_{z\to 2}(z-2)\frac{z^n}{(z-2)(z+1)(z+3)}$$

$$= \lim_{z \to 2} \frac{z^n}{(z+1)(z+3)}$$

$$= \frac{2^n}{15}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=-1} = \lim_{z \to 1} (z+1) \frac{z^n}{(z-2)(z+1)(z+3)}$$

$$= \frac{(-1)^n}{(-3)(2)} = \frac{(-1)^n}{-6}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=-3} = \lim_{z \to 1} (z+3) \frac{z^n}{(z-2)(z+1)(z+3)}$$

$$= \frac{(-3)^n}{(-5)(-2)} = \frac{(-3)^n}{10}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=-1} = \lim_{z \to 1} (z+1) \frac{z^n}{(z+1)(z+3)}$$

$$= \frac{(-1)^n}{(2)}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=-3} = \lim_{z \to 1} (z+3) \frac{z^n}{(z+1)(z+3)} = \frac{(-3)^n}{(-2)}$$

$$\text{Res}[\{z^{n-1} y(z)\} = \text{sum of residues}$$

$$= \frac{2^n}{15} + \frac{(-1)^n}{-6} + \frac{(-1)^n}{(2)} + \frac{(-3)^n}{10} + \frac{(-3)^n}{(-2)}$$

$$= \frac{2^n}{15} + \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=9sCw9kg021Q

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 5.77-5.95

Course Faculty

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **II-Z-Transforms And Difference Equations**

Date of Lecture:

Topic of Lecture: Solution of difference equations using Z - transforms

Introduction : Using the initial conditions, we get an algebraic equation of the form F(z) = f(z). By taking the inverse Z-transform, we get the required solution f_n of the given difference equation. Solve the difference equation $y_{n+1} + y_n = 1$, $y_0 = 0$, by Z - transform method. Let Y(z) be the Z -transform of $\{y_n\}$

Prerequisite knowledge for Complete understanding and learning of Topic:

4. Formation of Difference Equations
5. Poles and order
6. Residue Method

Detailed content of the Lecture:

1. **Solve y(n+3) – 3y(n+1)+2y(n)=0**

Solution :

Take z transform on both sides,

$$Z[y(n+3)]-3Z[y(n+1)]+2Z[y(n)]=0$$

$$[z^3y(z)-z^3y(0)-z^2y(1)-z(2)]-3(zy(z)-4z)+2y(z)=0$$

$$z^3y(z)-4z^3-8z-3(zy(z-4z)+2y(z)=0$$

$$(z^3-3Z+2)y(z)=4z^3-4z$$

$$Y(z)=\frac{4z(z^2-1)}{z^3-3z+2}$$

$$=\frac{4z(z+1)}{(z-1)(z+2)}$$

$$z^{n-1}\,Y(z)=\frac{4z^n(z^2-1)}{(z+1)(z+2)}$$

Poles : 1,-2 (order 1)

$$\text{Res}[z^{n-1}\, y(z)]_{z=1} = \lim_{z\to 1}(z-1)\frac{4z^n(z^2-1)}{(z+1)(z+2)} = \frac{8(1)^n}{3}$$

$$\text{Res}[z^{n-1}\, y(z)]_{z=-2} = \lim_{z\to -2}(z-2)\frac{4z^n(z^2-1)}{(z+1)(z+2)} = \frac{4(-2)^n}{3}$$

Sum of residues= y(n) = $\frac{8(1)^n}{3} + \frac{4(-2)^n}{3}$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=9sCw9kg021Q

Important Books/Journals for further learning including the page nos.:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 5.77-5.95

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II / III

Course Name with Code :Transforms and Partial Differential Equations/19BSS23

Course Faculty : M.Nazreen Banu

Unit : III-Fourier Series **Date of Lecture:**

Topic of Lecture: Dirichlet‟s conditions and General Fourier series

Introduction : To represent any periodic signal $f(x)$, Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthoganality condition.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. **Odd Function**
2. **Even Function**
3. **Fourier Series Formula**

Detailed content of the Lecture:

1 **Write the Dirichlet's conditions on the existence of Fourier series.**

Solution: Any function $f(x)$ can be developed as a Fourier series in any one period, provided

a. It is periodic, single valued, finite.
b. The number of discontinuities if any is finite.
c. The number of maxima and minima if any is finite.

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=blS_OImUJ-c

Important Books/Journals for further learning including the page nos.:

A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011,
Page.No : 2.36-2.43

Course Faculty

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LECTURE HANDOUTS | **L**

AI&DS | **II / III**

Course Name with Code :Transforms and Partial Differential Equations/19BSS23

Course Faculty : M.Nazreen Banu

Unit : III- Fourier Series **Date of Lecture:**

Topic of Lecture: General Fourier series in $(0,2\pi)$

Introduction : To represent any periodic signal $f(x)$, Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthoganality condition.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. **Odd Function**
2. **Even Function**
3. **Fourier Series Formula**

Detailed content of the Lecture:

1. $\boldsymbol{Find F.S for f(x) = x^2 in\ (0, 2\pi)\ \& also PT}$

$\boldsymbol{i)\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$

$\boldsymbol{ii)\ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}}$

Solution :

Fourier series $f(x) = \frac{a_\circ}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_\circ = \frac{1}{l}\int_0^{2\pi} f(x)dx$$

$$a_\circ = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$$

$$a_\circ = \frac{1}{\pi}\left[\frac{x^3}{3}\right]$$

$$a_\circ = \frac{1}{\pi}\left[\frac{(2\pi)^3 - 0}{3}\right]$$

$$a_0 = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos\ nxdx$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} - 2x\frac{-cosnx}{n^2} + 2\frac{-cosnx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[4\pi^2\frac{0}{n} + 4\pi\left(\frac{1}{n^2}\right) - 2\frac{0}{n^3}\right] - [0 - 0 - 0]\right\}$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sinnxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 sinnxdx$$

$$= \frac{1}{\pi}\left[\frac{[-x^2cosnx]}{n} - \int_0^{2\pi} -\frac{2xcosnxdx}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\left[-(2\pi)^2(1) - \left(-(0)(1)\right)\right]}{n} + \frac{[2xsinnx]}{n^2} - \int_0^{2\pi}\frac{2sinnxdx}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{4\pi^2}{n}\right] + [0 - 0] - \frac{[-2cosnx]}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{4\pi^2}{n}\right] + \frac{[2(1) - 2(1)]}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{4\pi^2}{n}\right] + 0\right]$$

$$= -\frac{4\pi}{n}$$

The Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

$$f(x) = \frac{\frac{4\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosnx - \frac{4\pi}{n}sinnx\right)$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosnx - \frac{4\pi}{n}sinnx\right)$$

Deduction : 1

Put $x = 0$ $(x = 0\ is\ a\ point\ of\ discomtinuity)$

$$f(x) = \frac{f(0) + f(2\pi)}{2} = \frac{4\pi^2}{2} = 2\pi^2$$

$$2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosn(0) - \frac{4\pi}{n}sinn(0)\right)$$

$$2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosn(0)\right)$$

$$\frac{2\pi^2}{3}\frac{1}{4} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Deduction : 2

Put $x = \pi$ $(x = \pi\ is\ a\ point\ of\ comtinuity)$

$$f(\pi) = \pi^2$$

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosn(\pi) - \frac{4\pi}{n}sinn(\pi)\right)$$

$$\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty}\left(\frac{4}{n^2}cosn(\pi)\right)$$

$$-\frac{\pi^2}{3}\frac{1}{4} = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right)$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=JAS57fyIbhA

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 1.12-0.30

Course Faculty

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code :**Transforms and Partial Differential Equations/19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **III-Fourier Series** **Date of Lecture:**

Topic of Lecture: General Fourier series in $(0,2l)$

Introduction : A Fourier series is an expansion of a periodic function f(x) individually, and then recombined to obtain the solution to the original problem or an approximation to solutions of a linear homogeneous ordinary differential equation, if such an equation can be take Similarly, the function is instead defined on the interval [0,2L]

Prerequisite knowledge for Complete understanding and learning of Topic:

1. **Fourier series formula**
2. **Bernoulli Formula**

Detailed content of the Lecture:

1. **If $f(x) = 2x$ in the interval $(0,4)$, find the value of a_2.**

   Solution:

Given $f(x) = 2x$ in $(0,4)$

$$\therefore a_2 = \frac{1}{2}\int_0^4 2x \cos\frac{2\pi x}{2}\, dx$$

$$= \frac{1}{2}\int_0^4 2x \cos\pi x\, dx = \int_0^4 x \cos\pi x\, dx$$

$$= \left[x\left[\frac{\sin\pi x}{\pi}\right] - (1)\left[\frac{-cos\pi x}{\pi^2}\right]\right]_0^4 = 0.$$

2. **Find Fourier Series to represent $f(x) = 2x - x^2$ with period 3 in the range $(0,3)$**

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$l = \frac{3}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_0 = \frac{2}{3}\int_0^{3} (\boldsymbol{2x - x^2})dx$$

$$a_0 = \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$a_0 = \frac{2}{3}[9-9]$$

$$\therefore a_0 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$a_n = \frac{2}{3}\int_0^{3} (\boldsymbol{2x - x^2})\cos\frac{2n\pi x}{3}dx$$

$$a_n = \frac{2}{3}\left[(\boldsymbol{2x - x^2})\frac{3}{2n\pi}\sin\frac{2n\pi x}{3} - (2-2x)\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi x}{3} + (-2)\frac{-27}{8n^3\pi^3}\sin\frac{2n\pi x}{3}\right]_0^3$$

$$a_n = \frac{2}{3}\left[\left((\boldsymbol{2.3 - 3^2})\frac{3}{2n\pi}\sin\frac{2n\pi 3}{3} - (2-2.3)\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi 3}{3} + (-2)\frac{-27}{8n^3\pi^3}\sin\frac{2n\pi 3}{3}\right)\right.$$
$$\left. - \left((\boldsymbol{2.0 - 0^2})\frac{3}{2n\pi}\sin\frac{2n\pi 0}{3} - (2-2.0)\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi 0}{3} + (-2)\frac{-27}{8n^3\pi^3}\sin\frac{2n\pi 0}{3}\right)\right]$$

$$sin0 = 0, cos0 = 1, sin2n\pi = 0, cos2n\pi = 1$$

$$a_n = \frac{2}{3}\left[\left(-(2-2.3)\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi 3}{3}\right) - \left(-(2)\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi 0}{3}\right)\right]$$

$$a_n = \frac{2}{3}\left[-4\frac{9}{4n^2\pi^2} - 2\frac{9}{4n^2\pi^2}\right]$$

$$a_n = \frac{2}{3}\left[-6\frac{9}{4n^2\pi^2}\right]$$

$$a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

$$b_n = \frac{2}{3}\int_0^{3} (\boldsymbol{2x - x^2})\sin\frac{2n\pi x}{3}dx$$

$$b_n = \frac{2}{3}\left[(\boldsymbol{2x - x^2})\frac{-3}{2n\pi}\cos\frac{2n\pi x}{3} - (2-2x)\frac{-9}{4n^2\pi^2}\sin\frac{2n\pi x}{3} + (-2)\frac{27}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right]_0^3$$

$$b_n = \frac{2}{3}\left[\left((\boldsymbol{2.3 - 3^2})\frac{-3}{2n\pi}\cos\frac{2n\pi 3}{3} - (2-2.3)\frac{-9}{4n^2\pi^2}\sin\frac{2n\pi 3}{3} + (-2)\frac{27}{8n^3\pi^3}\cos\frac{2n\pi 3}{3}\right)\right.$$
$$\left. - \left((\boldsymbol{2.0 - 0^2})\frac{-3}{2n\pi}\cos\frac{2n\pi 0}{3} - (2-2.0)\frac{-9}{4n^2\pi^2}\sin\frac{2n\pi 0}{3} + (-2)\frac{27}{8n^3\pi^3}\cos\frac{2n\pi 0}{3}\right)\right]$$

$$b_n = \frac{2}{3}\left[\left((\boldsymbol{-3})\frac{-3}{2n\pi}\cos 2n\pi + (-2)\frac{27}{8n^3\pi^3}\cos 2n\pi\right) - \left((-2)\frac{27}{8n^3\pi^3}\cos 0\right)\right]$$

$$b_n = \frac{2}{3}\left[\left((\mathbf{-3})\frac{-3}{2n\pi} + (-2)\frac{27}{8n^3\pi^3}\right) - \left((-2)\frac{27}{8n^3\pi^3}\right)\right]$$

$$b_n = \frac{2}{3}\left[\left(\frac{9}{2n\pi} - \frac{27}{4n^3\pi^3}\right) - \left(\frac{-27}{4n^3\pi^3}\right)\right]$$

$$b_n = \frac{2}{3}\left[\frac{9}{2n\pi} - \frac{27}{4n^3\pi^3} + \frac{27}{4n^3\pi^3}\right]$$

$$b_n = \frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$b_n = \frac{3}{n\pi}$$

The Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$$

$$f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{3}{n\pi}\sin\frac{2n\pi x}{3}\right)$$

$$f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{3}{n\pi}\sin\frac{2n\pi x}{3}\right)$$

$$\mathbf{2x - x^2} = \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \frac{3}{n\pi}\sin\frac{2n\pi x}{3}\right)$$

Video Content / Details of website for further learning (if any):

1. **https://www.youtube.com/watch?v=p3t233ZV5ok**
2. **https://www.youtube.com/watch?v=Dnf8vahAzDI**

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 1.47-1.61

Course Faculty

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code :**Transforms and Partial Differential Equations/19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **III-Fourier Series** **Date of Lecture:**

Topic of Lecture: Odd and even functions and General Fourier series in $(-\pi, \pi)$

Introduction :A function f(x) is said to have period P if f(x+P)=f(x) for all x. Let the function f(x) has period 2π. In this case, it is enough to consider behavior of the function on the interval [−π,π].

1. Suppose that the function f(x) with period 2π is absolutely integrable on [−π,π] so that the following so-called Dirichlet integral is finite.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. **Fourier series formula**
2. **Bernoulli Formula**

Detailed content of the Lecture:

1. Find the constant term in th expansion of cos^2x as a Fourier series in the interval $(-\pi\ ,\pi\ )$.

Solution: Given $f(x) = cos^2x$

The constant term

2. Obtain the first term of the Fourier series for the function $f(x) = x^2$, $(-\pi, \pi)$.

Solution:

Given $f(x) = x^2 , -\pi < x < \pi$is an even function

Hence $b_n = 0$ and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$...............(1)

First term of the Fourier series is$\frac{a_0}{2}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x^2dx = \frac{2}{\pi}(\frac{x^3}{3})_0^{\pi} = \frac{2}{\pi}[\frac{\pi^3}{3} - 0]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^2.$$

3. Determine the value of a_n in the Fourier series expansion of $(x) = x^3\ in - \pi < x < \pi$.

Solution:

Given:$f(x) = x^3 is\ an\ odd\ function\ in - \pi < x < \pi$

Hence $a_n = 0$.

Video Content / Details of website for further learning (if any): https://www.youtube.com/watch?v=cfxqDp-ks20
Important Books/Journals for further learning including the page nos.: **1.** 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 1.28-1.40

Course Faculty

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Course Name with Code :**Transforms and Partial Differential Equations/19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **III-Fourier Series** **Date of Lecture:**

Topic of Lecture: Odd and even functions and General Fourier series in $(-l, l)$

Introduction :A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical.

Prerequisite knowledge for Complete understanding and learning of Topic:

3. **Fourier series formula**
4. **Bernoulli Formula**

Detailed content of the Lecture:

1. **Give the expression for the Fourier Series co-efficient b_n for the function $f(x)$ defined in $(-2, 2)$.**

   Solution: $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx.$

2. **Find the Fourier Series for the function $f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$ in $(-l, l)$.**

Solution:

$$\text{Given : } f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

$$f(-x) = \begin{cases} 0 & -1 < -x < 0 \\ 1 & 0 < -x < 1 \end{cases}$$

$$= \begin{cases} 0 & 0 < x < 1 \\ 1 & -1 < x < 0 \end{cases}$$

$$f(-x) \neq -f(x) \neq f(x)$$

Therefore, $f(x)$ is neither even nor odd.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{l} + b_n sin\frac{n\pi x}{l}$$

Put $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosn\pi x + b_n sinn\pi x$$

To find a_0 :

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

Put $l = 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx$$

$$= \int_{0}^{1} (1)dx = [\mathrm{x}]_{0}^{1}$$

$$= [1 - 0] = 1$$

To find a_n :

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\frac{n\pi x}{l} dx$$

Put $l = 1$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)cosn\pi x \, dx$$

$$a_n = \int_{0}^{1} (1)cosn\pi x \, dx$$

$$= \left[\frac{sinn\pi x}{n\pi}\right]_{0}^{1}$$

$$= \left[\frac{sinn\pi}{n\pi} - \frac{sin0}{n\pi}\right] = 0$$

To find b_n :

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)sin\frac{n\pi x}{l} \, dx$$

Put $l = 1$

$$b_n = \frac{1}{1}\int_{-1}^{l} f(x)sinn\pi x \, dx$$

$$b_n = \int_{0}^{1} (1)sinn\pi x \, dx$$

$$= \left[\frac{-cosn\pi x}{n\pi}\right]_{0}^{1}$$

$$= \left[\frac{-cosn\pi}{n\pi} + \frac{cos0}{n\pi}\right]$$

$$= [-\frac{(-1)^n}{n\pi} + \frac{1}{n\pi}]$$

$$= \frac{1}{n\pi}[-(-1)^n + 1]$$

$$= \frac{1}{\pi n}\begin{cases} 2 & ,n = odd \\ 0 & ,n = even \end{cases}$$

$$b_n = \begin{cases} \dfrac{2}{\pi n} & ,n = odd \\ 0 & ,n = even \end{cases}$$

The Fourier Series is

$$f(x) = \frac{1}{2} + \sum_{n=odd}^{\infty} \big((0)\text{cosn}\pi\text{x}\big) + \sum_{n=odd}^{\infty} \frac{2}{n\pi}\, sinn\pi x$$

$$f(x) = \frac{1}{2} + \sum_{n=odd}^{\infty} \frac{2}{n\pi}\, sinn\pi x$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tNDvigipV5w

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2^{rd} Edition, 2011,
Page.No : 1.62-1.72

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LECTURE HANDOUTS | **L**

AI&DS | **II / III**

Course Name with Code :**Transforms and Partial Differential Equations/19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **III-Fourier Series** **Date of Lecture:**

Topic of Lecture: Half Range Fourier Sine Series and Parseval's identity

Introduction : If a function is defined over half the range, say 0 to *l*, instead of the full range from -l to l, it may be expanded in a series of sine terms only. The series produced is then called a **half range sine Fourier series**.Conversely, the Fourier Series of an odd function can be analysed using the half range sine definition.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Half Range sine Series
2. Parseval's identity
3. Bernoulli formula

Detailed content of the Lecture:

1. Find the Half Range Fourier Sine Series for the function of $f(x) = x(\pi - x)$ in (o, π) and hence deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^2}{32}$

Solution:

Given : $f(x) = x(\pi - x) = (\pi x - x^2)$

The Half Range Fourier Sine Series $f(x) = \sum_{n=1}^{\infty} b_n \, sinnx$

To find b_n :

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, sinnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) sinnxdx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{cosnx}{n}\right) - (1-2x)\left(-\frac{sinnx}{n^2}\right) + (-2)\left(\frac{cosnx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\begin{matrix}\left\{(\pi^2-\pi^2)\left(-\frac{cosn\pi}{n}\right)-(1-2\pi)\left(-\frac{sinn\pi}{n^2}\right)+(-2)\left(\frac{cosn\pi}{n^3}\right)\right\}- \\ \{(0-0)\left(-\frac{cos0}{n}\right)-(1-0)\left(-\frac{sin0}{n^2}\right)+(-2)\left(\frac{cos0}{n^3}\right)\}\end{matrix}\right]$$

$$= \frac{2}{\pi}\left[\left\{(0)+(0)-(2)\left(\frac{(-1)^n}{n^3}\right)\right\}-(0)+(0)+(2)\left(\frac{1}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[-(2)\left(\frac{(-1)^n}{n^3}\right)+\frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}.\frac{2}{n^3}[-(-1)^n+1]$$

$$= \frac{4}{\pi n^3}[-(-1)^n+1]$$

$$= \frac{4}{\pi n^3}\begin{cases} 2\,, n = odd \\ 0\,, n = even\end{cases}$$

$$b_n = \begin{cases}\frac{8}{\pi n^3}\,, n = odd \\ 0\quad, n = even\end{cases}$$

The Half Range Sine Series :

$$f(x) = \sum_{n=1}^{\infty} b_n sinnx$$

$$= \sum_{n=odd}^{\infty}\frac{8}{\pi n^3}sinnx$$

$$= \frac{8}{\pi}\sum_{n=odd}^{\infty}\frac{sinnx}{\pi n^3}$$

DEDUCTION :

$$Putx = \frac{\pi}{2}\ (x = \frac{\pi}{2} isapointof continuity)$$

$$Given: f(x) = (\pi x - x^2)$$

$$f(\frac{\pi}{2}) = (\frac{\pi^2}{2}-\frac{\pi^2}{4}) = (\frac{2\pi^2-\pi^2}{4}) = (\frac{\pi^2}{4})$$

$$Putx = \frac{\pi^2}{4} inf(x)$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=odd}^{\infty}\frac{\sin n\frac{\pi}{2}}{n^3}$$

$$\frac{\pi^2}{4}\cdot\frac{\pi}{8} = \sum_{n=odd}^{\infty}\frac{\sin n\frac{\pi}{2}}{n^3}$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} + \frac{(-1)}{3^3} + \frac{1}{5^3} + \frac{(-1)}{7^3} + \cdots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

2**. Obtain Half Range sine series for $\boldsymbol{f(x) = x}$in $\boldsymbol{(0, \pi)}$. Show that**

$$\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$

Solution :

Given $f(x) = x$

The Half Range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\pi\left(-\frac{\cos n\pi}{n}\right) - (1)\left(-\frac{\sin n\pi}{n^2}\right)\right\} - \left\{0\left(-\frac{\cos 0}{n}\right) - (1)\left(-\frac{\sin 0}{n^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$\therefore b_n = \frac{2(-1)^n}{\pi}$$

The Half Range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi} sinnx$$

DEDUCTION :

By Parseval's Identity

$$\frac{{a_0}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2) = \frac{1}{b-a}\int_a^b (f(x))^2\, dx$$

$$\frac{0}{4} + \frac{1}{2}\sum_{n=1}^{\infty} 0 + \left(\frac{2(-1)^n}{\pi}\right)^2 = \frac{1}{\pi - 0}\int_0^{\pi} x^2 dx$$

$$2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{1}{3\pi}[\pi^3 - 0]$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{2}\frac{1}{\pi}\frac{\pi^3}{3}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

Video Content / Details of website for further learning (if any):

1. **https://www.youtube.com/watch?v=jmg2Tsi3h_A**
2. **https://www.youtube.com/watch?v=XrWlr9BdzRQ**

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 1.72-1.87

Course Faculty

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Course Name with Code :Transforms and Partial Differential Equations/19BSS23

Course Faculty : M.Nazreen Banu

Unit : III-Fourier Series **Date of Lecture:**

Topic of Lecture:Half Range Cosine series and Parseval's Identity

Introduction : If a function is defined over half the range, say 0 to *l*, instead of the full range from -*l* to *l*, it may be expanded in a series of cosine terms only. The series produced is then called a **half range cosine Fourier series**.Conversely, the Fourier Series of an even function can be analysed using the half range cosine definition.

Prerequisite knowledge for Complete understanding and learning of Topic:

4. Half Range Cosine Series
5. Parseval's identity
6. Bernoulli formula

Detailed content of the Lecture:

1. **Without finding the value of a_0 , a_n&b_n for the function $f(x) = x^2$ in $(0, \pi)$, find the value of $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$**

   Solution: Given $f(x) = x^2$ in $(0, \pi)$

   By Parseval's Theorem

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[x^2]^2 dx = \frac{\pi^4}{5}.$$

2. **Obtain Half Range Cosine Series for the function $f(x) = x$in $(0, \pi)$. Use Parseval's identity and show that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$.**

Solution :

Given : $f(x) = x$

The Half Range Cosine Series : $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

To find a_0 :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}[\pi^2 - 0]$$

$$= \frac{1}{\pi}[\pi^2]$$

$$a_0 = \pi$$

To find a_n :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) cosnx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, cosnx\, dx$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi sinn\pi}{n} + \frac{cosn\pi}{n^2}\right\} - \left\{\frac{0}{n} + \frac{cos0}{n^2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{n^2}[(-1)^n - 1]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$= \frac{2}{\pi n^2}\begin{cases} -2 & n = odd \\ 0 & n = even \end{cases}$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2} & n = odd \\ 0 & n = even \end{cases}$$

The Half Range Cosine Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx$$

$$= \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{\pi n^2}\, cosnx$$

$$= \frac{\pi}{2} + \frac{-4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\, cosnx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\, cosnx$$

DEDUCTION :

By Parseval’s Identity

$$\frac{a_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2) = \frac{1}{b-a}\int_a^b (f(x))^2\, dx$$

$$\frac{\pi^2}{4} + \frac{1}{2}\sum_{n=odd}^{\infty}\frac{16}{\pi^2 n^4} + 0 = \frac{1}{\pi - 0}\int_0^{\pi} x^2 dx$$

$$\frac{\pi^2}{4} + \frac{1}{2}.\frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\frac{\pi^2}{4} + \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{1}{3\pi}[\pi^3 - 0]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{\pi^2}{3} - \frac{\pi^2}{4}$$

$$\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{4\pi^2 - 3\pi^2}{12}.\frac{\pi^2}{8}$$

$$= \frac{\pi^2}{12}.\frac{\pi^2}{8}$$

$$\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] = \frac{\pi^4}{96}$$

Video Content / Details of website for further learning (if any):
1. https://www.youtube.com/watch?v=gWXTyHO5NWg
2. https://www.youtube.com/watch?v=pjA4TAmNIzI

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 1.72-1.87

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MUTHAYAMMAL ENGINEERING COLLEGE

(An Autonomous Institution)

(Approved by AICTE, New Delhi, Accredited by NAAC & Affiliated to Anna University)
Rasipuram - 637 408, Namakkal Dist., Tamil Nadu

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II / III

Course Name with Code :Transforms and Partial Differential Equations/19BSS23

Course Faculty : M.Nazreen Banu

Unit : III-Fourier Series **Date of Lecture:**

Topic of Lecture:Harmonics Analysis

Introduction : Harmonics Analysis

The process of finding the fourier series for a function given by numerical value is known as Harmonic Analysis

$$f(x) = \frac{a_0}{2} + (a_1 cosx + b_1 sinx) + (a_2 cos2x + b_2 sin2x) + (a_3 cos3x + b_3 sin3x) + \cdots$$

Prerequisite knowledge for Complete understanding and learning of Topic:

1. **First Harmonic**
2. **Second Harmonic**
3. **Third Harmonic**

Detailed content of the Lecture:

1. Define Harmonic and write the first two harmonic

The process of finding the fourier series for a function given by numerical value is known as Harmonic Analysis

$$f(x) = \frac{a_0}{2} + (a_1 cosx + b_1 sinx) + (a_2 cos2x + b_2 sin2x)$$

2. What are the fundamental or First Harmonic

The term $(a_1 cosx + b_1 sinx)$ in the fourier series is called fundamental or First Harmonic
The term $(a_2 cos2x + b_2 sin2x)$ in the fourier series is called Second Harmonic

3. Find the Fourier Series upto one Harmonic

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{6}$	$\frac{5T}{6}$	T
$f(x)$	1.98	1.30	1.05	1.30	0.88	0.25	1.98

Solution:

Since the last value of y is a repetition of the first, only the first six values will be used .
The Fourier Series of first three harmonics is given by

$$f(x) = \frac{a_0}{2} + (a_1 cos\theta + b_1 sin\theta), \theta = \frac{2\pi x}{T}$$

x	$\theta = \frac{2\pi x}{T}$	$y = f(x)$	$ycos\theta$	$ysinx\theta$
0	0	1.98	1.980	0
$\frac{T}{6}$	$\frac{\pi}{3}$	1.30	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	1.05	-0.525	0.9093
$\frac{T}{2}$	π	1.30	-1.3	0
$\frac{2T}{6}$	$\frac{4\pi}{3}$	-0.85	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
		$\sum y = 4.5$	$\sum y\,cos\theta = 1.12$	$\sum y\,sin\theta = 3.013$

$$n = 6$$
$$a_0 = 2\left(\frac{\sum y}{n}\right) = 1.50$$
$$a_1 = 2\left(\frac{\sum ycos\theta}{n}\right) = 0.37$$
$$b_{1=}2\left(\frac{\sum ysin\theta}{n}\right) = 1.004$$
$$f(x) = \frac{1.5}{2} + (3.7cos\theta + 1.004sin\theta)$$
$$f(x) = 0.75 + (3.7cos\theta + 1.004sin\theta)$$

Video Content / Details of website for further learning (if any):

1. **https://www.youtube.com/watch?v=09BqFdQFCTg**

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 1.100-1.110

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Course Name with Code :Transforms and Partial Differential Equations/19BSS23

Course Faculty : M.Nazreen Banu

Unit : III-Fourier Series **Date of Lecture:**

Topic of Lecture:Harmonic Analysis

Introduction : Harmonics Analysis

The process of finding the fourier series for a function given by numerical value is known as Harmonic Analysis

$$f(x) = \frac{a_0}{2} + (a_1 cosx + b_1 sinx) + (a_2 cos2x + b_2 sin2x) + (a_3 cos3x + b_3 sin3x)$$

Prerequisite knowledge for Complete understanding and learning of Topic:

4. **First Harmonic**
5. **Second Harmonic**
6. **Third Harmonic**

Detailed content of the Lecture:

1.Find the Fourier series upto third harmonic

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.2

Solution:

x	$y = f(x)$	$ycosx$	$ysinx$	$ycos2x$	$ysin2x$	$ycos3x$	$ysin3x$
0	1.0	1	0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.7	1.212	-0.7	1.212	-1.4	0
$\frac{2\pi}{3}$	1.9	-0.95	1.65	-0.95	-1.645	1.9	0
π	1.7	-1.7	0	1.7	0	-1.7	0
$\frac{4\pi}{3}$	1.5	-0.75	-1.299	-0.75	1.299	1.5	0
$\frac{5\pi}{3}$	1.2	0.6	-1.039	-0.6	-1.039	-1.2	0
	$\sum y = 8.7$	$\sum y\,cosx = -1.1$	$\sum y\,sinx = 0.5196$	$\sum y\,cos2x = -0.3$	$\sum y\,sin2x = -0.1732$	$\sum y\,cos3x = 0.1$	$\sum y\,sin3x = 0$

Since the last value of y is 0 repetition of the first, only the first 6 value will be used. The Fourier series of first three harmonic is given by

$$f(x)=\frac{a_0}{2}+(a_1cosx+b_1sinx)+(a_2cos2x+b_2sin2x)+(a_3cos3x+b_3sin3x)$$

$$a_0=2\left(\frac{\sum y}{n}\right)=2.90$$

$$a_1=2\left(\frac{\sum ycosx}{n}\right)=-0.37$$

$$b_{1=}2\left(\frac{\sum ysinx}{n}\right)=0.17$$

$$a_2=2\left(\frac{\sum ycos2x}{n}\right)=-0.10$$

$$b_{2=}2\left(\frac{\sum ysin2x}{n}\right)=-0.06$$

$$a_3=2\left(\frac{\sum ycos3x}{n}\right)=0.03$$

$$b_3=2\left(\frac{\sum ysin3x}{n}\right)=0$$

$$f(x)=\frac{2.9}{2}+(-0.37cosx+0.17sinx)+(-0.1cos2x-0.06sin2x)+(0.03cos3x+0sin3x)$$

$$f(x)=1.45+(-0.37cosx+0.17sinx)+(-0.1cos2x-0.06sin2x)+(0.03cos3x+0sin3x)$$

Video Content / Details of website for further learning (if any):

1. **https://www.youtube.com/watch?v=09BqFdQFCTg**

Important Books/Journals for further learning including the page nos.:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2[rd] Edition, 2011, Page.No : 1.100-1.110

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II / III

Course Name with Code :**Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV - Boundary value problems** **Date of Lecture:**

Topic of Lecture: Classification of PDE

Introduction :A Boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Boundary Value Problem
2. Elliptic Function
3. Hyperbolic Function
4. Parabolic Function

Detailed content of the Lecture:

1. **Classification of Second order Quasi Linear Partial Differential Equations**

A general form of second order linear partial differential equation of two independent

variable x & y is

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+F=0$$

Where, A,B,C,D,E&F are either constants (or) functions of x &y .

$B^2-4AC<0$ Elliptic Function

$B^2-4AC>0$ Hyperbolic Function

$B^2-4AC=0$ Parabolic Function

2. **Classify the PDE: $3u_{xx}+4u_{xy}+6u_{yy}-2u_x+u_y-u=0$**

Solution: Given: $3u_{xx}+4u_{xy}+6u_{yy}-2u_x+u_y-u=0$

Here A= 3,B = 4, C = 6

B^2- 4AC = - 56 < 0

The nature of the PDE is elliptic equation.

3. **Classify the PDE: $3u_{xx}$+ $4u_{xy}$+$3u_y$-$2u_x$= 0.**

Solution: Given: $3u_{xx}$+ $4u_{xy}$+$3u_y$-$2u_x$= 0

Here A= 3, B = 4, C = 0

B^2-4AC = 16 > 0.

Hence, the given PDE is classified as hyperbolic equation.

4. **Classify the PDE** $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Solution: Given: $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$$Here\ A = 4, B = 0, C = 0 \quad \therefore B^2 - 4AC = 0$$

∴The given equation is parabolic equation

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=RsztUXnoDPk

Important Books/Journals for further learning including the page nos.:
1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 4.1-4.10

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LECTURE HANDOUTS

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II / III

Course Name with Code :**Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV- Boundary Value Problems** **Date of Lecture:**

Topic of Lecture: One dimension wave equation

Introduction : The wave equation in one space dimension can be written as follows:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable y may represent a second space dimension, if, for example, the displacement y takes place in y-direction, as in the case of a string that is located in the x–y plane.

Prerequisite knowledge for Complete understanding and learning of Topic:

1. One dimension wave equation
2. Boundary conditions
3. Half range Fourier sine series

Detailed content of the Lecture:

1. A string is stretched and fastened to two points x = 0 and x = l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$from which it is released at time t = 0. Find the displacement of any point of the string at a distance x from one end at any time t.

Solution :

Step : 1 One dimension wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$..................... (1)

Step : 2 Boundary conditions

1. $y(0,t) = 0 \quad for\ t \geq 0$
2. $y(l,t) = 0 \quad for\ t \geq 0$
3. $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0 \quad for\ 0 < x < l$

4. $y(x,0) = f(x) = k(lx - x^2) \quad for\ 0 < x < l$

Step : 3 The possible solutions is

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = (A \cos \lambda x + B \sin\lambda x)(C \cos \lambda at + D \sin\lambda at)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

Step : 4 The suitable solution is

$y(x,t) = (A \cos \lambda x + B \sin\lambda x)(C \cos \lambda at + D \sin\lambda at)$ ....(2)

Step : 5 Using Boundary condition (1) $y(0,t) = 0$ in (2)

Sub x = 0 in (2)

$$y(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (A+0)(C\cos\lambda at + D\sin\lambda at)$$

A = 0 since $C\cos\lambda at + D\sin\lambda at \neq 0$

sub A = 0 in (2)

$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ………………….(3)

Step : 6 Using Boundary condition (2) $y(l,t) = 0$ in (3)

Sub $x = l$ in (3)

$$y(l,t) = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0$ &$(C\cos\lambda at + D\sin\lambda at) \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$ ………………….(4)

Step : 7 Using Boundary condition (3) $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0$ in (4)

Differentiating (4) partially w.r.to t

$$\frac{\partial y}{\partial t} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}\left(-C\sin\frac{n\pi at}{l} + D\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C\sin 0 + D\cos 0)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C(0) + D(1))$$

$$D = 0, B \neq 0, \sin\frac{n\pi x}{l} \neq 0 \quad , \qquad \frac{n\pi at}{l} \neq 0$$

Sub D = 0 in (4)

$y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$………………….(5)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$, $B_n = BC$………………….(6)

Step : 8Using Boundary condition (4)$y(x,0) = f(x)$ in (6)

Sub t=0 in (6)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Step : 9 To find B_n

$$f(x) = k(lx - x^2) \ \ and \ \ l = l$$

$$B_n = \frac{2}{l}\int_0^l k(lx - x^2)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\,\sin\frac{n\pi x}{l}\,dx$$

$$u = (lx - x^2) dv = \sin\frac{n\pi x}{l}\,dx$$

$$u' = (l - 2x) v = \frac{-l}{n\pi}\,\cos\frac{n\pi x}{l}$$

$$u'' = -2 v_1 = \frac{-l^2}{n^2\pi^2}\,\sin\frac{n\pi x}{l}$$

$$v_2 = \frac{l^3}{n^3\pi^3}\,\cos\frac{n\pi x}{l}$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-l}{n\pi}\,\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(\frac{-l^2}{n^2\pi^2}\,\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\,\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-\frac{l}{n\pi}(lx - x^2)\left(\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2x)\left(\sin\frac{n\pi x}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{l}\left\{\begin{array}{c}\left[-\frac{l}{n\pi}(ll - l^2)\left(\cos\frac{n\pi l}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2l)\left(\sin\frac{n\pi l}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\cos\frac{n\pi l}{l}\right)\right] \\ -\left[-\frac{l}{n\pi}(l(0) - 0^2)\left(\cos\frac{n\pi 0}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2(0))\left(\sin\frac{n\pi 0}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\cos\frac{n\pi 0}{l}\right)\right]\end{array}\right\}$$

$$= \frac{2k}{l}\left\{\begin{array}{c}\left[-\frac{l}{n\pi}(l^2 - l^2)(\cos n\pi) + \frac{l^2}{n^2\pi^2}(-l)(\sin n\pi) - 2\frac{l^3}{n^3\pi^3}(\cos n\pi)\right] \\ -\left[-\frac{l}{n\pi}(0)(\cos 0) + \frac{l^2}{n^2\pi^2}(l)(\sin 0) - \frac{l^3}{n^3\pi^3}(2)(\cos 0)\right]\end{array}\right\}$$

$$= \frac{2k}{l}\left\{\left[-\frac{l}{n\pi}(0)(-1)^n + \frac{l^2}{n^2\pi^2}(-l)(0) - 2\frac{l^3}{n^3\pi^3}(-1)^n\right] - \left[0 + \frac{l^2}{n^2\pi^2}(l)(0) - 2\frac{l^3}{n^3\pi^3}(1)\right]\right\}$$

$\because [\sin n\pi = 0\ , \sin n\pi = (-1)^n\ , \sin 0 = 0\ , \cos 0 = 1]$

$$= \frac{2k}{l}\left\{\left[0 + 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right] - \left[0 + 0 - \frac{2l^3}{n^3\pi^3}\right]\right\}$$

$$= \frac{2k}{l}\left\{-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right\}$$

$$= \frac{2k}{l}\frac{2l^3}{n^3\pi^3}[-(-1)^n + 1]$$

$$= \frac{4kl^2}{n^4\pi^4}\begin{cases} 1 + 1 & n - odd \\ -1 + 1 & n - even\end{cases}$$

$$B_n = \begin{cases} \frac{8kl^2}{n^4\pi^4} & n - odd \\ 0 & n - even\end{cases}$$

Step : 10 Sub B_n in (6) , The required solution is

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^4\pi^4}\sin\frac{n\pi x}{l}\,\sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^2}{n^4\pi^4}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\,\sin\frac{n\pi at}{l}$$

Video Content / Details of website for further learning (if any): https://www.youtube.com/watch?v=1f6wR3FQCwg
Important Books/Journals for further learning including the page nos: **1.** 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 4.11-4.36

Course Faculty

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LECTURE HANDOUTS

AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV- Boundary Value Problems** **Date of Lecture:**

Topic of Lecture:One dimension wave equation

Introduction : The wave equation in one space dimension can be written as follows:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable y may represent a second space dimension, if, for example, the displacement y takes place in y-direction, as in the case of a string that is located in the x–y plane.

Prerequisite knowledge for Complete understanding and learning of Topic:

4. One dimension wave equation
5. Boundary conditions
6. Half range Fourier sine series

Detailed content of the Lecture:

2. A string is stretched and fastened to two points $x = 0 and x = l$ apart. Motion is started by displacing the string into the form $(x, 0) = y_0 \sin^3(\frac{\pi x}{\ell})$. If it is released from this position find the displacement y at any distance x from one end at any time t.

Solution :

Step : 1 One dimension wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$..................... (1)

Step : 2 Boundary conditions

1. $y(0, t) = 0 \quad for\ t \geq 0$
2. $y(l, t) = 0 \quad for\ t \geq 0$
3. $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0 \quad for\ 0 < x < l$

4. $y(x, 0) = f(x) = y_0 \sin^3(\frac{\pi x}{\ell})\ for\ 0 < x < l$

Step : 3 The possible solutions is

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = (A \cos \lambda x + B \sin\lambda x)(C \cos \lambda at + D \sin\lambda at)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

Step : 4 The suitable solution is

$y(x,t) = (A \cos \lambda x + B \sin\lambda x)(C \cos \lambda at + D \sin\lambda at)$ ....(2)

Step : 5 Using Boundary condition (1) $y(0, t) = 0$ in (2)

Sub x = 0 in (2)

$$y(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (A+0)(C\cos\lambda at + D\sin\lambda at)$$

A = 0 since $C\cos\lambda at + D\sin\lambda at \neq 0$

sub A = 0 in (2)

$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ………………….(3)

Step : 6 Using Boundary condition (2) $y(l,t) = 0$ in (3)

Sub $x = l$ in (3)

$$y(l,t) = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0\ \&(C\cos\lambda at + D\sin\lambda at) \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right)$ ………………….(4)

Step : 7 Using Boundary condition (3) $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0$ in (4)

Differentiating (4) partially w.r.to t

$$\frac{\partial y}{\partial t} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}\left(-C\sin\frac{n\pi at}{l} + D\cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C\sin 0 + D\cos 0)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C(0) + D(1))$$

$$D = 0, B \neq 0, \sin\frac{n\pi x}{l} \neq 0 \quad , \qquad \frac{n\pi at}{l} \neq 0$$

Sub D = 0 in (4)

$y(x,t) = BC\sin\frac{n\pi x}{l}\ \cos\frac{n\pi at}{l}$………………….(5)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\ \cos\frac{n\pi at}{l}$, $\quad B_n = BC$………………….(6)

Step : 8Using Boundary condition (4)$y(x,0) = f(x)$ in (6)

Sub t=0 in (6)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

Step : 9 To find B_n

$$f(x) = y_0\sin^3\left(\frac{\pi x}{\ell}\right) \ \ and \ \ l = l$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$y_0\sin^3\left(\frac{\pi x}{\ell}\right) = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + \cdots$$

$$\because \sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$y_0 \frac{1}{4}\left(3\sin\left(\frac{\pi x}{l}\right) - \sin 3\left(\frac{\pi x}{l}\right)\right) = B_1\, sin\frac{\pi x}{l} + B_2\, sin\frac{2\pi x}{l} + B_3\, sin\frac{3\pi x}{l} + \cdots$$

$$\frac{3y_0}{4} 3\sin\left(\frac{\pi x}{l}\right) - \frac{y_0}{4}\sin 3\left(\frac{\pi x}{l}\right) = B_1\, sin\frac{\pi x}{l} + B_2\, sin\frac{2\pi x}{l} + B_3\, sin\frac{3\pi x}{l} + \cdots$$

Equating coefficient of $in\frac{\pi x}{l}, sin\frac{2\pi x}{l}$, $sin\frac{3\pi x}{l}$,…….

$B_1 = \frac{3y_0}{4}$, $B_2 = 0$, $B_3 = -\frac{y_0}{4}$, $B_4 = 0$,……...

Step : 10 Sub B_n in (6) , The required solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\ cos\frac{n\pi at}{l}$$

$$y(x,t) = B_1\, sin\frac{\pi x}{l} cos\frac{\pi at}{l} + B_2\, sin\frac{2\pi x}{l} cos\frac{2\pi at}{l} + B_3\, sin\frac{3\pi x}{l} cos\frac{3\pi at}{l} + \cdots$$

$$y(x,t) = \frac{3y_0}{4}\ sin\frac{\pi x}{l} cos\frac{\pi at}{l} + (0)\ sin\frac{2\pi x}{l} cos\frac{2\pi at}{l} - \frac{y_0}{4}\ sin\frac{3\pi x}{l} cos\frac{3\pi at}{l} + \cdots$$

$$y(x,t) = \frac{3y_0}{4}\ sin\frac{\pi x}{l} cos\frac{\pi at}{l} - \frac{y_0}{4}\ sin\frac{3\pi x}{l} cos\frac{3\pi at}{l} + \cdots$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=g9ASIMnLdNM

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 4.11-4.36

Course Faculty

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LECTURE HANDOUTS

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV-Boundary Value Problems** **Date of Lecture:**

Topic of Lecture:One dimension wave equation

Introduction : The wave equation in one space dimension can be written as follows:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable y may represent a second space dimension, if, for example, the displacement y takes place in y-direction, as in the case of a string that is located in the x–y plane.

Prerequisite knowledge for Complete understanding and learning of Topic:

7. One dimension wave equation
8. Boundary conditions
9. Half range Fourier sine series

Detailed content of the Lecture:

1. A String is tightly stretched and its ends are fastened to two points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$. Find the displacement.

Solution:

Step : 1 One dimension wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ .... (1)

Step : 2 Boundary conditions

1. $y(0, t) = 0 \quad for\ t \geq 0$
2. $y(l, t) = 0 \quad for\ t \geq 0$
3. $y(x, 0) = 0 \quad for\ 0 < x < l$
4. $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = f(x) = 3(lx - x^2) \quad for\ 0 < x < l$

Step : 3 The possible solutions is

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

Step : 4 The suitable solution is

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ....(2)

Step : 5 Using Boundary condition (1) $y(0, t) = 0$ in (2)

Sub x = 0 in (2)

$$y(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (A+0)(C\cos\lambda at + D\sin\lambda at)$$

A = 0 since $C\cos\lambda at + D\sin\lambda at \neq 0$

sub A = 0 in (2)

$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ………………….(3)

Step : 6 Using Boundary condition (2) $y(l,t) = 0$ in (3)

Sub x = l in (3)

$$y(l,t) = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0 \;\&(C\cos\lambda at + D\sin\lambda at) \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi}{l}at + D\sin\frac{n\pi}{l}at\right)$ ………………….(4)

Step : 7 Using Boundary condition (3) $y(x,0) = 0$ in (4)

Sub t = 0 in (4)

$$y(x,0) = \left(B\sin\frac{n\pi x}{l}\right)(C\cos 0 + D\sin 0)$$

$$0 = \left(B\sin\frac{n\pi x}{l}\right)(C\,(1) + D\,(0))$$

$$C = 0, B \neq 0\,, \sin\frac{n\pi x}{l} \neq 0$$

Sub C = 0 in (4)

$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(D\sin\frac{n\pi at}{l}\right)$ ………………….(5)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, \sin\frac{n\pi at}{l}\,, \quad B_n = BD$ ………………….(6)

Step : 8 Differentiating (6) partially w.r.to t

$$\frac{\partial y}{\partial t} = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\,\cos\frac{n\pi at}{l}$$

Using Boundary condition (4) $\left(\frac{\partial y}{\partial t}\right)_{at=0} = f(x)$

$$\left(\frac{\partial y}{\partial t}\right)_{at=0} = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos 0$$

$$f(x) = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n\frac{n\pi a}{l} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n = \frac{2}{n\pi a}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Step : 9 To find B_n

$$f(x) = 3(lx - x^2) \ \ and \ \ l = l$$

$$B_n = \frac{2}{n\pi a}\int_0^l 3(lx - x^2)\, sin\frac{n\pi x}{l}dx$$

$$= \frac{6}{n\pi a}\int_0^l (lx - x^2)\, sin\frac{n\pi x}{l}dx$$

$$u = (lx - x^2) dv = sin\frac{n\pi x}{l}dx$$

$$u' = (l - 2x) v = \frac{-l}{n\pi}\ cos\frac{n\pi x}{l}$$

$$u'' = -2 v_1 = \frac{-l^2}{n^2\pi^2}\ sin\frac{n\pi x}{l}$$

$$v_2 = \frac{l^3}{n^3\pi^3}\ cos\ \frac{n\pi x}{l}$$

$$= \frac{6}{n\pi a}\left[(lx - x^2)\left(\frac{-l}{n\pi}\ cos\frac{n\pi x}{l}\right) - (l - 2x)\left(\frac{-l^2}{n^2\pi^2}\ sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\ cos\ \frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{6}{n\pi a}\left[-\frac{l}{n\pi}(lx - x^2)\left(cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2x)\left(sin\frac{n\pi x}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\ cos\ \frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{6}{n\pi a}\left\{\begin{matrix} \left[-\frac{l}{n\pi}(ll - l^2)\left(cos\frac{n\pi l}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2l)\left(sin\frac{n\pi l}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\ cos\ \frac{n\pi l}{l}\right)\right] \\ - \left[-\frac{l}{n\pi}(l(0) - 0^2)\left(cos\frac{n\pi 0}{l}\right) + \frac{l^2}{n^2\pi^2}(l - 2(0))\left(sin\frac{n\pi 0}{l}\right) - \frac{l^3}{n^3\pi^3}(2)\left(\ cos\ \frac{n\pi 0}{l}\right)\right]\end{matrix}\right\}$$

$$= \frac{6}{n\pi a}\left\{\begin{matrix}\left[-\frac{l}{n\pi}(l^2 - l^2)(\cos n\pi) + \frac{l^2}{n^2\pi^2}(-l)(\sin n\pi) - 2\frac{l^3}{n^3\pi^3}(cos\ n\pi)\right] \\ - \left[-\frac{l}{n\pi}(0)(\cos 0) + \frac{l^2}{n^2\pi^2}(l)(\sin 0) - \frac{l^3}{n^3\pi^3}(2)(\ cos\ 0)\right]\end{matrix}\right\}$$

$$= \frac{6}{n\pi a}\left\{\left[-\frac{l}{n\pi}(0)(-1)^n + \frac{l^2}{n^2\pi^2}(-l)(0) - 2\frac{l^3}{n^3\pi^3}(-1)^n\right] - \left[0 + \frac{l^2}{n^2\pi^2}(l)(0) - 2\frac{l^3}{n^3\pi^3}(1)\right]\right\}$$

$$\because [\sin n\pi = 0\ , \sin n\pi = (-1)^n\ \ , \sin 0 = 0\ , cos\ 0 = 1]$$

$$= \frac{6}{n\pi a}\left\{\left[0 + 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right] - \left[0 + 0 - \frac{2l^3}{n^3\pi^3}\right]\right\}$$

$$= \frac{6}{n\pi a}\left\{-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right\}$$

$$= \frac{6}{n\pi a}\frac{2l^3}{n^3\pi^3}[-(-1)^n + 1]$$

$$= \frac{12l^3}{an^4\pi^4}\begin{cases} 1 + 1 & n - odd \\ -1 + 1 & n - even\end{cases}$$

$$B_n = \begin{cases}\frac{24l^3}{an^4\pi^4} & n - odd \\ 0 & n - even\end{cases}$$

Step : 10 SubB_n in (6) , The required solution is

$$y(x,t) = \sum_{n=odd}^{\infty}\frac{24l^3}{an^4\pi^4}sin\frac{n\pi x}{l}\ sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{24l^3}{a\pi^4} \sum_{n=odd}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \, \sin\frac{n\pi at}{l}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=g9ASIMnLdNM

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 4.11-4.36

Course Faculty

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MUTHAYAMMAL ENGINEERING COLLEGE

(An Autonomous Institution)

(Approved by AICTE, New Delhi, Accredited by NAAC & Affiliated to Anna University)
Rasipuram - 637 408, Namakkal Dist., Tamil Nadu

Estd. 2000

LECTURE HANDOUTS

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV -Boundary Value Problems** **Date of Lecture:**

Topic of Lecture:One dimension wave equation

Introduction : The wave equation in one space dimension can be written as follows:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable y may represent a second space dimension, if, for example, the displacement y takes place in y-direction, as in the case of a string that is located in the x–y plane.

Prerequisite knowledge for Complete understanding and learning of Topic:

10. One dimension wave equation
11. Boundary conditions
12. Half range Fourier sine series
13. Bernoulli's formula

Detailed content of the Lecture:

1. A String is tightly stretched and its ends are fastened to two points $x = 0$ & $x = 2l$. The midpoint of the strings is displaced transversely through a small distance 'b" and the string is released from rest in that position . Find the displacement at any point on the string.

Solution :

Equation of OB

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{l - 0}$$

$$y = \frac{b}{l}x \qquad 0 < x < l$$

Equation of AB

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 2l}{l - 2l}$$

$$\frac{y}{b} = \frac{x - 2l}{-l}$$

$$\frac{y}{b} = \frac{2l - x}{l}$$

$$\frac{y}{b} = \frac{1}{l}(2l - x)$$

$$y = \frac{b}{l}(2l - x) \quad 0 < x < 2l$$

$$y = f(x) = \frac{b}{l}\begin{cases} x & 0 < x < l \\ (2l - x) & 0 < x < 2l \end{cases}$$

Step : 1 One dimension wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$………………… (1)

Step : 2 Boundary conditions

1. $y(0,t) = 0 \quad for\ t \geq 0$
2. $y(l,t) = 0 \quad for\ t \geq 0$
3. $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0 \quad for\ 0 < x < l$

4. $y(x,0) = f(x) = \frac{b}{l}\begin{cases} x & 0 < x < l \\ (2l - x) & 0 < x < 2l \end{cases}$

Step : 3 The possible solutions is

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = (A\ cos\ \lambda x + B\ sin\lambda x)(C\ cos\ \lambda at + D\ sin\lambda at)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

Step : 4 The suitable solution is

$y(x,t) = (A\ cos\ \lambda x + B\ sin\lambda x)(C\ cos\ \lambda at + D\ sin\lambda at)$ ………………….(2)

Step : 5 Using Boundary condition (1) $y(0,t) = 0$ in (2)

Sub x = 0 in (2)

$$y(0,t) = (A\ cos\ \lambda 0 + B\ sin\lambda 0)(C\ cos\ \lambda at + D\ sin\lambda at)$$

$$0 = (A + 0)(C\ cos\ \lambda at + D\ sin\lambda at)$$

A = 0 since $C\ cos\ \lambda at + D\ sin\lambda at \neq 0$

sub A = 0 in (2)

$y(x,t) = (B\ sin\lambda x)(C\ cos\ \lambda at + D\ sin\lambda at)$ ………………….(3)

Step : 6 Using Boundary condition (2) $y(l,t) = 0$ in (3)

Sub $x = l$ in (3)

$$y(l,t) = (B\ sin\lambda l)(C\ cos\ \lambda at + D\ sin\lambda at)$$

$$0 = (B\ sin\lambda l)(C\ cos\ \lambda at + D\ sin\lambda at)$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0\ \&(C\ cos\ \lambda at + D\ sin\lambda at) \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$y(x,t) = \left(B\ sin\frac{n\pi x}{l}\right)\left(C\ cos\ \frac{n\pi at}{l} + D\ sin\frac{n\pi at}{l}\right)$ ………………….(4)

Step : 7 Using Boundary condition (3) $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = 0$ in (4)

Differentiating (4) partially w.r.to t

$$\frac{\partial y}{\partial t} = \left(B\ sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}\left(-C\ sin\frac{n\pi at}{l} + D\ cos\frac{n\pi at}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\ sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C\sin 0 + D\cos 0)$$

$$\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = \left(B\ sin\frac{n\pi x}{l}\right)\frac{n\pi a}{l}(-C(0)+D(1))$$

$$D=0, B\neq 0\ , sin\frac{n\pi x}{l}\neq 0\quad ,\qquad \frac{n\pi at}{l}\neq 0$$

Sub D = 0 in (4)

$y(x,t) = BC\ sin\frac{n\pi x}{l}\ cos\frac{n\pi at}{l}$………………….(5)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\ cos\frac{n\pi at}{l}\ ,\quad B_n = BC$ ………………….(6)

Step : 8Using Boundary condition (4)$\boldsymbol{y(x,0)=f(x)}$ in (6)

Sub t=0 in (6)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n = \frac{2}{l}\int_0^l f(x) sin\frac{n\pi x}{l}dx$$

Step : 9 To find $\boldsymbol{B_n}$

$$f(x) = \boxed{\frac{b}{l}\begin{cases} x & 0<x<l \\ (2l-x) & l<x<2l\end{cases} \text{and } l\to 2l}$$

$$B_n = \frac{2}{2l}\left\{\int_0^l \frac{b}{l}(x) sin\frac{n\pi x}{2l}dx + \int_l^{2l}\frac{b}{l}(2l-x)sin\frac{n\pi x}{2l}dx\right\}$$

$$B_n = \frac{2}{2l}\frac{b}{l}\left\{\int_0^l (x) sin\frac{n\pi x}{2l}dx + \int_0^l (2l-x)sin\frac{n\pi x}{2l}dx\right\}$$

$$B_n = \frac{b}{l^2}\left\{\int_0^l (x) sin\frac{n\pi x}{2l}dx + \int_l^{2l} (2l-x)sin\frac{n\pi x}{2l}dx\right\}$$

$$B_n = \frac{b}{l^2}\left\{\left[(x)\left(-\frac{2l}{n\pi}cos\frac{n\pi x}{2l}\right)-(1)\left(-\frac{4l^2}{n^2\pi^2}sin\frac{n\pi x}{2l}\right)\right]_0^l + \left[((2l-x))\left(-\frac{2l}{n\pi}cos\frac{n\pi x}{2l}\right)-(-1)\left(-\frac{4l^2}{n^2\pi^2}sin\frac{n\pi x}{2l}\right)\right]_l^{2l}\right\}$$

$$B_n = \frac{b}{l^2}\left\{\left[\left(-\frac{2l^2}{n\pi}cos\frac{n\pi}{2}\right)+\left(\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}\right)-0-0\right]+\left[0+0+\left(\frac{2l^2}{n\pi}cos\frac{n\pi}{2}\right)+\left(\frac{4l^2}{n^2\pi^2}sin\frac{n\pi x}{2}\right)\right]\right\}$$

$$B_n = \frac{b}{l^2}\left\{\left[\left(-\frac{2l^2}{n\pi}cos\frac{n\pi}{2}\right)+\left(\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}\right)\right]+\left[\left(\frac{2l^2}{n\pi}cos\frac{n\pi}{2}\right)+\left(\frac{4l^2}{n^2\pi^2}sin\frac{n\pi x}{2}\right)\right]\right\}$$

$$B_n = \frac{b}{l^2}\left[-\frac{2l^2}{n\pi}cos\frac{n\pi}{2}+\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}+\frac{2l^2}{n\pi}cos\frac{n\pi}{2}+\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{b}{l^2}\left[\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}+\frac{4l^2}{n^2\pi^2}sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2} sin \frac{n\pi}{2}\right]$$

$$B_n = \frac{8b}{n^2\pi^2} sin \frac{n\pi}{2}$$

Step : 10 **Sub B_n in (6),The required solution is**

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2} sin \frac{n\pi}{2}\, sin \frac{n\pi x}{2l}\ cos \frac{n\pi at}{2l}$$

$$y(x,t) = \frac{8b}{n^2\pi^2} \sum_{n=1}^{\infty} sin \frac{n\pi}{2}\, sin \frac{n\pi x}{2l}\ cos \frac{n\pi at}{2l}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=1f6wR3FQCwg

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 4.11-4.36

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV - Boundary Value Problems** **Date of Lecture:**

Topic of Lecture:One dimension wave equation

Introduction : The wave equation in one space dimension can be written as follows:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable y may represent a second space dimension, if, for example, the displacement y takes place in y-direction, as in the case of a string that is located in the x–y plane.

Prerequisite knowledge for Complete understanding and learning of Topic:

14. One dimension wave equation
15. Boundary conditions
16. Half range Fourier sine series
17. Bernoulli's formula

Detailed content of the Lecture:

2. A String is tightly stretched and its ends are fastened to two points $x = 0$ and $x = 2l$ is initially at rest in its equilibrium position. If the initial velocity is given by $v = \begin{cases} \frac{c}{l}x & 0 < x < l \\ \frac{c}{l}(2l - x) & l < x < 2l \end{cases}$. Find the displacement.

Solution:

Step : 1 One dimension wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.................... (1)

Step : 2 Boundary conditions

1. $y(0,t) = 0 \quad for\ t \geq 0$
2. $y(l,t) = 0 \quad for\ t \geq 0$
3. $y(x,0) = 0 \quad for\ 0 < x < l$
4. $\left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = f(x) = \begin{cases} \frac{c}{l}x & 0 < x < l \\ \frac{c}{l}(2l - x) & l < x < 2l \end{cases}$

Step : 3 The possible solutions is

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$$

$$y(x,t) = (Ax + B)(Cx + D)$$

Step : 4 The suitable solution is

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ………………….(2)

Step : 5 Using Boundary condition (1) $\boldsymbol{y(0,t) = 0}$ in (2)

Sub x = 0 in (2)

$$y(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (A+0)(C\cos\lambda at + D\sin\lambda at)$$

A = 0 since $C\cos\lambda at + D\sin\lambda at \neq 0$

sub A = 0 in (2)

$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ ………………….(3)

Step : 6 Using Boundary condition (2) $\boldsymbol{y(l,t) = 0}$ in (3)

Sub $x = l$ in (3)

$$y(l,t) = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$$0 = (B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at)$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0$ &$(C\cos\lambda at + D\sin\lambda at) \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi}{l}at + D\sin\frac{n\pi}{l}at\right)$ ………………….(4)

Step : 7 Using Boundary condition (3) $\boldsymbol{y(x,0) = 0}$ in (4)

Sub t = 0 in (4)

$$y(x,0) = \left(B\sin\frac{n\pi x}{l}\right)(C\cos 0 + D\sin 0)$$

$$0 = \left(B\sin\frac{n\pi x}{l}\right)(C(1) + D(0))$$

$$C = 0, B \neq 0, \sin\frac{n\pi x}{l} \neq 0$$

Sub C = 0 in (4)

$y(x,t) = \left(B\sin\frac{n\pi x}{l}\right)\left(D\sin\frac{n\pi at}{l}\right)$ ………………….(5)

The most general solution is

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$, $B_n = BD$ ………………….(6)

Step : 8 Differentiating (6) partially w.r.to t

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{t}} = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Using Boundary condition (4) $\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{t}}\right)_{\boldsymbol{at=0}} = \boldsymbol{f(x)}$

$$\left(\frac{\partial y}{\partial t}\right)_{at=0} = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos 0$$

$$f(x) = \frac{n\pi a}{l}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n\frac{n\pi a}{l} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n = \frac{2}{n\pi a}\int_0^l f(x)\, sin\frac{n\pi x}{l}dx$$

Step : 9 To find B_n

$$f(x) = \begin{cases} \frac{c}{l}x & 0 < x < l \\ \frac{c}{l}(2l - x) & l < x < 2l \end{cases} \quad and \;\; l = 2l$$

$$B_n = \frac{2}{n\pi a}\int_0^{2l} f(x)\, sin\frac{n\pi x}{2l}dx$$

$$= \frac{2}{n\pi a}\left\{\int_0^l \frac{c}{l}x\, sin\frac{n\pi x}{2l}dx + \int_l^{2l} \frac{c}{l}(2l - x)\, sin\frac{n\pi x}{2l}dx\right\}$$

$$= \frac{2}{n\pi a}\frac{c}{l}\left\{\int_0^l x\, sin\frac{n\pi x}{2l}dx + \int_l^{2l} (2l - x)\, sin\frac{n\pi x}{2l}dx\right\}$$

$$= \frac{2c}{ln\pi a}\left\{\int_0^l x\, sin\frac{n\pi x}{2l}dx + \int_l^{2l} (2l - x)\, sin\frac{n\pi x}{2l}dx\right\}$$

$$B_n = \frac{2c}{ln\pi a}[I_1 + I_2]$$

$$I_1 I_2$$

$$u = x dv = sin\frac{n\pi x}{2l}dx u = (2l - x)dv = sin\frac{n\pi x}{2l}dx$$

$$u' = 1 v = \frac{-2l}{n\pi}\; cos\frac{n\pi x}{2l} u' = -1 v = \frac{-2l}{n\pi}\; cos\frac{n\pi x}{2l}$$

$$v_1 = -\frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi x}{2l} v_1 = -\frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi x}{2l}$$

$\int udv = uv - u'v_1 + u''v_2 -$........

$$= \frac{2c}{ln\pi a}\left\{\begin{matrix} \left[(x)\left(\frac{-2l}{n\pi}\; cos\frac{n\pi x}{2l}\right) - (1)\left(-\frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi x}{2l}\right)\right]_0^l \\ + \left[(2l - x)\left(\frac{-2l}{n\pi}\; cos\frac{n\pi x}{2l}\right) - (-1)\left(-\frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi x}{2l}\right)\right]_l^{2l} \end{matrix}\right\}$$

$$= \frac{2c}{ln\pi a}\left\{\left[\frac{-2l}{n\pi}(x)\left(\; cos\frac{n\pi x}{2l}\right) + \left(\frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi x}{2l}\right)\right]_0^l + \left[\frac{-2l}{n\pi}(2l - x)\left(cos\frac{n\pi x}{2l}\right) - \frac{4l^2}{n^2\pi^2}\left(sin\frac{n\pi x}{2l}\right)\right]_l^{2l}\right\}$$

$$= \frac{2c}{ln\pi a}\left\{\begin{matrix} \left[\left(\frac{-2l}{n\pi}(l)\; cos\frac{n\pi l}{2l} + \frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi l}{2l}\right) - \left(\frac{-2l}{n\pi}(0)\cos(0) + \frac{4l^2}{n^2\pi^2}\sin(0)\right)\right] \\ + \left[\left(\frac{-2l}{n\pi}(2l - 2l)\; cos\frac{n\pi 2l}{2l} - \frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi 2l}{2l}\right) - \left(\frac{-2l}{n\pi}(2l - l)\; cos\frac{n\pi l}{2l} - \frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi l}{2l}\right)\right] \end{matrix}\right\}$$

$$= \frac{2c}{ln\pi a}\left\{\begin{matrix} \left[\left(\frac{-2l^2}{n\pi}\; cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi}{2}\right) - \left(0 + \frac{4l^2}{n^2\pi^2}(0)\right)\right] \\ + \left[\left(\frac{-2l}{n\pi}(0)\cos n\pi - \frac{4l^2}{n^2\pi^2}\; \sin n\pi\right) - \left(\frac{-2l}{n\pi}(l)\; cos\frac{n\pi}{2} - \frac{4l^2}{n^2\pi^2}\; sin\frac{n\pi}{2}\right)\right] \end{matrix}\right\}$$

$$= \frac{2c}{ln\pi a}\left\{\left[\left(\frac{-2l^2}{n\pi}\ cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2}\right) - (0+0)\right] + \left[(0-0) - \left(\frac{-2l}{n\pi}(l)\ cos\frac{n\pi}{2} - \frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2}\right)\right]\right\}$$

$\because [\sin n\pi = 0\ , \sin n\pi = (-1)^n\ \ , \sin 0 = 0\ , cos\ 0 = 1]$

$$= \frac{2c}{ln\pi a}\left(\frac{-2l^2}{n\pi}\ cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\ cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2}\right)$$

$$= \frac{2c}{ln\pi a}\left(\frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\ sin\frac{n\pi}{2}\right)$$

$$= \frac{2c}{ln\pi a}\left(\frac{8l^2}{n^2\pi^2}\ sin\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{2c}{ln\pi a}\frac{8l^2}{n^2\pi^2}\ sin\frac{n\pi}{2} & n-odd \\ 0 & n-even \end{cases}$$

$$B_n = \begin{cases} \frac{2c}{ln\pi a}\frac{8l^2}{n^2\pi^2}\ sin\frac{n\pi}{2} & n-odd \\ 0 & n-even \end{cases}$$

Step : 10 Sub$\boldsymbol{B_n}$ in (6) , The required solution is

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{16cl}{an^3\pi^3} sin\frac{n\pi}{2} sin\frac{n\pi x}{2l}\ sin\frac{n\pi at}{2l}$$

$$y(x,t) = \frac{16cl}{a\pi^3}\sum_{n=odd}^{\infty} \frac{1}{n^3} sin\frac{n\pi}{2} sin\frac{n\pi x}{2l}\ sin\frac{n\pi at}{2l}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=1f6wR3FQCwg

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 4.11-4.36

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV - Boundary Value Problems** **Date of Lecture:**

Topic of Lecture:One dimension heat equation

Introduction : The heat equation models the flow of heat in a rod that is insulated everywhere except at the two ends. Solutions of this equation are functions of two variables - one spatial variable (position along the rod) and time. Let u(x,t) represent the temperature at the point x meters along the rod at time t (in seconds).

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

Prerequisite knowledge for Complete understanding and learning of Topic:

18. One dimension heat equation
19. Boundary conditions
20. Half range Fourier sine series
21. Bernoulli's formula

Detailed content of the Lecture:

1. A rod 30cm long has its end A and B kept at 20° C and 80° C respectively, until steady state conditions prevails. The temperature at each end is then suddenly reduced to 0° C and kept so. Find the resulting temperature function u(x, t) at any point x from one end of the rod and at time t seconds

Solution:

The initial temperature distribution is $u = \left(\frac{b-a}{l}\right)x + a$

$$a = 20°\,\mathrm{C}\,;\mathrm{b} = 80°\,\mathrm{C}\,;\ \ell = 30\ \mathrm{cm}$$

$$u = \left(\frac{80-20}{30}\right)x + 20 \therefore u = 2x + 20 \quad 0 < x < 30$$

$$u(x,0) = 2x + 20 \quad 0 < x < 30$$

Step : 1 One dimension heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$..................... (1)

Step : 2 Boundary conditions

1. $u(0,t) = 0 \quad for\ t \geq 0$
2. $u(l,t) = 0 \quad for\ t \geq 0$
3. $u(x,0) = f(x) = 2x + 20 \quad for\ 0 < x < 30$

Step : 3 The possible solutions is

$$u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Ce^{-\alpha^2\lambda^2 t}$$

$$u(x,t) = (A\cos\lambda x + B\,sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$$

$$u(x,t) = (Ax + B)C$$

Step : 4 The suitable solution is

$u(x,t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$ ....(2)

Step : 5 Using Boundary condition (1) $u(0,t) = 0$ in (2)

Sub x = 0 in (2)

$$u(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)Ce^{-\alpha^2\lambda^2 t}$$

$$0 = (A + 0)Ce^{-\alpha^2\lambda^2 t}$$

A = 0 since $Ce^{-\alpha^2\lambda^2 t} \neq 0$

sub A = 0 in (2)

$u(x,t) = (B\sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$ ....(3)

Step : 6 Using Boundary condition (2) $u(l,t) = 0$ in (3)

Sub $x = l$ in (3)

$$u(l,t) = (B\sin\lambda l)Ce^{-\alpha^2\lambda^2 t}$$

$$0 = (B\sin\lambda l)Ce^{-\alpha^2\lambda^2 t}$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0$ & $Ce^{-\alpha^2\lambda^2 t} \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$u(x,t) = \left(B\sin\frac{n\pi}{l}x\right)Ce^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$ ....(4)

The most general solution is

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$, $B_n = BC$ ....(5)

Step : 7 Using Boundary condition (3) $u(x,0) = 0$ in (5)

Sub t = 0 in (5)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^0$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Step : 8 To find B_n

$$f(x) = 2x + 20 \quad for \;\; 0 < x < 30 \;\; and \;\; l = 30$$

$$B_n = \frac{2}{30}\int_0^{30} (2x + 20)\sin\frac{n\pi x}{30}dx$$

$$= \frac{1}{15}\int_0^{30} (2x + 20)\sin\frac{n\pi x}{30}dx$$

$$u = 2x + 20dv = \sin\frac{n\pi x}{30}dx$$

$$u' = 2v = \frac{-30}{n\pi}\ cos\frac{n\pi x}{30}$$

$$u'' = 0v_1 = \frac{-900}{n^2\pi^2}\ sin\frac{n\pi x}{30}$$

$$= \frac{1}{15}\left[(2x+20)\left(\frac{-30}{n\pi}\ cos\frac{n\pi x}{30}\right) - (2)\left(\frac{-900}{n^2\pi^2}\ sin\frac{n\pi x}{30}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left[\frac{-30}{n\pi}(2x+20)cos\frac{n\pi x}{30} + 2\frac{900}{n^2\pi^2}\ sin\frac{n\pi x}{30}\right]_0^{30}$$

$$= \frac{1}{15}\left\{\left[\frac{-30}{n\pi}(2(30)+20)cos\frac{n\pi 30}{30} + 2\frac{900}{n^2\pi^2}\ sin\frac{n\pi 30}{30}\right] - \left[\frac{-30}{n\pi}(2(0)+20)\cos 0 + 2\frac{900}{n^2\pi^2}\sin 0\right]\right\}$$

$$= \frac{1}{15}\left\{\left[-\frac{30}{n\pi}(80)\cos n\pi + 2\frac{900}{n^2\pi^2}\sin n\pi\right] - \left[-\frac{30}{n\pi}(20)(1) + 2\frac{900}{n^2\pi^2}(0)\right]\right\}$$

$$= \frac{1}{15}\left\{\left[-\frac{2400}{n\pi}(-1)^n + 2\frac{900}{n^2\pi^2}(0)\right] - \left[-\frac{600}{n\pi} + (0)\right]\right\}$$

$\because [\sin n\pi = 0\ , \sin n\pi = (-1)^n\ , \sin 0 = 0\ , cos\ 0 = 1]$

$$= \frac{1}{15}\left[-\frac{2400}{n\pi}(-1)^n + \frac{600}{n\pi}\right]$$

$$= \frac{1}{15}\frac{600}{n\pi}[-4(-1)^n + 1]$$

$$B_n = \frac{40}{n\pi}[1 - 4(-1)^n]$$

Step : 9SubB_n in (5) , The required solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \ell = 30$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}[1 - 4(-1)^n]\, sin\frac{n\pi x}{30} e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=JASw8fJKoyI

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 4.37-4.58

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit - IV : **Boundary Value Problems** **Date of Lecture:**

Topic of Lecture: One dimension heat equation

Introduction : The heat equation models the flow of heat in a rod that is insulated everywhere except at the two ends. Solutions of this equation are functions of two variables - one spatial variable (position along the rod) and time. Let u(x,t) represent the temperature at the point x meters along the rod at time t (in seconds).

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

Prerequisite knowledge for Complete understanding and learning of Topic:

22. One dimension heat equation
23. Boundary conditions
24. Half range Fourier sine series
25. Bernoulli's formula

Detailed content of the Lecture:

1. A bar 10 cm long , with insulated sides has its end A & B kept at 20˚C and 40˚ C respectively until the steady state condition prevails. The temperature at A is suddenly raised to 50˚ C and B is lowered to 10˚C . Find the subsequent temperature function u (x ,t)

Solution:

The initial temperature distribution is $u = \left(\frac{b-a}{l}\right)x + a$

$$a = 20°\,\mathrm{C}\,;\mathrm{b} = 40°\,\mathrm{C}\,;\ \ell = 10\text{ cm}$$

$$u = \left(\frac{40-20}{10}\right)x + 20 \therefore u = 2x + 20 \ \ 0 < x < 10$$

$$u(x,0) = 2x + 20 \ \ 0 < x < 10$$

The one dimension heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ .... (i)

Boundary conditions

1. $u(0,t) = 50^\circ\,\mathrm{C} \quad for\ t \geq 0$
2. $u(l,t) = 10^\circ\,\mathrm{C} \quad for\ t \geq 0$
3. $u(x,0) = f(x) = 2x + 20 \quad for\ 0 < x < 10$

Here, we have no non zero boundary conditions. So we cannot find the values of A and B.

Therefore, we split u (x ,t) in to two parts.

$u(x,t) = u_s(x) + u_t(x,t)$ .... (ii)

Where $u_s(x)$ is a solution of the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ and is a function of x alone and satisfying the conditions

$u_s(0) = 50$, $u_s(l) = 10$

Where $u_t(x,t)$ is a transient solution satisfying (ii) which decrease at t increases.

$u_s(x)$ is a steady state solution and $u_t(x,t)$ is a transient solution.

To find : $u_s(x)$

$$u_s = \left(\frac{b-a}{l}\right)x + a$$

$$a = 50°\text{ C} ; \text{b} = 10°\text{ C} ; \ \ell = 10 \text{ cm}$$

$$u = \left(\frac{10-50}{10}\right)x + 50 \therefore u_s = -4x + 50 \ \ 0 < x < 10$$

$$u_s(x) = 50 - 4x \ \ 0 < x < 10$$

To find: $u_t(x,t)$

From (ii) $\quad u(x,t) = u_s(x) + u_t(x,t)$

$u_t(x,t) = u(x,t) - u_s(x)$ ………………… (iii)

Sub x = 0 in (iii)

$$u_t(0,t) = u(0,t) - u_s(0) \because u_s(0) = 50 , u(0,t) = 50$$

$u_t(0,t) = 50 - 50 = 0$ ………………… (iv)

Sub x = 10 in (iii)

$$u_t(10,t) = u(10,t) - u_s(10) \because u_s(10) = 10 , u(10,t) = 10$$

$u_t(10,t) = 10 - 10 = 0$ ………………… (v)

Sub t = 0 in (iii)

$$u_t(x,0) = u(x,0) - u_s(x) \because u_s(x) = 50 - 4x , u(x,0) = 2x - 4x$$

$u_t(x,0) = 2x + 20 - 50 + 4x = 6x - 30$ ………………… (vi)

Now, we have to the solution for $u_t(x,t)$

Step : 1 One dimension heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ ………………… (1)

Step : 2 Boundary conditions

1. $u_t(0,t) = 0 \ \ for\ t \geq 0$
2. $u_t(l,t) = 0 \ \ for\ t \geq 0$
3. $u_t(x,0) = f(x) = 6x + 30 \ \ for\ 0 < x < 10$

Step : 3 The possible solutions is

$$u_t(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})Ce^{-\alpha^2\lambda^2 t}$$

$$u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$$

$$u_t(x,t) = (Ax + B)C$$

Step : 4 The suitable solution is

$u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$ …………………(2)

Step : 5 Using Boundary condition (1) $u_t(0,t) = 0$ in (2)

Sub x = 0 in (2)

$$u_t(0,t) = (A\cos\lambda 0 + B\sin\lambda 0)Ce^{-\alpha^2\lambda^2 t}$$

$$0 = (A + 0)Ce^{-\alpha^2\lambda^2 t}$$

A = 0 since $Ce^{-\alpha^2\lambda^2 t} \neq 0$

sub A = 0 in (2)

$u_t(x,t) = (B\ sin\lambda x)Ce^{-\alpha^2\lambda^2 t}$ ………………….(3)

Step : 6 Using Boundary condition (2)$u_t(l,t) = 0$ in (3)

Sub $x = l$ in (3)

$$u_t(l,t) = (B\ sin\lambda l)Ce^{-\alpha^2\lambda^2 t}$$

$$0 = (B\ sin\lambda l)Ce^{-\alpha^2\lambda^2 t}$$

$\lambda = \frac{n\pi}{l}$ since $B \neq 0$ & $Ce^{-\alpha^2\lambda^2 t} \neq 0$

Sub $\lambda = \frac{n\pi}{l}$ in (3)

$u_t(x,t) = \left(B\ sin\frac{n\pi}{l}x\right)Ce^{-\alpha^2\left(\frac{n\pi}{l}\right)^2 t}$ ………………….(4)

The most general solution is

$u_t(x,t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$, $B_n = BC$ ………………….(5)

Step : 7 Using Boundary condition (3) $u_t(x,0) = 0$ in (5)

Sub t = 0 in (5)

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\, e^0$$

$$f(x) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}$$

which is of the form of half range Fourier Sine series,

$$B_n = \frac{2}{l}\int_0^l f(x)\ sin\frac{n\pi x}{l}\, dx$$

Step : 8 To find B_n

$$f(x) = 6x + 10 \quad for\ \ 0 < x < 10\ \ and\ \ l = 10$$

$$B_n = \frac{2}{10}\int_0^{10} (6x + 20)\ sin\frac{n\pi x}{10}\, dx$$

$$= \frac{1}{5}\int_0^{10} (6x + 20)\ sin\frac{n\pi x}{10}\, dx$$

$$u = 6x + 30 dv = sin\frac{n\pi x}{10}\, dx$$

$$u' = 6 v = \frac{-10}{n\pi}\ cos\frac{n\pi x}{10}$$

$$u'' = 0 v_1 = \frac{-100}{n^2\pi^2}\ sin\frac{n\pi x}{10}$$

$\int u dv = uv - u'v_1 + u''v_2 -$……..

$$= \frac{1}{5}\left[(6x + 30)\left(\frac{-10}{n\pi}\ cos\frac{n\pi x}{10}\right) - (6)\left(\frac{-100}{n^2\pi^2}\ sin\frac{n\pi x}{10}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[\frac{-10}{n\pi}(6x+30)cos\frac{n\pi x}{10} + 6\frac{100}{n^2\pi^2}\ sin\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5}\left\{\left[\frac{-10}{n\pi}(6(10)+30)cos\frac{n\pi 10}{10} + 6\frac{100}{n^2\pi^2}\ sin\frac{n\pi 10}{10}\right] - \left[\frac{-10}{n\pi}(6(0)+30)cos0 + 6\frac{100}{n^2\pi^2}\ sin0\right]\right\}$$

$$= \frac{1}{5}\left\{\left[-\frac{10}{n\pi}(30)\cos n\pi + 6\frac{100}{n^2\pi^2}\sin n\pi\right] - \left[-\frac{10}{n\pi}(-30)(1) + 6\frac{100}{n^2\pi^2}(0)\right]\right\}$$

$$= \frac{1}{5}\left\{\left[-\frac{300}{n\pi}(-1)^n + 6\frac{100}{n^2\pi^2}(0)\right] - \left[\frac{300}{n\pi} + (0)\right]\right\}$$

$\because [\sin n\pi = 0\ , \sin n\pi = (-1)^n\ , \sin 0 = 0\ , cos\ 0 = 1]$

$$= \frac{1}{5}\left[-\frac{300}{n\pi}(-1)^n - \frac{300}{n\pi}\right]$$

$$= \frac{1}{5}\frac{-300}{n\pi}[(-1)^n + 1]$$

$$= \frac{-60}{n\pi}\begin{cases}-1+1 & n-odd\\ 1+1 & n-even\end{cases}$$

$$= \frac{-60}{n\pi}\begin{cases}0 & n-odd\\ 2 & n-even\end{cases}$$

$$B_n = \begin{cases}0 & n-odd\\ \frac{-120}{n\pi} & n-even\end{cases}$$

Step : 9SubB_n in (5)

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}\ \ell = 10$$

$$u_t(x,t) = \sum_{n=odd}^{\infty}\frac{-120}{n\pi}sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

Step : 10 From (ii)

$$u(x,t) = u_s(x) + u_t(x,t)$$

The required solution is

$$u(x,t) = 50 - 4x + \sum_{n=odd}^{\infty}\frac{-120}{n\pi}sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=PbucCMGDuao

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 4.37-4.58

Course Faculty

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **IV - Boundary Value Problems** **Date of Lecture:**

Topic of Lecture: Steady state solutionof twodimensionalequationofheatconduction (excluding insulatededges) on finite square plates (excluding circular plates).

Introduction : When the heat flow is along curves, instead straight lines, the curves lying in parallel planes, the flow is called two dimensional . The twodimensional heat flow equations $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ which is known as Laplace's equation in two dimensions

Prerequisite knowledge for Complete understanding and learning of Topic:

26. Two dimension heat equation
27. Boundary conditions
28. Half range Fourier sine series
29. Bernoulli's formula

Detailed content of the Lecture:

1. The boundary value problem governing the steady state temperature distribution in a flat, thin, square plate is given by $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad 0<x<a, 0<y<a$

(i) $u(x,0) = 0 \; for all t \geq 0$

(ii) $u(x,a) = 4\sin^3\left(\frac{\pi x}{a}\right) \quad 0<x<a$

(iii) $u(0,y) = 0$

(iv) $u(a,y) = 0 \quad 0<y<a$

Find the steady-state temperature distribution in the plate.

Solution:

Step : 1 The Two dimension flow equation in steady state is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \ldots\ldots (1)$

Step : 2 Boundary conditions

1. $u(0,y)=0 \quad for\ 0<y<a$
2. $u(a,y)=0 \quad for\ 0<y<a$
3. $u(x,0)=0 \quad for\ 0<x<a$
4. $u(x,a)=4\sin^3\left(\frac{\pi x}{a}\right) \quad for\ 0<x<a$

Step : 3 The possible solutions is

$$u(x,y)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)(C\cos\lambda y+D\sin\lambda y)$$

$$u(x,y)=(A\cos\lambda x+B\sin\lambda x)\left(Ce^{\lambda y}+De^{-\lambda y}\right)$$

$$u(x,y)=(Ax+B)(Cx+D)$$

Step : 4 The suitable solution is

$u(x,y) = (A\cos\lambda x + B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$ …………………(2)

Step : 5 Using Boundary condition (1) $u(0,y) = 0$ in (2)

Sub x = 0 in (2)

$$u(x,y) = (A\cos\lambda 0 + B\sin\lambda 0)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$0 = (A+0)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

A = 0 since $\left(Ce^{\lambda y} + De^{-\lambda y}\right) \neq 0$

sub A = 0 in (2)

$u(x,y) = (B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$ …………………(3)

Step : 6 Using Boundary condition (2) $u(a,y) = 0$ in (3)

Sub $x = a$ in (3)

$$u(a,y) = (B\sin\lambda a)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$0 = (B\sin\lambda a)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$\lambda = \frac{n\pi}{a}$ since $B \neq 0$ &$\left(Ce^{\lambda y} + De^{-\lambda y}\right) \neq 0$

Sub $\lambda = \frac{n\pi}{a}$ in (3)

$u(x,y) = \left(B\sin\frac{n\pi}{a}x\right)\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right)$ …………………(4)

Step : 7 Using Boundary condition (3) $u(x,0) = 0$ in (4)

Sub y = 0 in (4)

$$u(x,0) = \left(B\sin\frac{n\pi}{a}x\right)\left(Ce^{\frac{n\pi 0}{a}} + De^{-\frac{n\pi 0}{a}}\right)$$

$$u(x,0) = \left(B\sin\frac{n\pi}{a}x\right)(C+D)$$

Here, $\sin\frac{n\pi}{a}x \neq 0$& $B \neq 0$

$$(C+D) = 0 \quad \therefore D = -C$$

$$u(x,y) = \left(B\sin\frac{n\pi}{a}x\right)\left(Ce^{\frac{n\pi y}{a}} - Ce^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = BC\sin\frac{n\pi}{a}x\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$u(x,y) = 2BC\sin\frac{n\pi}{a}x\sin h\frac{n\pi y}{a}$ …………………(5)

$$u(x,y) = 2B_n\sin\frac{n\pi}{a}x\sin h\frac{n\pi y}{a}\,, \quad Where\ B_n = BC$$

The most general solution is

$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\ \sin h\frac{n\pi y}{a}\,,$ …………………(6)

Step : 8 Using Boundary condition (4) $u(x,a) = 4\sin^3\left(\frac{\pi x}{a}\right)$ in (6)

$u(x,y) = 4\sin^3\left(\frac{\pi x}{a}\right) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\ \sin h\, n\pi$

$$3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\ \sin h\, n\pi$$

$$= B_1\sin\frac{\pi x}{a}\ \sin h\,\pi + B_2\sin\frac{2\pi x}{a}\ \sin h\,2\pi + B_3\sin\frac{3\pi x}{a}\ \sin h\,3\pi + \cdots$$

Equating the like terms, we get

$$B_1\sin h\,\pi = 3, B_2 = 0\,, B_3\sin h\,3\pi = -1, B_4 = 0\,, B_5 = 0 = \cdots = 0$$

$$B_1 = \frac{3}{\sin h\, \pi}, B_3 = \frac{-1}{\sin h\, 3\pi}$$

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi y}{a}$$

$$u(x, y) = \frac{3}{\sin h\, \pi} \sin \frac{\pi x}{a} \sin h\, \pi + \frac{-1}{\sin h\, 3\pi} \sin \frac{3\pi x}{a} \sin h\, 3\pi$$

$$u(x, y) = 3\, cosec\, hx\, \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} - cosec\, h3x\, \sin \frac{3\pi x}{a} \sin \frac{3\pi y}{a}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=PbucCMGDuao

Important Books/Journals for further learning including the page nos:

1. 1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 4.59-4.63

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LECTURE HANDOUTS | **L**

AI&DS | **II / III**

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Formation of partial differential equations

Introduction : A Partial differential equation is one which involves partial derivatives. The order of a PDE is the order of the highest derivative occurring in it. A PDE is said to be linear, if the dependent variable and partial derivatives occur in the first degree only and separately.

Notations : $z = f(x, y, z)$, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Partial differential equations
2. Arbitrary constants
3. Order
4. Degree

Detailed content of the Lecture:

1. **Form the PDE by eliminating the arbitrary constants a and b from**

$z = (x^2 + a^2)(y^2+b^2)$

Solution: Given: $z = (x^2 + a^2)(y^2+b^2)$ $\rightarrow$ (I)

Diff (I) par.w.r.to x and y,

$p = \frac{\partial z}{\partial x} = 2x(y^2+b^2)$ $\rightarrow$ (II) $q = \frac{\partial z}{\partial y} = 2y(x^2 + a^2)$ $\rightarrow$ (III)

From (II), $\frac{p}{2x} = (y^2+b^2)$ $\rightarrow$ (IV) From (III), $\frac{q}{2y} = (x^2 + a^2)$ $\rightarrow$ (V)

Sub (IV) and (V) in (I) we get

$z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$ (or) $pq = 4xyz$

2. **Form the PDE by eliminating the arbitrary constants from $z = a^2 x + ay^2 + b$**

Solution:Given: $z = a^2 x + ay^2 + b$ $\rightarrow$ (I)

Diff (I) par.w.r.to x and y,

$p = \frac{\partial z}{\partial x} = a^2$ $\rightarrow$ (II), $q = \frac{\partial z}{\partial y} = 2ay$ $\rightarrow$ (III)

From (III) , $y = \frac{q}{2a}$ $\rightarrow$ (IV)

$y^2 = \frac{q^2}{4a^2}$ (or) $y^2 p = q^2$

3. **Form the PDE by eliminating from the relation $z = f(x^2 + y^2) + x + y$**

Solution: Given $z = x + y + f(x^2 + y^2)$ $\rightarrow$ (I)

Diff. (I) p.w.r.to x: $p = 1 + f'(x^2 + y^2).2x$ i.e. $p - 1 = 2x f'(x^2 + y^2)$ $\rightarrow$ (II)

Diff. (I) p.w.r.to y: $q = 1 + f'(x^2 + y^2).2y$ i.e. $q - 1 = 2y f'(x^2 + y^2)$ $\rightarrow$ (III)

$\frac{(II)}{(III)} \Rightarrow \frac{(p-1)}{(q-1)} = \frac{2x f'(x2 + y2)}{2y f'(x2 + y2)} = \frac{x}{y}$

i.e. $qx - py = x + y$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=xydJU0CUR6o

Important Books/Journals for further learning including the page nos.:

1.K.Sankara Rao – Introduction to partial differential Equations , 3rd Edition, Jan 2012, Page.No : 7 -11

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations Date of Lecture:**

Topic of Lecture: Singular integrals and Solutions of standard types of first order partial differential equations

Introduction :A singular integral is an integral whose integrand reaches an infinite value at one or more points in the domain of integration. Even so, such integrals can converge, in which case, they are said to exist. (If they do not converge, they are said not to exist.)

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Partial Differential Equations
2. Claimant's Form
3. Singular Integral
4. Complete Integral

Detailed content of the Lecture:

1. **Solve $p+q=pq$**

Solution:

$p+q=pq$ …….(1) This is the form of $F(p,q)=0$

Sub $p=a\&q=b$ in (1)

$$a+b=ab \Rightarrow a=ab-b \Rightarrow a=b(a-1)$$

$b=\frac{a}{a-1}\&$

Sub b in $z=ax+by+c$

i.e. $\boldsymbol{z=ax+\frac{a}{a-1}y+c}$

2. **Solve $\boldsymbol{z=px+qy+p^2+q^2}$**

Solution: Given $z=px+qy+p^2+q^2$ …………………. (1)

Which is the Claimant's Form

Complete Integral

Sub p = a , q = b in (1)

$z=ax+by+a^2+b^2$ …………………. (2)

Which is the Complete Integral

Singular Integral

Diff (2) partially w.r.to a

$$0 = x + 2a$$

$x = -2a$ …………………. (3)

Diff (2) partially w.r.to b

$$0 = y - 2b$$

$y = 2b$ …………………. (4)

To find a & b From (3) & (4)

(3) ⇒ $x = -2a$

$$a = \frac{-x}{2}$$

(4) ⇒ $y = 2b$

$$b = \frac{y}{2}$$

Sub a & b in (2)

(2) ⇒ $z = ax + by + a^2 + b^2$

$$z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 + \left(\frac{y}{2}\right)^2$$

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{2} - \frac{y^2}{2}$$

$$z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$$\mathbf{4z = y^2 - x^2}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ehDMLRVNGrk

Important Books/Journals for further learning including the page nos:

1.K.Sankara Rao – Introduction to partial differential Equations , 3rd Edition, Jan 2012, Page.No : 11-18

Course Faculty

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MUTHAYAMMAL ENGINEERING COLLEGE

(An Autonomous Institution)

(Approved by AICTE, New Delhi, Accredited by NAAC & Affiliated to Anna University)
Rasipuram - 637 408, Namakkal Dist., Tamil Nadu

Estd. 2000

LECTURE HANDOUTS

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential EquationsDate of Lecture:**

Topic of Lecture: Solutions of standard types of first order partial differential equations

Introduction :A singular integral is an integral whose integrand reaches an infinite value at one or more points in the domain of integration. Even so, such integrals can converge, in which case, they are said to exist. (If they do not converge, they are said not to exist.)

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Partial Differential Equations
2. Claimant's Form
3. Singular Integral
4. Complete Integral

Detailed content of the Lecture:

Solve$z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution: Given $z = px + qy + \sqrt{1 + p^2 + q^2}$ …………………. (1)

Which is the Claimant's Form

Complete Integral

Sub p = a , q = b in (1)

$z = ax + by + \sqrt{1 + a^2 + b^2}$ …………………. (2)

Which is the Complete Integral

Singular Integral

Diff (2) partially w.r.to a

$$0 = x + 0 + \frac{2a}{2\sqrt{1 + a^2 + b^2}} \because d(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$x = -\frac{a}{\sqrt{1+a^2+b^2}}$ …………………. (3)

Diff (2) partially w.r.to b

$$0 = y + 0 + \frac{2b}{2\sqrt{1 + a^2 + b^2}} \because d(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$y = -\frac{b}{\sqrt{1+a^2+b^2}}$ …………………. (4)

To find a & b From (3) & (4)

$$(3)^2 + (4)^2 = x^2 + y^2 = \left(-\frac{a}{\sqrt{1+a^2+b^2}}\right)^2 + \left(-\frac{b}{\sqrt{1+a^2+b^2}}\right)^2$$

$$x^2 + y^2 = \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - (x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$$

$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$ …………………. (5)

$\sqrt{1-a^2-b^2} = \frac{1}{\sqrt{1+x^2+y^2}}$ …………………. (6)

$(3) \Rightarrow x = -\frac{a}{\sqrt{1+a^2+b^2}}$

$$a = -x\sqrt{1+a^2+b^2}$$

$a = -\frac{x}{\sqrt{1-x^2-y^2}}$ …………………. (7)

$$\because \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$(4) \Rightarrow y = -\frac{b}{\sqrt{1+a^2+b^2}}$

$$b = -y\sqrt{1+a^2+b^2}$$

$b = -\frac{y}{\sqrt{1-x^2-y^2}}$ …………………. (8)

$$\because \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

Sub a & b in (2)

$(2) \Rightarrow z = ax + by + \sqrt{1+a^2+b^2}$

$$z = x\frac{-x}{\sqrt{1-x^2-y^2}} + y\frac{-y}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1 - x^2 - y^2$$

$$\boldsymbol{x^2 + y^2 + z^2 = 1}$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=ehDMLRVNGrk

Important Books/Journals for further learning including the page nos:

1.K.Sankara Rao – Introduction to partial differential Equations , 3rd Edition, Jan 2012, Page.No : 11-18

Course Faculty

Verified by HOD

MUTHAYAMMAL ENGINEERING COLLEGE

(An Autonomous Institution)

(Approved by AICTE, New Delhi, Accredited by NAAC & Affiliated to Anna University)
Rasipuram - 637 408, Namakkal Dist., Tamil Nadu

Estd. 2000

LECTURE HANDOUTS

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II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations Date of Lecture:**

Topic of Lecture: Lagrange's linear equation

Introduction : The equation of the form $Pp + Qq = R$ is known as Lagrange's equation when P, Q, R are function of x , y, z .
The auxiliary equation can be solved in two ways

1. Method of grouping
2. Method of Multipliers

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Lagrange's linear equation
2. Auxiliary equation
3. Choosing Multipliers
4. Integration

Detailed content of the Lecture:

Solve $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

Solution:

Given : $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$ ______________ (1)

Which is of the form $Pp + Qq = R$

$$P = (x^2 - yz) \qquad Q = (y^2 - zx) \qquad R = (z^2 - xy)$$

Auxiliary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - zx)} = \frac{dz}{(z^2 - xy)}$$

Choosing (x, y, z)&$(1,1,1)$ as Multipliers, we get

$$\frac{xdx + ydy + zdz}{x(x^2 - yz) + y(y^2 - zx) + z(z^2 - xy)} = \frac{dx + dy + dz}{(x^2 - yz) + (y^2 - zx) + (z^2 - xy)}$$

$$\frac{xdx + ydy + zdz}{x^3 - xyz + y^3 - xyz + z^3 - xyz} = \frac{dx + dy + dz}{x^2 - yz + y^2 - zx + z^2 - xy}$$

$$\frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

$$\frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2 - yz - zx - xy)(x + y + z)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

$$\frac{xdx + ydy + zdz}{(x+y+z)} = \frac{dx+dy+dz}{1}$$

$$xdx + ydy + zdz = (x+y+z)d(x+y+z)$$

Integrating , we get

$$\int xdx + \int ydy + \int zdz = \int (x+y+z)d(x+y+z) + C_1$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + C_1$$

$$x^2 + y^2 + z^2 = (x+y+z)^2 + 2C_1$$

$$x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 2C_1$$

$$0 = 2xy + 2yz + 2zx + 2C_1$$

$$xy + yz + zx = -C_1$$

i.e., $u = xy + yz + zx$

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\frac{dx - dy}{x^2 - y^2 - yz + zx} = \frac{dy - dz}{y^2 - z^2 - zx + xy}$$

$$\frac{dx - dy}{x^2 - y^2 + z(x-y)} = \frac{dy - dz}{y^2 - z^2 + x(y-z)}$$

$$\frac{dx - dy}{(x-y)(x+y) + z(x-y)} = \frac{dy - dz}{(y-z)(y+z) + x(y-z)}$$

$$\frac{dx - dy}{(x-y)[x+y+z]} = \frac{dy - dz}{(y-z)[x+y+z]}$$

$$\frac{dx - dy}{(x-y)} = \frac{dy - dz}{(y-z)}$$

Integrating, we get

$$\int \frac{d(x-y)}{(x-y)} = \int \frac{d(y-z)}{(y-z)} + \log C_2$$

$$\log(x-y) = \log(y-z) + \log C_2$$

$$\log(x-y) - \log(y-z) = \log C_2$$

$$\log \frac{(x-y)}{(y-z)} = \log C_2$$

$$\frac{(x-y)}{(y-z)} = C_2$$

$$v = \frac{(x-y)}{(y-z)}$$

The solution of PDE is $f(u,v) = 0$

$$f\left(xy + yz + zx, \frac{(x-y)}{(y-z)} \right) = 0$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=qWHNxKgO15g
Important Books/Journals for further learning including the page nos:
1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 3.79-3.96

Course Faculty

Verified by HOD

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Lagrange's linear equation

Introduction : The equation of the form $Pp + Qq = R$ is known as Lagrange's equation when P, Q, R are function of x , y, z .

The auxiliary equation can be solved in two ways

3. Method of grouping
4. Method of Multipliers

Prerequisite knowledge for Complete understanding and learning of Topic:

5. Lagrange's linear equation
6. Auxiliary equation
7. Choosing Multipliers
8. Integration

Detailed content of the Lecture:

Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution:

Given $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ ____________(1)

Which is of the form $Pp + Qq = R$

$$P = x(y^2 - z^2) Q = y(z^2 - x^2) R = z(x^2 - y^2)$$

Auxialiary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

Choosing

$$\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$$

as multipliers , we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y^2 - z^2) + \frac{1}{y}y(z^2 - x^2) + \frac{1}{z}\,z(x^2 - y^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$i.e., \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = logc_1$$

$$logx + logy + logz = logc_1$$

$$\log(xyz) = logc_1$$

$$xyz = c_1$$

$$i.e., u = xyz$$

Choosing (x, y, z)as Multipliers, we get

$$\frac{xdx + ydy + zdz}{xx(y^2 - z^2) + yy(z^2 - x^2) + z\,z(x^2 - y^2)} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$$

$$= \frac{xdx + ydy + zdz}{x^2y^2 - x^2z^2 + y^2z^2 - y^2x^2 + z^2x^2 - z^2y^2}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Integrating, we get

$$\int xdx + \int ydy + \int zdz = c_2$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$x^2 + y^2 + z^2 = 2c_2$$

$$x^2 + y^2 + z^2 = v$$

The solution of the given PDE is$f(u.v) = 0$

$$f(xyz, x^2 + y^2 + z^2) = 0$$

Video Content / Details of website for further learning (if any):

https://www.youtube.com/watch?v=qWHNxKgO15g

Important Books/Journals for further learning including the page nos:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 3.79-3.96

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Linear partial differential equations of second and higher order with constant coefficients of homogeneous when the R.H.S is e^{ax+by}

Introduction : A Partial differential equation is one which involves partial derivatives.
A PDE is said to be linear, if the dependent variable and partial derivatives occur in the first degree only and separately.

Two Types:

1. Homogeneous Linear partial differential equations with constant
2. Non Homogeneous Linear partial differential equations with constant

Prerequisite knowledge for Complete understanding and learning of Topic:

1. Linear partial differential equations
2. Homogeneous and Non Homogeneous
3. Auxiliary Equation
4. Complementary Function
5. Particular Integral

Detailed content of the Lecture:

1. Solve: $(D^2 + 2DD' + 2D'^2)z = \sinh(x+y) + e^{x+2y}$

<u>Solution:</u>

Given $(D^2 + 2DD' + 2D'^2)z = \sinh(x+y) + e^{x+2y}$

$$(D^2 + 2DD' + 2D'^2)z = \frac{e^{x+y} - e^{-(x+y)}}{2} + e^{x+2y}$$

$$(D^2 + 2DD' + 2D'^2)z = \frac{1}{2}e^{x+y} - \frac{1}{2}e^{-(x+y)} + e^{x+2y}$$

$$(D^2 + 2DD' + 2D'^2)z = \mathrm{PI}_1 + \mathrm{PI}_2 + \mathrm{PI}_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Sub $D = m$ & $D' = 1$ in (1)

Auxiliary Equation

$$m^2 + 2m + 1 = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

Complementary Function

$$C.F = f_1(y-x) + xf_2(y-x)$$

Particular Integral

$$\text{PI}_1 = \frac{1}{D^2 + 2DD' + 2D'^2}\frac{1}{2}e^{x+y}$$

$$= \frac{1}{2}\frac{1}{(D + D')^2}e^{x+y} \qquad \text{D} = \text{a} = 1\ \&D' = b = 1$$

$$= \frac{1}{2}\frac{1}{(1 + 1)^2}e^{x+y}$$

$$\text{PI}_1 = \frac{1}{8}e^{x+y}$$

$$\text{PI}_2 = \frac{1}{D^2 + 2DD' + 2D'^2}\frac{1}{2}e^{-(x+y)}$$

$$= \frac{1}{2}\frac{1}{(D + D')^2}e^{-(x+y)}$$

$$= \frac{1}{2}\frac{1}{(-1 - 1)^2}e^{-x-y} \qquad \text{D} = \text{a} = -1\ \&D' = b = -1$$

$$\text{PI}_2 = -\frac{1}{8}e^{-x-y}$$

$$\text{PI}_3 = \frac{1}{D^2 + 2DD' + 2D'^2}e^{x+2y}$$

$$= \frac{1}{(D + D')^2}e^{x+2y}$$

$$= \frac{1}{(1 + 2)^2}e^{x+2y} \qquad \text{D} = \text{a} = 1\ \&D' = b = 2$$

$$\text{PI}_3 = \frac{1}{9}e^{x+2y}$$

$\text{PI} = \text{PI}_1 + \text{PI}_2 + \text{PI}_3$

$$\text{PI} = \frac{1}{8}e^{x+y} - \frac{1}{8}e^{-x-y} + \frac{1}{9}e^{x+2y}$$

Complete Solution

$$z = C.F + PI$$

$$z = f_1(y - x) + xf_2(y - x) + \frac{1}{8}e^{x+y} - \frac{1}{8}e^{-x-y} + \frac{1}{9}e^{x+2y}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tHqx1qxA8q4

Important Books/Journals for further learning including the page nos:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011, Page.No : 3.97-3.121

Course Faculty

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Linear partial differential equations of second and higher order with constant coefficients of homogeneous when the R.H.S is $x^m y^n$ m,n>0

Introduction : A Partial differential equation is one which involves partial derivatives.
A PDE is said to be linear, if the dependent variable and partial derivatives occur in the first degree only and separately.

Two Types:

3. Homogeneous Linear partial differential equations with constant
4. Non Homogeneous Linear partial differential equations with constant

Prerequisite knowledge for Complete understanding and learning of Topic:

6. Linear partial differential equations
7. Homogeneous and Non Homogeneous
8. Auxiliary Equation
9. Complementary Function
10. Particular Integral

Detailed content of the Lecture:

2. Solve: $(D^2 + DD' - 6D'^2)z = e^{3x+y} + x^2 y$

Solution:

Given $(D^2 + DD' - 6D'^2)z = e^{3x+y} + x^2 y$

$$(D^2 + DD' - 6D'^2)z = \mathrm{PI}_1 + \mathrm{PI}_2 \qquad \text{……………………(1)}$$

Sub $D = m$ & $D' = 1$ in (1)

Auxiliary Equation

$$m^2 + m - 6 = 0$$

$$m^2 + 3m - 2m - 6 = 0$$

$$m(m+3) - 2(m-2) = 0$$

$$(m+3)(m-2) = 0$$

$$m = 2, -3$$

Complementary Function

$$C.F = f_1(y+2x) + f_2(y-3x)$$

Particular Integral

$$\mathrm{PI}_1 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y}$$

$$= \frac{1}{(3)^2+(3)(1)-6(1)^2}e^{3x+y} \qquad \mathrm{D}=\mathrm{a}=3\ \&D^{'}=b=1$$

$$=\frac{1}{9+3-6}e^{3x+y}$$

$$\mathrm{PI}_1=\frac{1}{6}e^{3x+y}$$

$$\mathrm{PI}_2=\frac{1}{D^2+DD'-6D'^2}x^2y$$

$$=\frac{1}{D^2\left(1+\frac{DD'}{D^2}-\frac{6D'^2}{D^2}\right)}x^2y$$

$$=\frac{1}{D^2\left(1+\frac{D'}{D}-\frac{6D'^2}{D^2}\right)}x^2y$$

$$=\frac{1}{D^2}\left[1+\frac{D'}{D}-\frac{6D'^2}{D^2}\right]^{-1}x^2y$$

$$=\frac{1}{D^2}\left[1-\left(\frac{D'}{D}-\frac{6D'^2}{D^2}\right)+\left(\frac{D'}{D}-\frac{6D'^2}{D^2}\right)^2-\cdots\right]x^2y$$

$D'(x^2y)=xD'^2(x^2y)=0$ ∴ Omitting D'^2 and higher power of D'^2

$$=\frac{1}{D^2}\left[1-\frac{D'}{D}\right]x^2y$$

$$=\frac{1}{D^2}\left[x^2y-\frac{D'(x^2y)}{D}\right]D' \text{ Differentiate with respect to y}$$

$$=\frac{1}{D^2}\left[x^2y-\frac{x^2}{D}\right]\frac{1}{D} \text{ Differentiate with respect to x}$$

$$=\frac{1}{D^2}\left[x^2y-\frac{x^3}{3}\right]\frac{1}{D} \text{ Differentiate with respect to x}$$

$$=\frac{1}{D}\left[\frac{x^3}{3}y-\frac{1}{3}\frac{x^4}{4}\right]\frac{1}{D} \text{ Differentiate with respect to x}$$

$$=\left[\frac{1}{3}\frac{x^4}{4}y-\frac{1}{12}\frac{x^5}{5}\right]$$

$$\mathrm{PI}_2=\frac{x^4y}{12}-\frac{x^5}{60}$$

$\mathrm{PI}=\mathrm{PI}_1+\mathrm{PI}_2$

$$\mathrm{PI}=\frac{1}{6}e^{3x+y}+\frac{x^4y}{12}-\frac{x^5}{60}$$

Complete Solution

$$z=C.F+PI$$

$$z=f_1(y+2x)+f_2(y-3x)+\frac{1}{6}e^{3x+y}+\frac{x^4y}{12}-\frac{x^5}{60}$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tHqx1qxA8q4

Important Books/Journals for further learning including the page nos:

1.A.Neel Armstrong – Transform and partial differential Equations , 2nd Edition, 2011, Page.No : 3.97-3.121

Course Faculty

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LECTURE HANDOUTS

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AI&DS

II / III

Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Linear partial differential equations of second and higher order with constant coefficients of homogeneous when the R.H.S is sin (ax + by)

Introduction : A Partial differential equation is one which involves partial derivatives.
A PDE is said to be linear, if the dependent variable and partial derivatives occur in the first degree only and separately.

Two Types:

5. Homogeneous Linear partial differential equations with constant
6. Non Homogeneous Linear partial differential equations with constant

Prerequisite knowledge for Complete understanding and learning of Topic:

11. Linear partial differential equations
12. Homogeneous and Non Homogeneous
13. Auxiliary Equation
14. Complementary Function
15. Particular Integral

Detailed content of the Lecture:

1. Solve: $(D^3 - 7DD'^2 - 6D'^3)z = \mathrm{e}^{3x+y} + \sin(x + 2y)$

Solution:

Given $(D^3 - 7DD'^2 - 6D'^3)z = \mathrm{e}^{3x+y} + \sin(x + 2y)$……………………..(1)

$$(D^3 - 7DD'^2 - 6D'^3)z = \mathrm{PI}_1 + \mathrm{PI}_2$$

Sub $D = m$ &$D' = 1$ in (1)

Auxiliary Equation

$$m^3 - 7m - 6 = 0$$

$m = -1$ is one of the root

By Synthetic Division Method

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

Remaining Equation

$$m^2 - 2m - 6 = 0$$

$$m^2 - 3m + 2m - 6 = 0$$

$$m(m - 3) + 2(m - 3) = 0$$

$$(m-3)(m+2)=0$$

$$m=-2,3$$

$$\therefore m=-1,-2,3$$

Complementary Function

$$C.F=f_1(y-x)+xf_2(y-2x)+f_3(y+3x)$$

Particular Integral

$$\mathrm{PI}_1=\frac{1}{D^3-7DD'^2-6D'^3}\mathrm{e}^{3x+y}\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$$=\frac{1}{(3)^3-7(3)(1)'^2-6(1)'^3}\mathrm{e}^{3x+y}\qquad \mathrm{D}=\mathrm{a}=3\ \&D'=b=1$$

$$=\frac{1}{27-21-6}\mathrm{e}^{3x+y}$$

$$\mathrm{PI}_1=\frac{1}{0}\mathrm{e}^{3x+y}\qquad Demominator\ Zer$$

Diff (2) partially with respect to D

$$\mathrm{PI}_1=\mathrm{x}\frac{1}{3D^2-7D'^2}\mathrm{e}^{3x+y}$$

$$\mathrm{PI}_1=\mathrm{x}\frac{1}{3(3)^2-7(1)^2}\mathrm{e}^{3x+y}\qquad \mathrm{D}=\mathrm{a}=3\ \&D'=b=1$$

$$\mathrm{PI}_1=\mathrm{x}\frac{1}{27-7}\mathrm{e}^{3x+y}$$

$$\mathrm{PI}_1=\frac{\mathrm{x}}{20}\mathrm{e}^{3x+y}$$

$$\mathrm{PI}_2=\frac{1}{D^3-7DD'^2-6D'^3}sin(x+2y)$$

$$=\frac{1}{D^2D-7DD'^2-6D'D'^2}sin(x+2y)$$

$$D^2=-a^2=-1\,,DD'=-ab=-2\,,D'^2=-b^2-4$$

$$=\frac{1}{(-1)D-7D(-4)-6(-4)D'^2}sin(x+2y)$$

$$=\frac{1}{-D+28D+24D'}sin(x+2y)$$

$$=\frac{1}{27D+24D'}sin(x+2y)$$

$$=\frac{1}{3}\frac{1}{(9D+8D')}sin(x+2y)$$

$$=\frac{1}{3}\frac{\mathrm{D}}{D(9D+8D')}sin(x+2y)$$

$$=\frac{1}{3}\frac{\mathrm{D}\,sin(x+2y)}{(9D^2+8DD')}$$

$$=\frac{1}{3}\frac{cos(x+2y)}{9(-1)+8(-2)}$$

$$D^2=-a^2=-1\,,DD'=-ab=-2\,,D'^2=-b^2-4$$

$$= \frac{1}{3} \frac{cos(x + 2y)}{-9 - 16}$$

$$= \frac{1}{3} \frac{cos(x + 2y)}{-25}$$

$$\mathrm{PI}_2 = -\frac{1}{75} cos(x + 2y)$$

$$\mathrm{PI} = \mathrm{PI}_1 + \mathrm{PI}_2$$

$$\mathrm{PI} = \frac{\mathrm{x}}{20} \mathrm{e}^{3\mathrm{x}+\mathrm{y}} - \frac{1}{75} cos(x + 2y)$$

Complete Solution

$$z = C.F + PI$$

$$z = f_1(y - x) + x f_2(y - 2x) + f_3(y + 3x) + \frac{\mathrm{x}}{20} \mathrm{e}^{3\mathrm{x}+\mathrm{y}} - \frac{1}{75} cos(x + 2y)$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tHqx1qxA8q4

Important Books/Journals for further learning including the page nos:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 3.97-3.121

Course Faculty

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Course Name with Code : **Transforms and Partial Differential Equations / 19BSS23**

Course Faculty : **M.Nazreen Banu**

Unit : **V – Partial Differential Equations** **Date of Lecture:**

Topic of Lecture: Linear partial differential equations of second and higher order with constant coefficients of homogeneous when the R.H.S is cos (ax + by)

Introduction :A Partial differential equation is one which involves partial derivatives.
A PDE is said to be linear, if the dependent variable and partial derivatives occur in the first degree only and separately.
Two Types:

7. Homogeneous Linear partial differential equations with constant
8. Non Homogeneous Linear partial differential equations with constant

Prerequisite knowledge for Complete understanding and learning of Topic:

16. Linear partial differential equations
17. Homogeneous and Non Homogeneous
18. Auxiliary Equation
19. Complementary Function
20. Particular Integral

Detailed content of the Lecture:

3. Solve: $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + cos(x + y)$

Solution:

Given $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + cos(x + y)$

$$(D^3 + D^2D' - DD'^2 - D'^3)z = PI_1 + PI_2 \qquad \ldots\ldots\ldots\ldots(1)$$

Sub $D = m$ &$D' = 1$ in (1)

Auxiliary Equation

$$m^3 + m^2 - m - 1 = 0$$

$m = 1$ is one of the root

By Synthetic Division Method

$$\begin{array}{c|cccc} & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

Remaining Equation

$$m^2 + 2m + 1 = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m + 1)^2 = 0$$

$$m = -1, -1$$

$$\therefore m = -1, -1, 1$$

Complementary Function

$$C.F = f_1(y-x) + xf_2(y-2x) + f_3(y+3x)$$

Particular Integral

$$\mathrm{PI}_1 = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} \mathrm{e}^{2\mathrm{x}+\mathrm{y}}$$

$$= \frac{1}{(2)^3 + (2)^2(1) - (2)(1)^2 - (1)^3} \mathrm{e}^{2\mathrm{x}+\mathrm{y}} \qquad \mathrm{D} = \mathrm{a} = 2 \,\& D' = b = 1$$

$$= \frac{1}{8 + 4 - 2 - 1} \mathrm{e}^{2\mathrm{x}+\mathrm{y}}$$

$$\mathrm{PI}_1 = \frac{1}{9} \mathrm{e}^{2\mathrm{x}+\mathrm{y}}$$

$$\mathrm{PI}_2 = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} cos(x\ +\ y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$$= \frac{1}{D^2D + D^2D' - DD'^2 - D'D'^2} cos(x\ +\ y)$$

$$D^2 = -a^2 = -1\,, DD' = -ab = -1\,, D'^2 = -b^2 - 1$$

$$= \frac{1}{(-1)D + (-1)D' - D(-1) - D'(-1)} cos(x\ +\ y)$$

$$= \frac{1}{-D - D' + D + D'} cos(x\ +\ y) \qquad Demominator\ Zero$$

Diff (2) partially with respect to D

$$= \mathrm{x} \frac{1}{3D^2 + 2DD' - D'^2} cos(x\ +\ y)$$

$$= \mathrm{x} \frac{1}{3(-1) + 2(-1) - (-1)} cos(x\ +\ y)$$

$$D^2 = -a^2 = -1\,, DD' = -ab = -1\,, D'^2 = -b^2 - 1$$

$$= \mathrm{x} \frac{1}{-3 - 2 + 1} cos(x\ +\ y)$$

$$\mathrm{PI}_2 = -\frac{\mathrm{x}}{4} cos(x\ +\ y)$$

$\mathrm{PI} = \mathrm{PI}_1 + \mathrm{PI}_2$

$$\mathrm{PI} = \frac{1}{9} \mathrm{e}^{2\mathrm{x}+\mathrm{y}} - \frac{\mathrm{x}}{4} cos(x\ +\ y)$$

Complete Solution

$$z = C.F + PI$$

$$z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \frac{1}{9} \mathrm{e}^{2\mathrm{x}+\mathrm{y}} - \frac{\mathrm{x}}{4} cos(x\ +\ y)$$

Video Content / Details of website for further learning (if any):
https://www.youtube.com/watch?v=tHqx1qxA8q4

Important Books/Journals for further learning including the page nos:

1.A.Neel Armstrong – Transform and partial differential Equations , 2rd Edition, 2011,
Page.No : 3.97-3.121

Course Faculty

Verified by HOD